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CALCULATIONS OF RIEMANN PROBLEMS FOR 2-D SCALAR CONSERVATION LAWS BY SECOND ORDER ACCURATE MmB SCHEME *1)

Yang Shu-li
(*Institute of Applied Mathematics, Academia Sinica, Beijing, China*)

Abstract

Numerical solutions of Riemann problems for 2-D scalar conservation law are given by a second order accurate MmB (locally Maximum-minimum Bounds preserving) scheme which is non-splitting. The numerical computations show that the scheme has high resolution and non-oscillatory properties. The results are completely in accordance with the theoretical solutions and all cases are distinguished efficiently.

1. Introduction

The initial value problem for 2-D scalar conservation law is

$$u_t + f(u)_x + g(u)_y = 0 \tag{1}$$
$$u(x, y, 0) = u_0(x, y) \tag{2}$$

Define the region $\pi_T = [0, T) \times \mathbb{R}^2$, then the weak form and entropy condition of problem (1) (2) are

$$\int_{\pi_T} [u\phi_t + f(u)\phi_x + g(u)\phi_y] dxdydt + \int_{t=0} \phi u_0(x, y) dxdy = 0, \quad \forall \phi \in C_0^\infty(\pi_T) \tag{3}$$

and

$$\int_{\pi_T} sign(u-k)\{(u-k)\phi_t + (f(u)-f(k))\phi_x + (g(u)-g(k))\phi_y\} dxdydt \geq 0$$

$$\forall k \in \mathbb{R}, \quad \forall \phi \in C_0^\infty(\pi_T), \quad \phi \geq 0. \tag{4}$$

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We all know that in [1], existence and uniqueness of solution to problem (1)(2) have been obtained by using (3) and (4), and in [2], the two dimensional Riemann problem for (1)(2) has been solved analytically under the assumption

$$f''g''(f''/g'')' \neq 0.$$

There are many numerical methods for solving initial value problems of one dimensional conservation laws and practical problems. For all linear difference schemes, there exist two shortcomings that the solutions of first order accurate schemes are smoothed and the solutions of second order (or higher order) accurate schemes have oscillatory phenomena near discontinuities. In order to eliminate the shortcomings, several modified difference schemes (or nonlinear difference schemes) have been presented: Harten[3] proposed a second order accurate TVD difference scheme and gave a sufficient condition for the scheme to be TVD; Sweby[4] unified the work of van Leer[5], Roe[6] and Osher[7] and gave a class of second order accurate TVD schemes in the form of limiters; van Leer[8] constructed MUSCL scheme which generalized Godunov scheme to second order accuracy. In the previous papers, the TVD property of these schemes is valid only for one dimensional cases and has got great success to calculate problems of fluid dynamics. Unfortunately, in [9], it has proved that 2-D TVD scheme is at most of first order accuracy, although the splitting methods by using higher order one dimensional TVD schemes seem to work quite well for practical problems. In [10] [11] we presented a new MmB difference scheme which has high resolution, second order accurate and nonoscillatory properties, for initial value problems of 2-D scalar conservation laws.

In this paper, we briefly describe the theoretical solutions of two dimensional Riemann problems for scalar conservation law in section 2, and present the concept of MmB scheme and construct a class of second order accurate MmB schemes for 2-D conservation law in section 3; Finally, we show you all the configurations by numerical experiments.

2. Theoretical solutions

Consider problem (1)(2) and take initial data as follows,

$$u_0(x,y) = \begin{cases} u_1, & x>0,\ y>0 \\ u_2, & x<0,\ y>0 \\ u_3, & x<0,\ y<0 \\ u_4, & x>0,\ y<0 \end{cases}. \tag{5}$$

In [2], in order to construct the solutions of Riemann problem of (1)(5), the similarity transformation was used

$$\xi = x/t, \quad \eta = y/t$$

then $u(x,y,t) = \tilde{u}(\xi,\eta)$, (1) is exchanged as

$$(f'(\tilde{u}) - \xi)\tilde{u}_\xi + (g'(\tilde{u}) - \eta)\tilde{u}_\eta = 0.$$

Correspondingly, the initial data exchange to boundary data

$$\lim_{\substack{\xi/\eta = const \\ \xi^2+\eta^2 \to \infty}} \widetilde{u}\xi, \eta = \begin{cases} u_1, & \xi > 0, \ \eta > 0 \\ u_2, & \xi < 0, \ \eta > 0 \\ u_3, & \xi < 0, \ \eta < 0 \\ u_4, & \xi > 0, \ \eta < 0 \end{cases}$$

.

Here we assume that $f'' > 0$, $g'' > 0$, $(f''/g'') < 0$ and $u_2 < u_4$, otherwise, the same configuration can be obtained by some transformations. Therefore we get all the configurations of solution structure.

From the analysis of paper [2], five cases are classified as follows,

(a) none shock wave $u_3 \le u_2 \le u_4 \le u_1$.

(b) none rarefaction wave $u_1 \le u_2 \le u_4 \le u_3$.

(c) exactly one shock wave

$$c_1 \quad u_3 \le u_2 \le u_1 < u_4, \qquad c_2 \quad u_2 < u_3 \le u_4 \le u_1.$$

(d) exactly one rarefaction wave

$$d_1 \quad u_1 < u_2 < u_3 < u_4, \qquad d_2 \quad u_2 < u_1 < u_4 < u_3.$$

(e) exactly two shock waves and two rarefaction waves

$$e_{11} \quad u_2 < u_3 \le u_1 < u_4, \qquad e_{12} \quad u_2 < u_1 < u_3 < u_4,$$

$$e_{21} \quad u_3 \le u_1 < u_2 \le u_4, \qquad e_{22} \quad u_2 \le u_4 < u_3 \le u_1.$$

The descriptions in detail can be seen in [2].

3. MmB schemes

3.1. One dimensional case

Consider the initial value problem for 1-D scalar conservation law

$$u_t + f(u)_x = 0 \tag{6}$$

and the inconservative form

$$u_t + a(u)u_x = 0$$

where $a(u) = f_u$.

A difference scheme $u_j^{n+1} = Lu_j^n$ is called a MmB scheme if

$$min(u_{j-1}^n, u_j^n) \le u_j^{n+1} \le max(u_{j-1}^n, u_j^n) \tag{8a}$$

$$or \quad min(u_j^n, u_{j+1}^n) \le u_j^{n+1} \le max(u_j^n, u_{j+1}^n). \tag{8b}$$

Condition (8) is equivalent to the equation

$$
\begin{aligned}
&u_j^{n+1} = u_j^n - c_{j-\frac{1}{2}}(u_j^n - u_{j-1}^n) \\
or \quad &u_j^{n+1} = u_j^n + d_{j+\frac{1}{2}}(u_{j+1}^n - u_j^n)
\end{aligned}
$$

where $0 \le c_{j-\frac{1}{2}}, d_{j+\frac{1}{2}} \le 1$.

Take f(u)=au, the following upwind scheme is

$$
\begin{aligned}
&u_j^{n+1} = u_j^n - \lambda(u_j^n - u_{j-1}^n) \quad a > 0 \\
or \quad &u_j^{n+1} = u_j^n - \lambda(u_{j+1}^n - u_j^n) \quad a < 0.
\end{aligned}
$$

The above scheme is MmB if $0 \le |\lambda| \le 1$, where $\lambda = a\Delta t/\Delta x$.

Second order accurate MmB schemes are written to

$$
u_j^{n+1} = u_j^n - \lambda(u_j^n - u_{j-1}^n) - \frac{1}{2}\lambda(1-\lambda)[\phi_{j+\frac{1}{2}}^- \Delta u_{j+\frac{1}{2}}^n - \phi_{j-\frac{1}{2}}^- \Delta u_{j-\frac{1}{2}}^n], \quad a > 0
$$

or

$$
u_j^{n+1} = u_j^n - \lambda(u_{j+1}^n - u_j^n) + \frac{1}{2}\lambda(1+\lambda)[\phi_{j+\frac{1}{2}}^+ \Delta u_{j+\frac{1}{2}}^n - \phi_{j-\frac{1}{2}}^+ \Delta u_{j-\frac{1}{2}}^n]. \quad a < 0
$$

where $\Delta u_{j+\frac{1}{2}} = u_{j+1} - u_j$. In this case, MmB and TVD schemes are almost identical[4][10].

3.2. Two dimensional case

Consider the initial value problem for 2-D scalar conservation law

$$
u_t + f(u)_x + g(u)_y = 0
$$

and inconservative form

$$
u_t + a(u)u_x + b(u)u_y = 0
$$

where $a(u) = f_u$ and $b(u) = g_u$.

If a difference scheme $u_{i,j}^{n+1} = Lu_{i,j}^n$ satisfies

$$
\begin{aligned}
&min(u_{i,j}^n, u_{i-1,j}^n, u_{i,j-1}^n) \le u_{i,j}^{n+1} \le min(u_{i,j}^n, u_{i-1,j}^n, u_{i,j-1}^n) \\
or \quad &min(u_{i,j}^n, u_{i+1,j}^n, u_{i,j-1}^n) \le u_{i,j}^{n+1} \le min(u_{i,j}^n, u_{i+1,j}^n, u_{i,j-1}^n) \\
or \quad &min(u_{i,j}^n, u_{i-1,j}^n, u_{i,j+1}^n) \le u_{i,j}^{n+1} \le min(u_{i,j}^n, u_{i-1,j}^n, u_{i,j+1}^n) \\
or \quad &min(u_{i,j}^n, u_{i+1,j}^n, u_{i,j+1}^n) \le u_{i,j}^{n+1} \le min(u_{i,j}^n, u_{i+1,j}^n, u_{i,j+1}^n)
\end{aligned}
\tag{9}
$$

we call the scheme as a MmB scheme based on three points; If a scheme $u_{i,j}^{n+1} = Lu_{i,j}^n$ satisfies

$$
\begin{aligned}
&min(u_{i,j}^n, u_{i-1,j}^n, u_{i,j-1}^n, u_{i-1,j-1}^n) \le u_{i,j}^{n+1}) \le min(u_{i,j}^n, u_{i-1,j}^n, u_{i,j-1}^n, u_{i-1,j-1}^n) \\
or \quad &min(u_{i,j}^n, u_{i-1,j}^n, u_{i,j-1}^n, u_{i-1,j-1}^n) \le u_{i,j}^{n+1}) \le min(u_{i,j}^n, u_{i-1,j}^n, u_{i,j-1}^n, u_{i-1,j-1}^n) \\
or \quad &min(u_{i,j}^n, u_{i-1,j}^n, u_{i,j-1}^n, u_{i-1,j-1}^n) \le u_{i,j}^{n+1}) \le min(u_{i,j}^n, u_{i-1,j}^n, u_{i,j-1}^n, u_{i-1,j-1}^n) \\
or \quad &min(u_{i,j}^n, u_{i-1,j}^n, u_{i,j-1}^n, u_{i-1,j-1}^n) \le u_{i,j}^{n+1}) \le min(u_{i,j}^n, u_{i-1,j}^n, u_{i,j-1}^n, u_{i-1,j-1}^n)
\end{aligned}
\tag{10}
$$

we call the scheme as a MmB scheme based on four points.

Take f(u)=au and g(u)=bu, for $a > 0$ and $b > 0$, we derive the following schemes,

(I) Three point upwind scheme

$$u_{i,j}^{n+1} = u_{i,j}^n - \lambda^x \Delta^x u_{i-\frac{1}{2},j}^n - \lambda \Delta^y u_{i,j-\frac{1}{2}}^n \tag{11}$$

when $\lambda^x + \lambda^y \leq 1$, scheme (11) satisfies (9) and is MmB scheme based on three points. Where $\lambda^x = a\Delta t/\Delta x$ and $\lambda^y = \Delta t/\Delta y$.

(II) Four point upwind scheme

$$u_{i,j}^{n+1} = u_{i,j}^n - \lambda^x \Delta^x u_{i-\frac{1}{2},j}^n - \lambda^y \Delta u_{i,j-\frac{1}{2}}^n + \lambda^x \lambda^y \Delta^x \Delta^y u_{i-\frac{1}{2},j-\frac{1}{2}}^n \tag{12}$$

when $0\lambda^x, \lambda^y \leq 1$, scheme (12) satisfies condition (10) and is MmB scheme based on four points.

(III) Second order accurate non-splitting MmB scheme

$$\begin{aligned} u_{i,j}^{n+1} = \quad & u_{i,j}^n - \lambda^x \Delta^x u_{i-\frac{1}{2},j}^n - \frac{1}{2}\lambda^x(1-\lambda^x)\{\phi_{i+\frac{1}{2},j}^{x,-} \Delta^x u_{i+\frac{1}{2},j}^n - \phi_{i-\frac{1}{2},j}^{x,-} \Delta^x u_{i-\frac{1}{2},j}^n\} \\ & -\lambda^y \Delta u_{i,j-\frac{1}{2}}^n - \lambda^y(1-\lambda^y)\{\phi_{i,j+\frac{1}{2}}^{y,-} \Delta^y u_{i,j+\frac{1}{2}}^n - \phi_{i,j-\frac{1}{2}}^{y,-} \Delta^y u_{i,j-\frac{1}{2}}^n\} \\ & +\lambda^x \lambda^y \Delta^x \Delta^y u_{i-\frac{1}{2},j-\frac{1}{2}}^n. \end{aligned} \tag{13}$$

where $\Delta^x u_{i+\frac{1}{2},j} = u_{i+1,j} - u_{i,j}$, $\Delta^y u_{i,j+\frac{1}{2}} = u_{i,j+1} - u_{i,j}$ and $\phi_{i+\frac{1}{2},j}^{x,-} = \phi(r_{i+\frac{1}{2},j}^{x,-})$, $\phi_{i,j+\frac{1}{2}}^{y,-} = \phi(r_{i,j+\frac{1}{2}}^{y,-})$. When $r^x, r^y \leq 0$, let $\phi^x, \phi^y \equiv 0$, scheme (13) is MmB if

$$0 \leq (\phi^x/r^x, \phi^x) \leq min(\frac{1}{\nu_x} - \frac{1-\alpha\nu_y}{1-\nu_x}, \frac{2(1-2\nu_y)}{1-\nu_x})$$
$$0 \leq (\phi^y/r^y, \phi^y) \leq min(\frac{1}{\nu_y} - \frac{1-\alpha\nu_x}{1-\nu_y}, \frac{2(1-2\nu_x)}{1-\nu_y})$$

where $\nu_x = a\frac{\Delta t}{\Delta x}$, $\nu_y = b\frac{\Delta t}{\Delta y}$, and $\nu_x, \nu_y \leq \frac{1}{4}$.

(IV) Second order accurate splitting MmB schemes

Theorem 1. *If L_x, L_y are 1-D MmB operators, then*

$$L_x L_y, \quad L_y L_x, \quad L_y L_x L_x L_y, \quad L_x L_y L_y L_x$$

are 2-D MmB operators.

Proof of theorem 1 can be seen in [10][12]. According to the theorem, the splitting schemes by the combinations of 1-D MmB operators L_x and L_y are all MmB schemes based on four points.

For general cases of problem (1)(2), nonsplitting second order accurate MmB schemes can be written to the following conservative form,

$$u_{i,j}^{n+1} = u_{i,j}^n - \lambda^x(f_{i+\frac{1}{2},j}^n - f_{i-\frac{1}{2},j}^n) - \lambda^y(g_{i,j+\frac{1}{2}}^n - g_{i,j-\frac{1}{2}}^n), \tag{14}$$

where $\lambda^x = \Delta t/\Delta x$, $\lambda^y = \Delta t/\Delta y$.

$$\begin{aligned} f_{i+\frac{1}{2},j}^n = \quad & \frac{1}{2}[f_{i,j}^n + f_{i+1,j}^n - \frac{1}{\lambda^x}\overline{\mu}_{i+\frac{1}{2},j}^x \Delta^x u_{i+\frac{1}{2},j}^n \\ & -\frac{\alpha}{\lambda^x}(\mu_{i,j-\frac{1}{2}}^{x,+}\mu_{i,j-\frac{1}{2}}^{y,+}\Delta^y u_{i,j-\frac{1}{2}}^n + \mu_{i,j+\frac{1}{2}}^{x,+}\mu_{i,j+\frac{1}{2}}^{y,-}\Delta^y u_{i,j+\frac{1}{2}}^n \\ & +\mu_{i+1,j-\frac{1}{2}}^{x,-}\mu_{i+1,j-\frac{1}{2}}^{y,-}\Delta^y u_{i+1,j-\frac{1}{2}}^n + \mu_{i+1,j+\frac{1}{2}}^{x,-}\mu_{i+1,j+\frac{1}{2}}^{y,-}\Delta^y u_{i+1,j+\frac{1}{2}}^n)] \end{aligned}$$

$$
\begin{aligned}
g^n_{i,j+\frac{1}{2}} = \; & \frac{1}{2}[g^n_{i,j} + g^n_{i,j+1} - \frac{1}{\lambda^y}\overline{\mu}^y_{i,j+\frac{1}{2}}\Delta^y u^n_{i,j+\frac{1}{2}} \\
& -\frac{\alpha}{\lambda^y}(\mu^{y,+}_{i-\frac{1}{2},j}\mu^{x,-}_{i-\frac{1}{2},j}\Delta^x u^n_{i-\frac{1}{2},j} + \mu^{y,+}_{i+\frac{1}{2},j}\mu^{x,-}_{i+\frac{1}{2},j}\Delta^x u^n_{i+\frac{1}{2},j} \\
& +\mu^{y,-}_{i-\frac{1}{2},j+1}\mu^{x,-}_{i-\frac{1}{2},j+1}\Delta^x u^n_{i-\frac{1}{2},j+1} + \mu^{y,-}_{i+\frac{1}{2},j+1}\mu^{x,-}_{i+\frac{1}{2},j+1}\Delta^x u^n_{i+\frac{1}{2},j+1})]
\end{aligned}
$$

$$
\begin{aligned}
&\overline{\mu}^x_{i+\frac{1}{2},j} = (\mu^x_{i+\frac{1}{2},j})^2 + |\mu^x_{i+\frac{1}{2},j}|(1 - |\mu^x_{i+\frac{1}{2},j}|)(1 - \phi^x_{i+\frac{1}{2},j}) \\
&\overline{\mu}^y_{i,j+\frac{1}{2}} = (\mu^y_{i,j+\frac{1}{2}})^2 + |\mu^y_{i,j+\frac{1}{2}}|(1 - |\mu^y_{i,j+\frac{1}{2}}|)(1 - \phi^y_{i,j+\frac{1}{2}}) \\
&\mu^x_{i+\frac{1}{2},j} = a^x_{i+\frac{1}{2},j}\lambda^x, \qquad \mu^y_{i,j+\frac{1}{2}} = b^y_{i,j+\frac{1}{2}}\lambda^y \\
&a^x_{i+\frac{1}{2},j} = \begin{cases} \dfrac{f_{i+1,j} - f_{i,j}}{u_{i+1,j} - u_{i,j}}, & u_{i,j} \neq u_{i+1,j} \\ (f_u)_{i,j} & u_{i,j} = u_{i+1,j} \end{cases} \\
&b^x_{i,j+\frac{1}{2}} = \begin{cases} \dfrac{g_{i,j+1} - g_{i,j}}{u_{i,j+1} - u_{i,j}}, & u_{i,j} \neq u_{i,j+1} \\ (g_u)_{i,j} & u_{i,j} = u_{i+1,j} \end{cases} \\
&\mu^{x,\pm}_{i+\frac{1}{2},j} = a^{x,\pm}_{i+\frac{1}{2},j}\lambda^x, \qquad \mu^{y,\pm}_{i,j+\frac{1}{2}} = b^{y,\pm}_{i,j+\frac{1}{2}}\lambda^y
\end{aligned}
$$

$$
a^+ = \frac{1}{2}(a + |a|) \qquad a^- = \frac{1}{2}(a - |a|)
$$

$$
\begin{aligned}
&\phi^x_{i+\frac{1}{2},j} = \phi(r^{x,-}_{i+\frac{1}{2},j}, r^{x,+}_{i+\frac{1}{2},j}), \quad \phi^y_{i,j+\frac{1}{2}} = \phi(r^{y,-}_{i,j+\frac{1}{2}}, r^{y,+}_{i,j+\frac{1}{2}}) \\
&r^{x,+}_{i+\frac{1}{2},j} = \frac{\Delta u_{i+\frac{3}{2},j}}{\Delta u_{i+\frac{1}{2},j}}, \quad r^{x,-}_{i+\frac{1}{2},j} = \frac{\Delta u_{i-\frac{1}{2},j}}{\Delta u_{i+\frac{1}{2},j}} \\
&r^{y,+}_{i,j+\frac{1}{2}} = \frac{\Delta u_{i,j+\frac{3}{2}}}{\Delta u_{i,j+\frac{1}{2}}}, \quad r^{y,-}_{i,j+\frac{1}{2}} = \frac{\Delta u_{i,j-\frac{1}{2}}}{\Delta u_{i,j+\frac{1}{2}}}.
\end{aligned}
$$

As the study of MmB schemes for the above linear equation, when $\alpha = 0$, schemes (14) are MmB schemes based on three points. In this case, MmB properties were proved in [10]; When $\alpha = 1$, schemes (14) are MmB schemes based on four points. Since it is very difficult to construct a two dimensional total variation norm and make a high order accurate scheme to be TVD, we must consider the local properties of schemes, Hence it is very reasonable for us to present MmB schemes.

4. Numerical experiments

We calculate the two dimensional Riemann problems of the scalar conservation law

$$
u_t + (u^2)_x + (u^3)_y = 0.
$$

The initial data are as (5)

Here, we choose $\Delta x = \Delta y = 0.1$ and $\Delta t = 0.01$ and give all configurations by using a nonsplitting MmB scheme with

$$
\phi(r) = max(0, min(1, r))
$$

and the time step of the results is n=60, S (or R)-(i-j) expresses the discontinuous line of i state and j state. From the calculations, the results are completely in accordance with the theoretical solutions and the MmB schemes have high resolution and non-oscillatory properties.

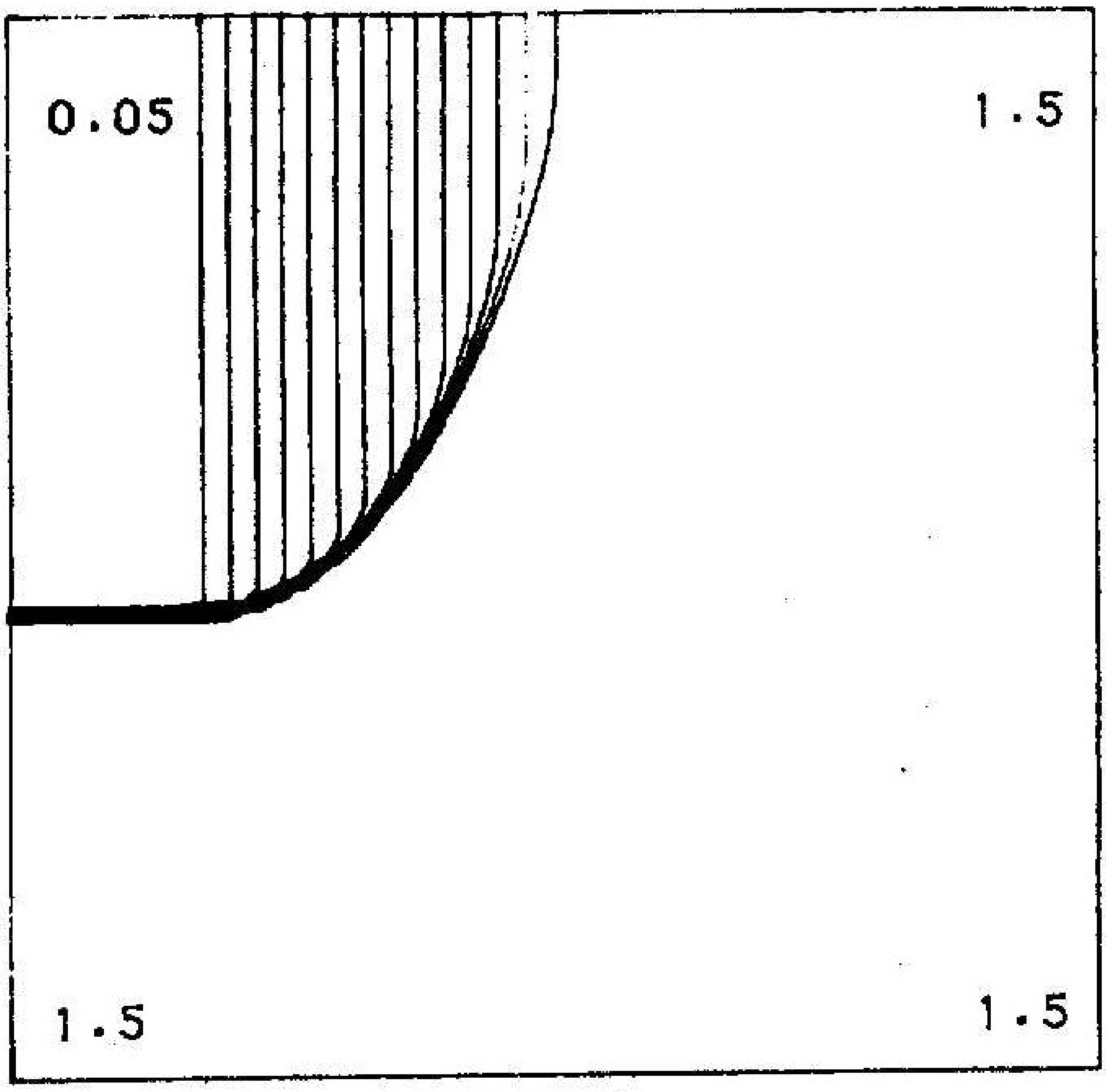

Fig.1 R-(1-2), S-(2-3)

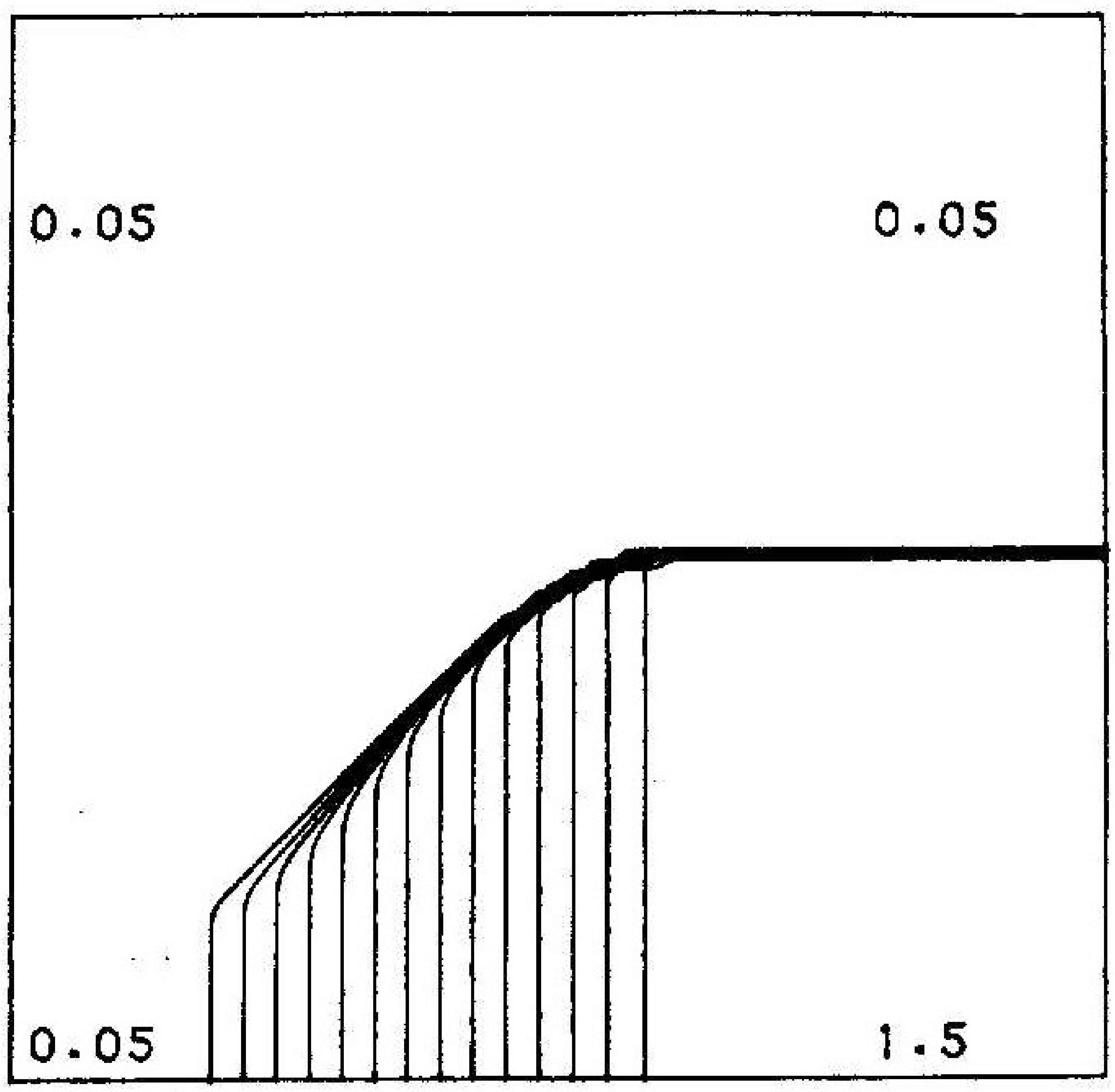

Fig.2 R-(3-4), S-(4-1)

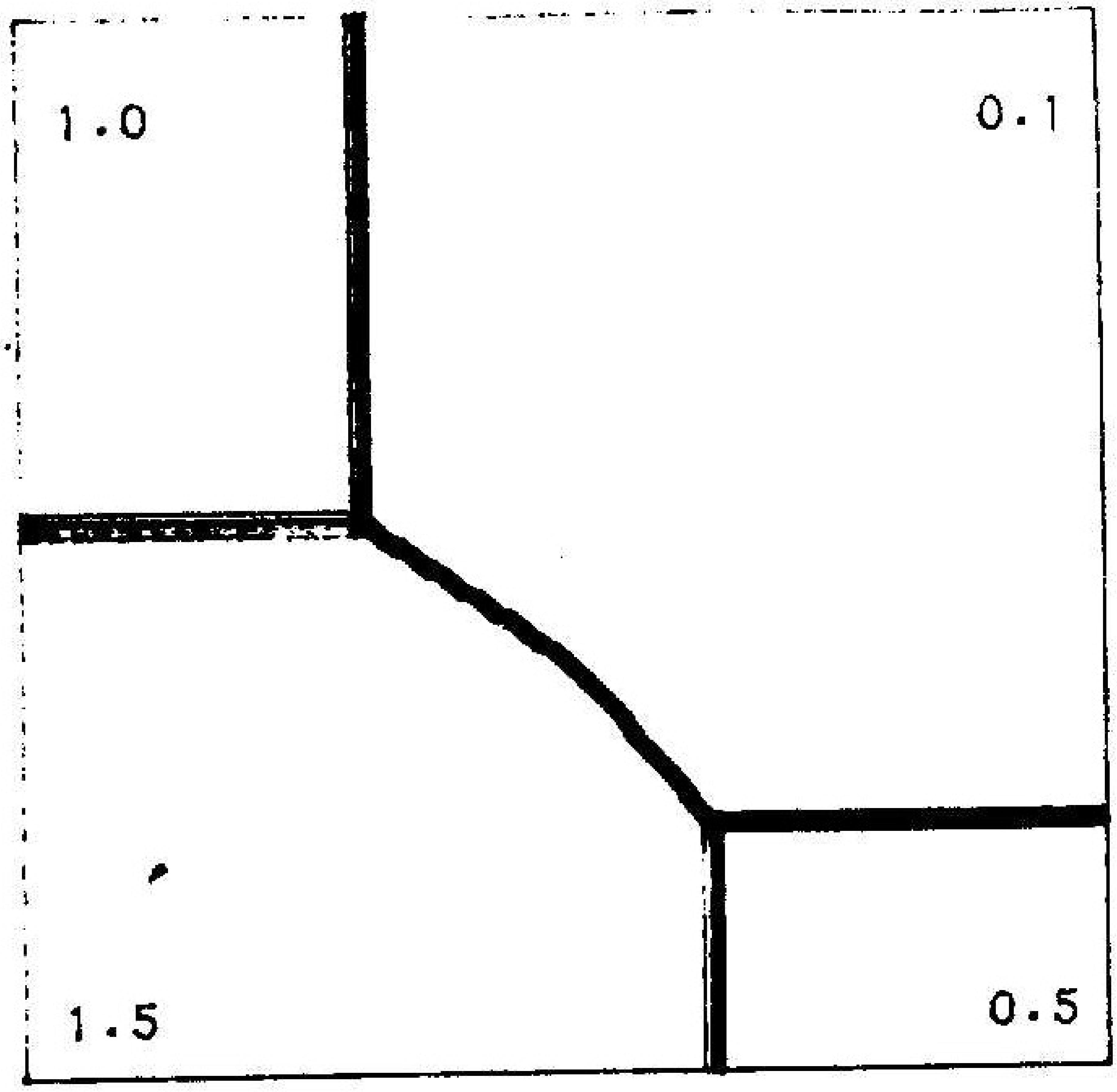

Fig.3 S-(1-2), S-(2-3), S-(3-4), S-(4-1)

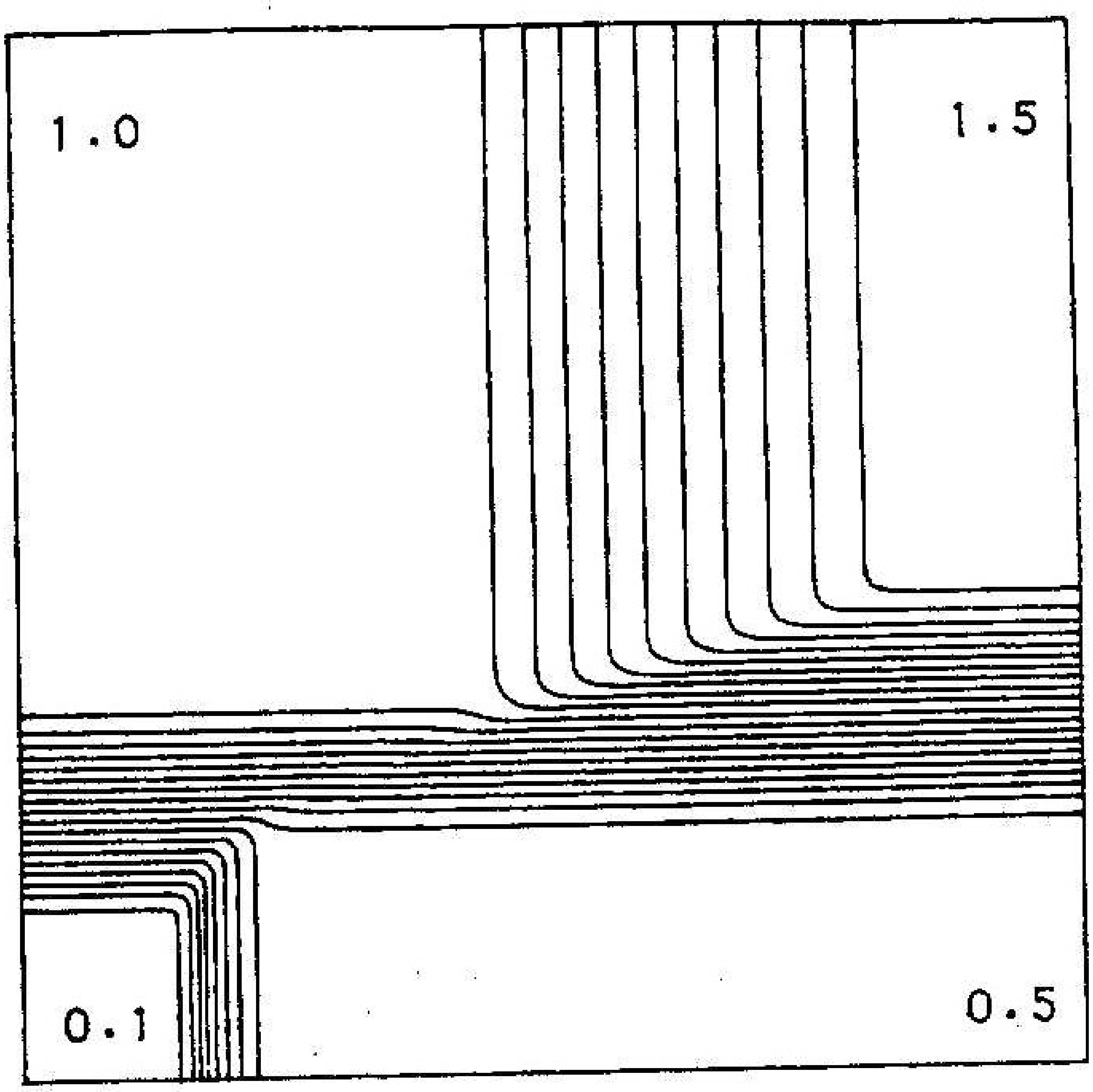

Fig.4 R-(1-2), R-(2-3), R-(3-4), R-(4-1)

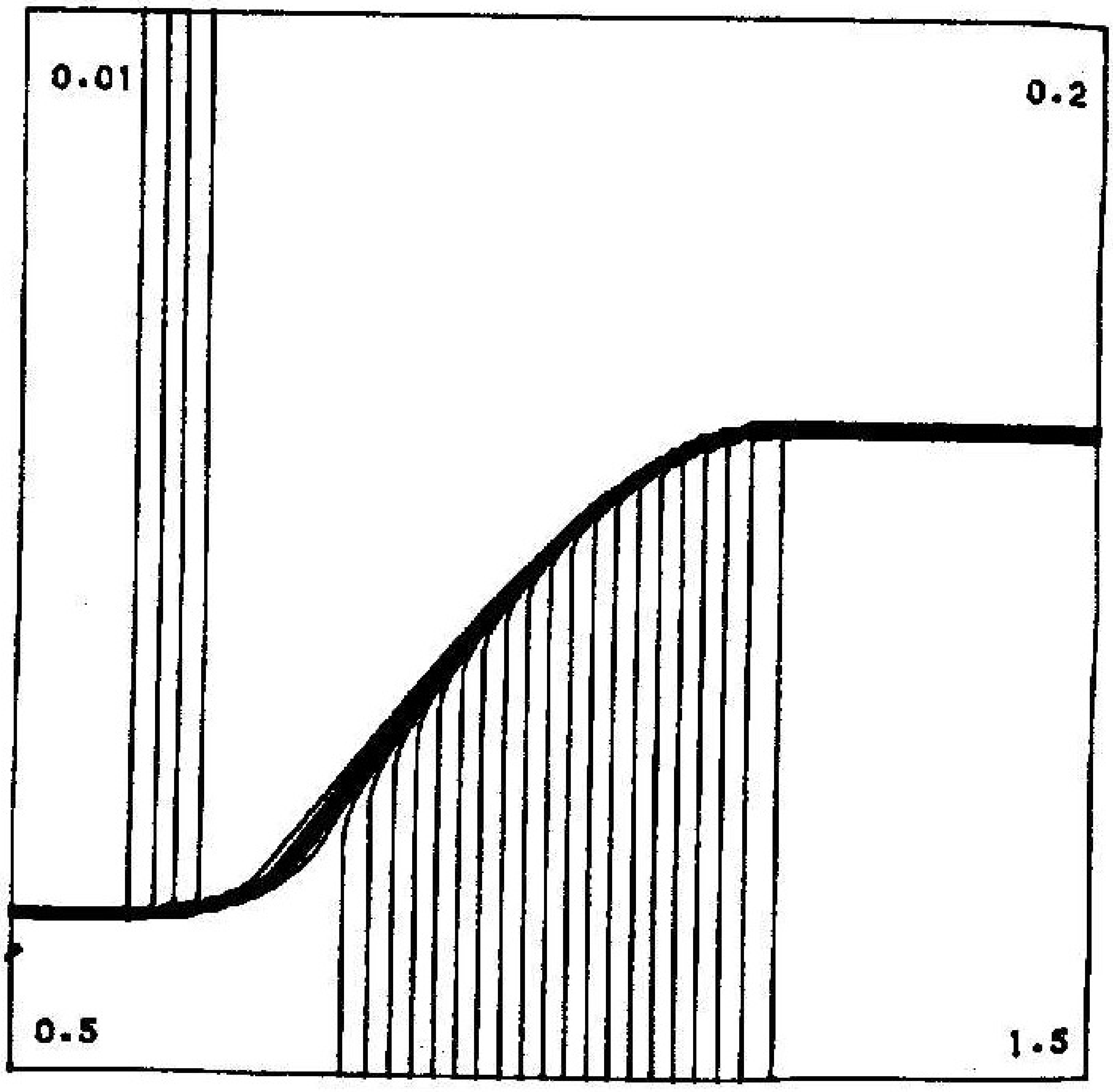

Fig.5a R-(1-2), R-(2-3), R-(3-4), S-(4-1)

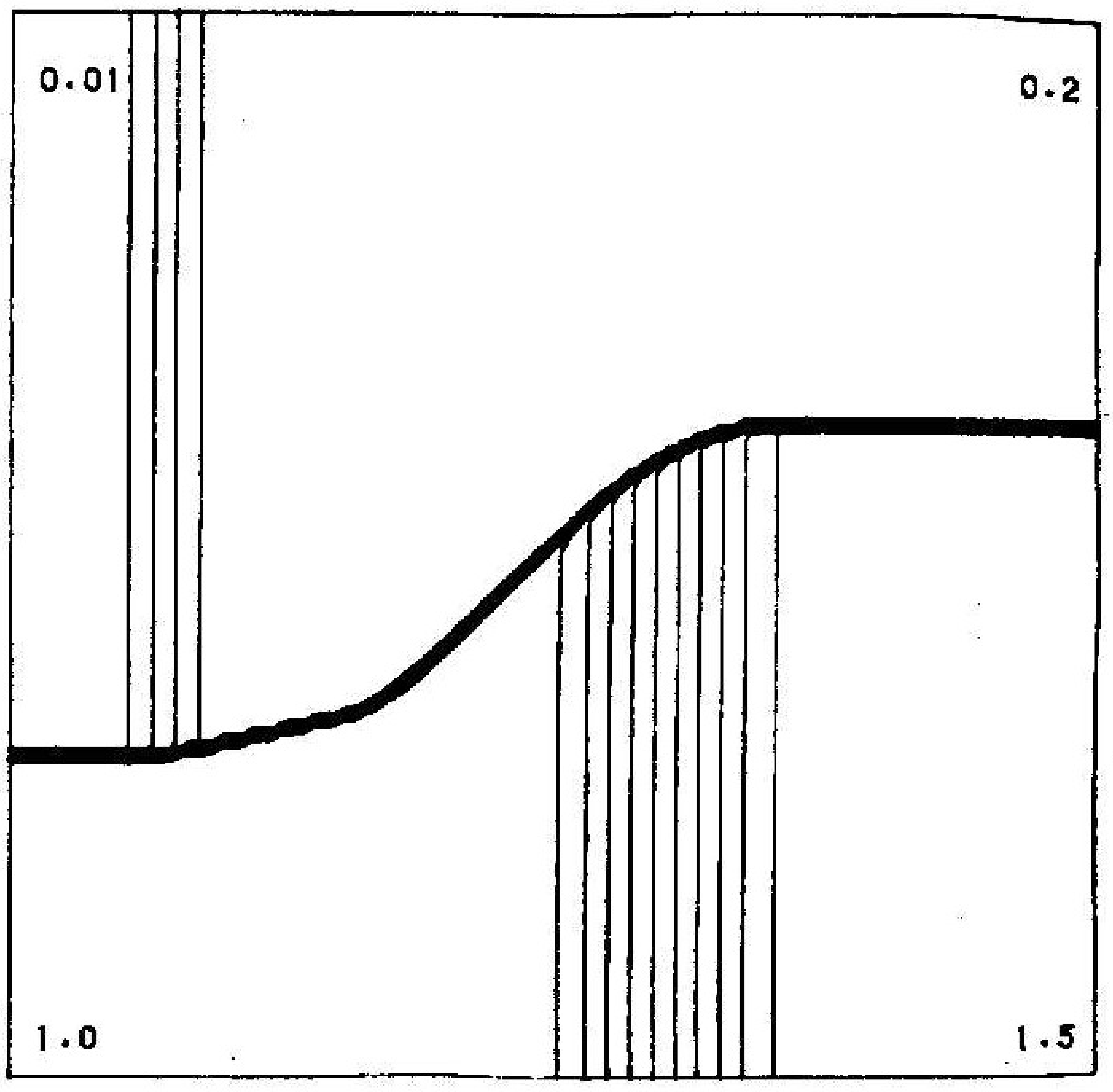

Fig.5b R-(1-2), S-(2-3), R-(3-4), S-(4-1)

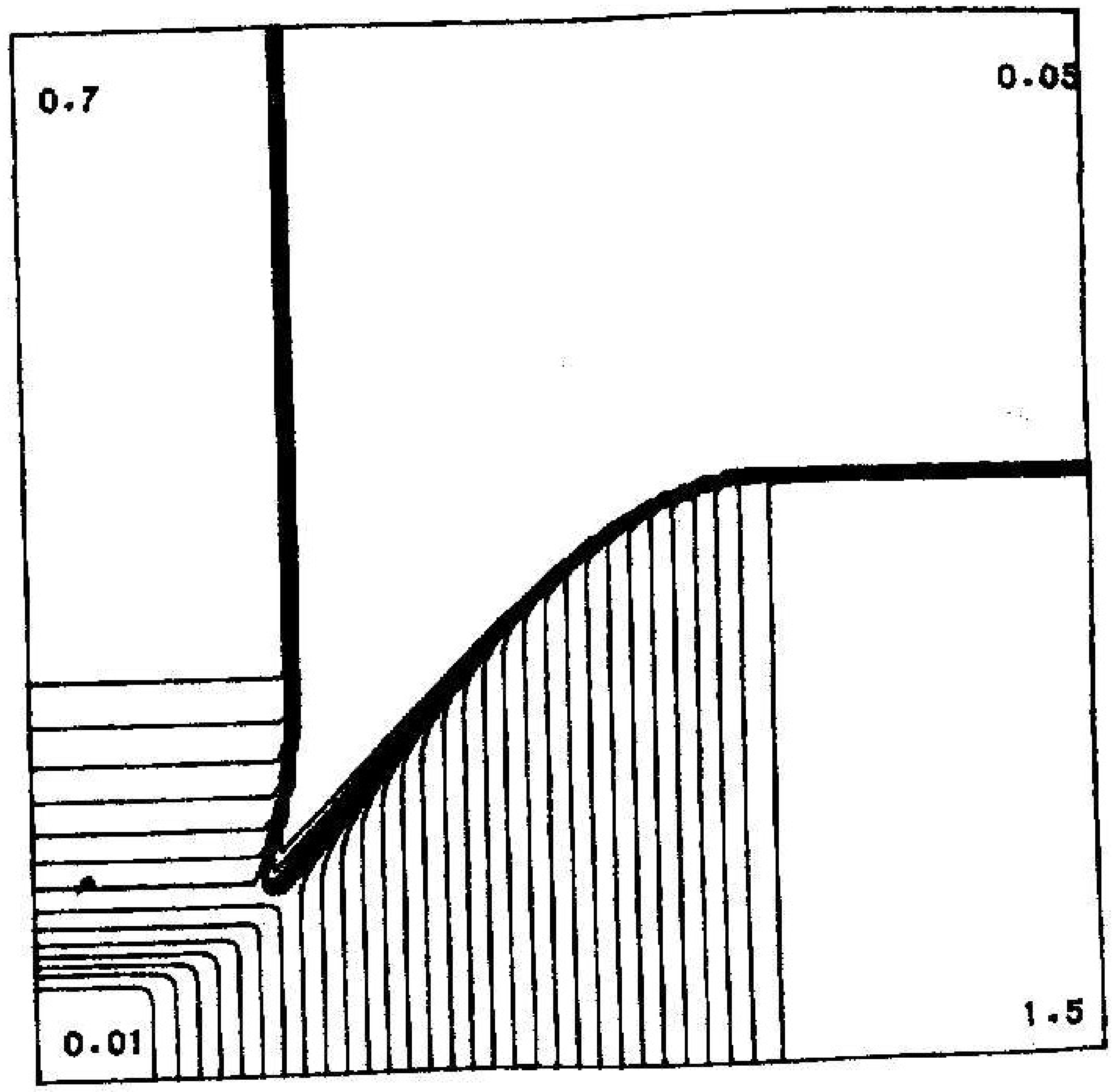

Fig.6a S-(1-2), R-(2-3), R-(3-4), S-(4-1)

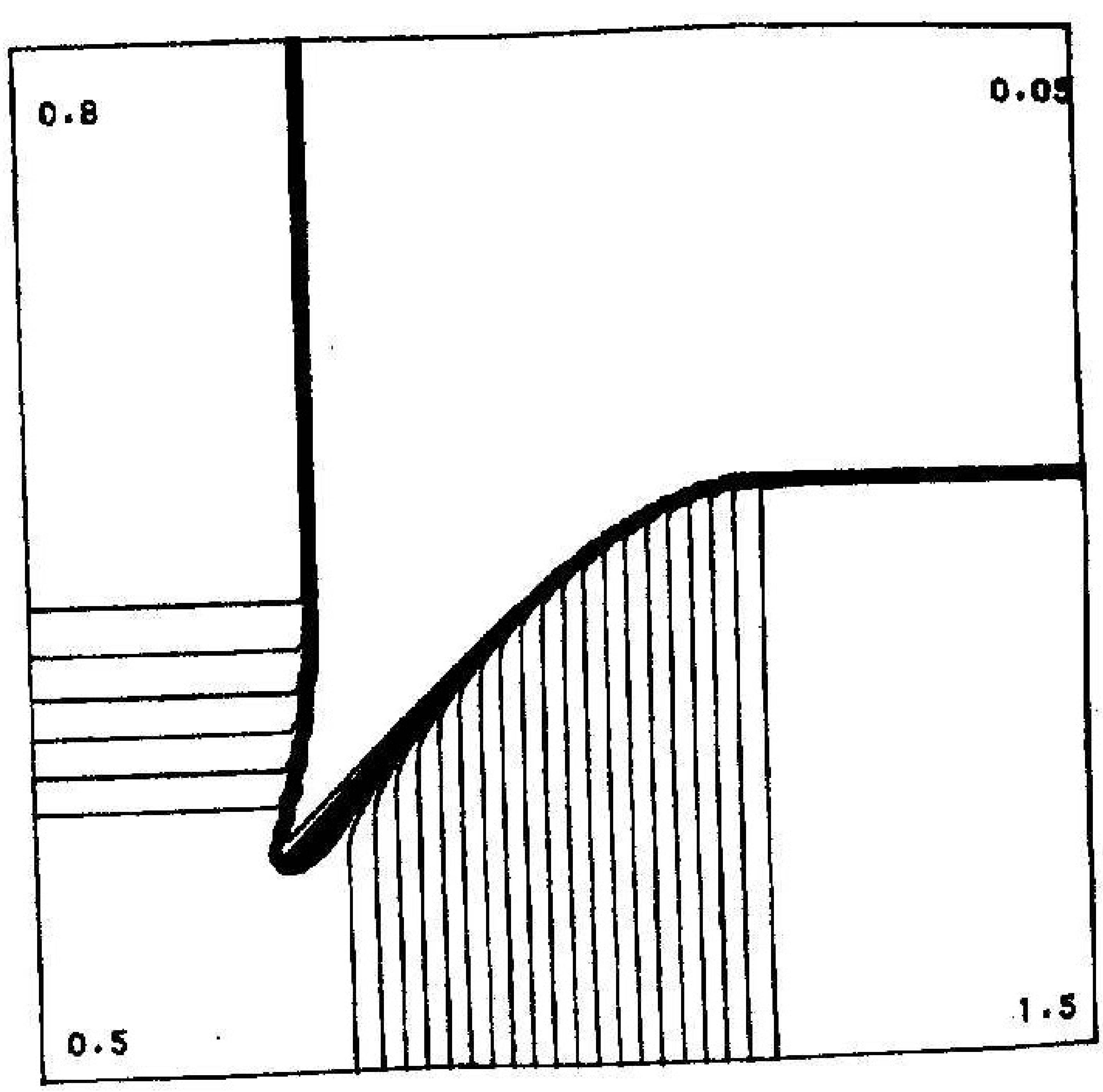

Fig.6b S-(1-2), R-(2-3), R-(3-4), S-(4-1)

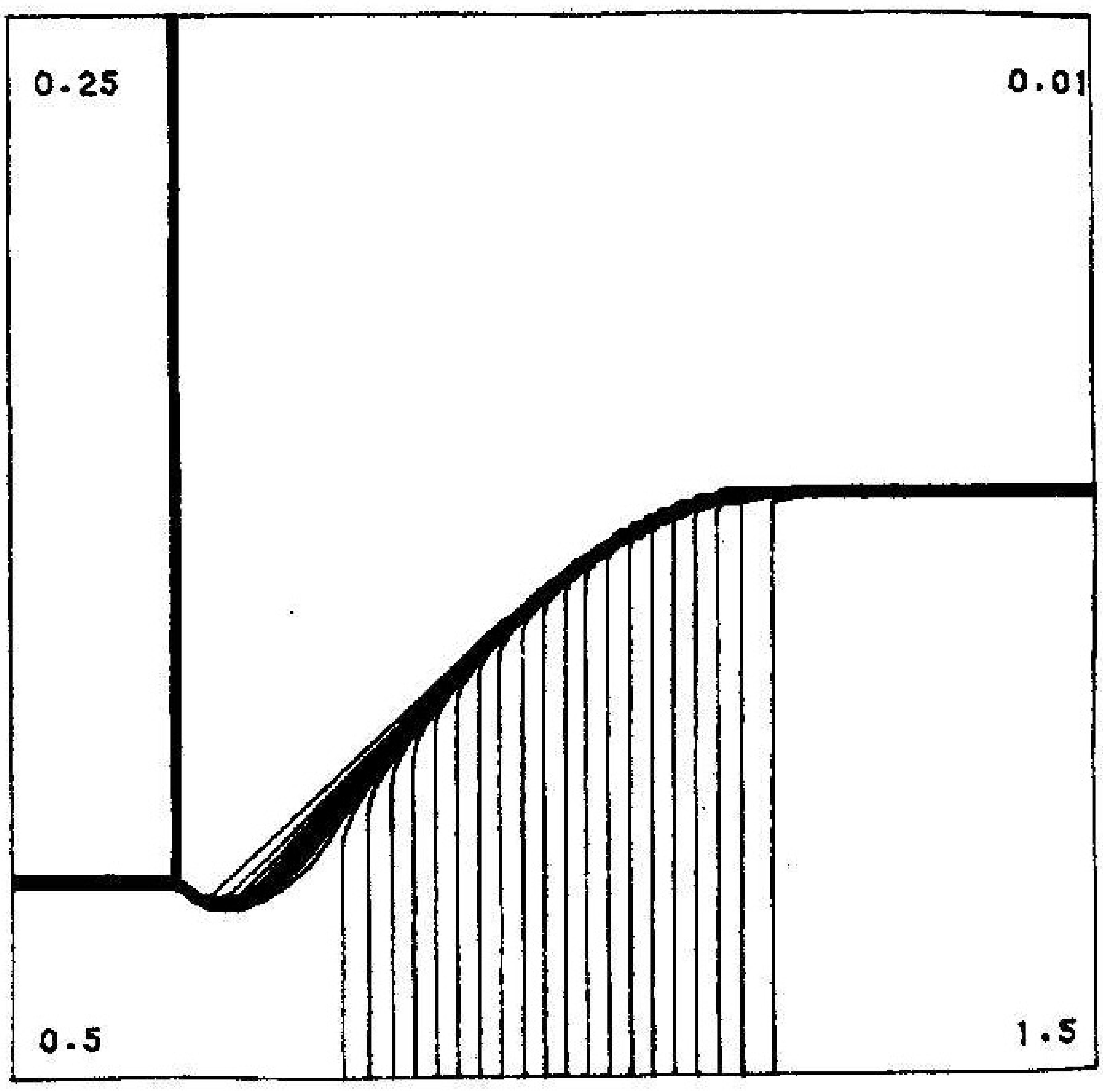

Fig.7a S-(1-2), S-(2-3), R-(3-4), S-(4-1)

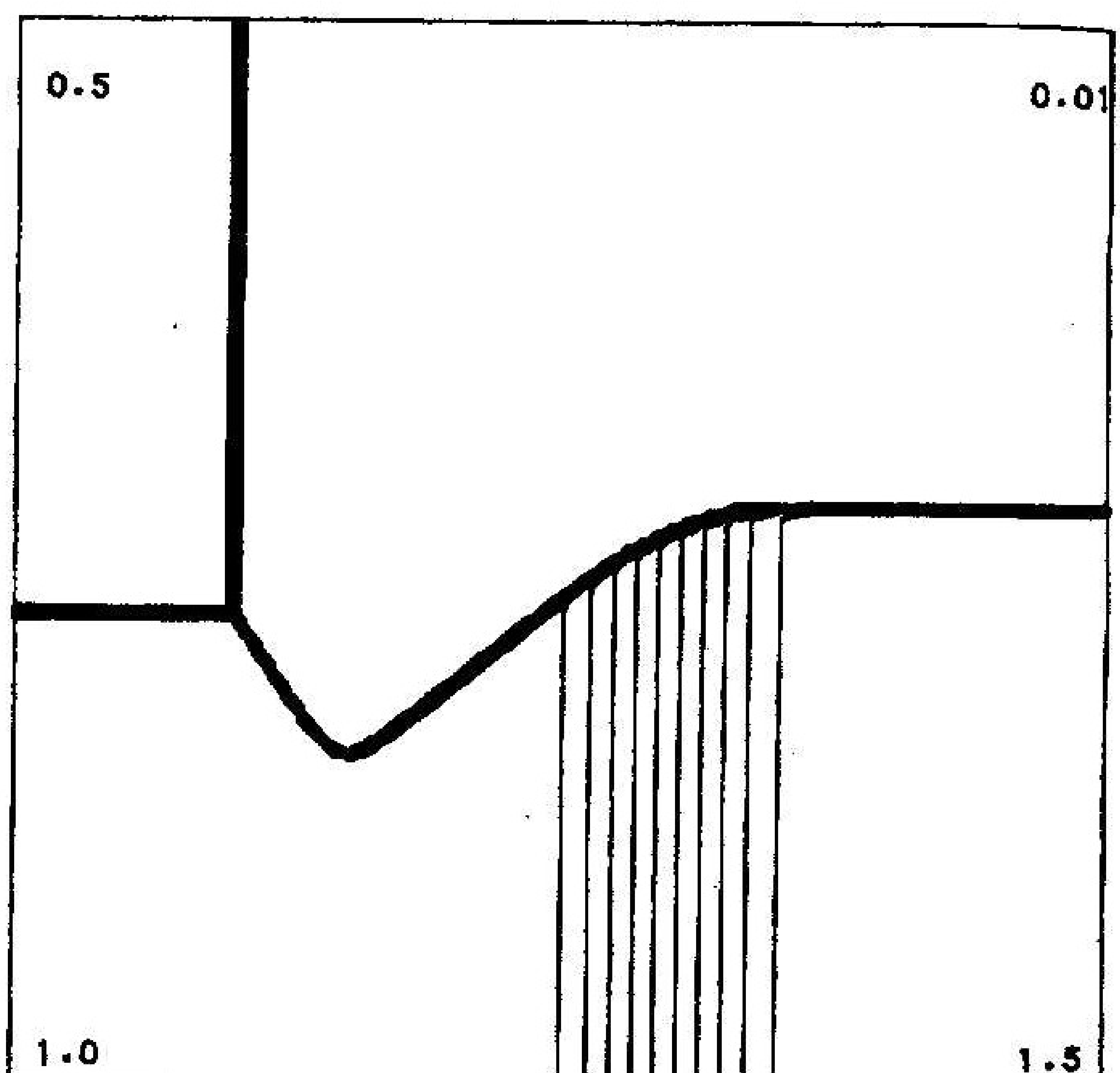

Fig.7b S-(1-2), S-(2-3), R-(3-4), S-(4-1)

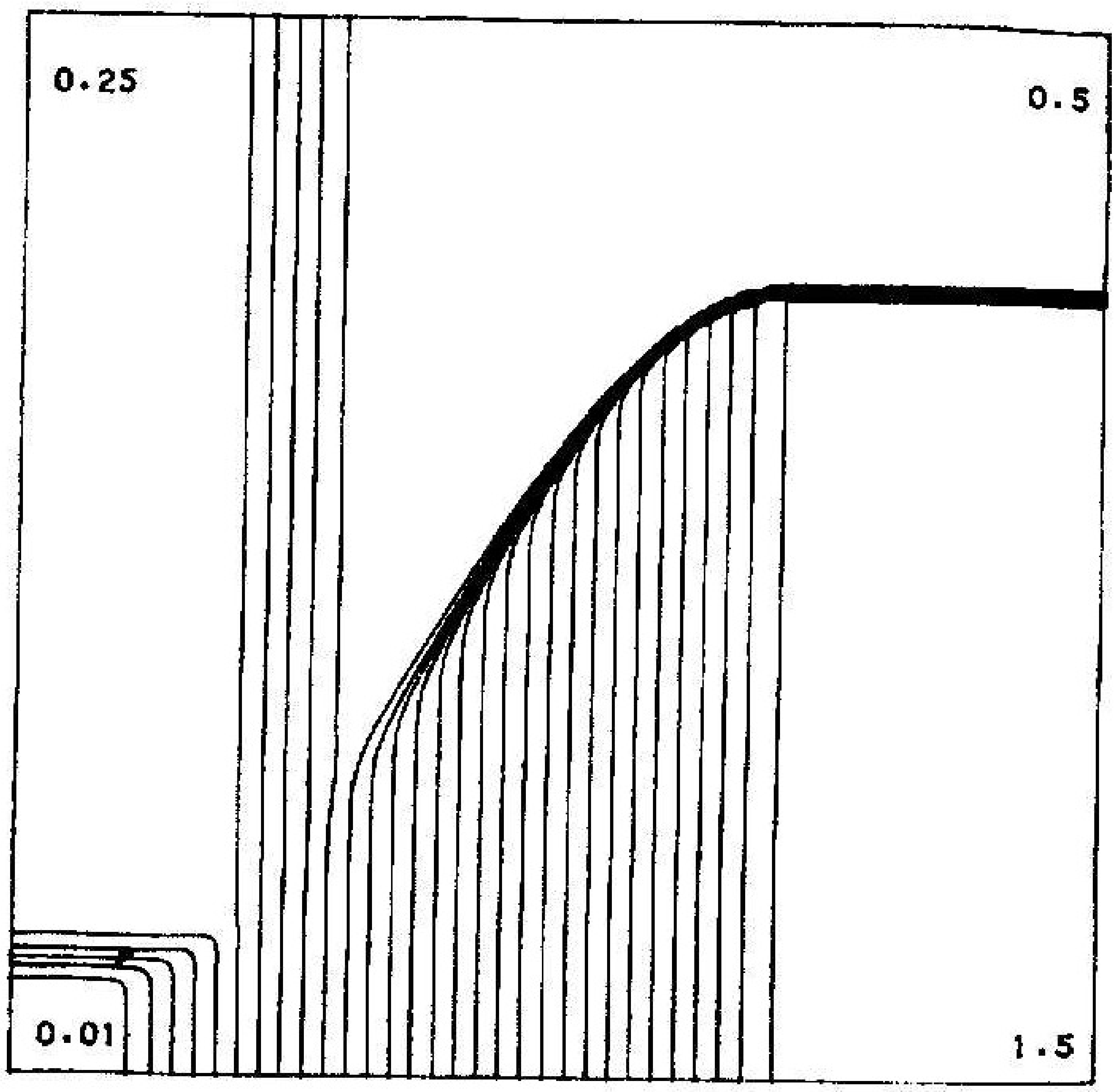

Fig.8a R-(1-2), R-(2-3), R-(3-4), S-(4-1)

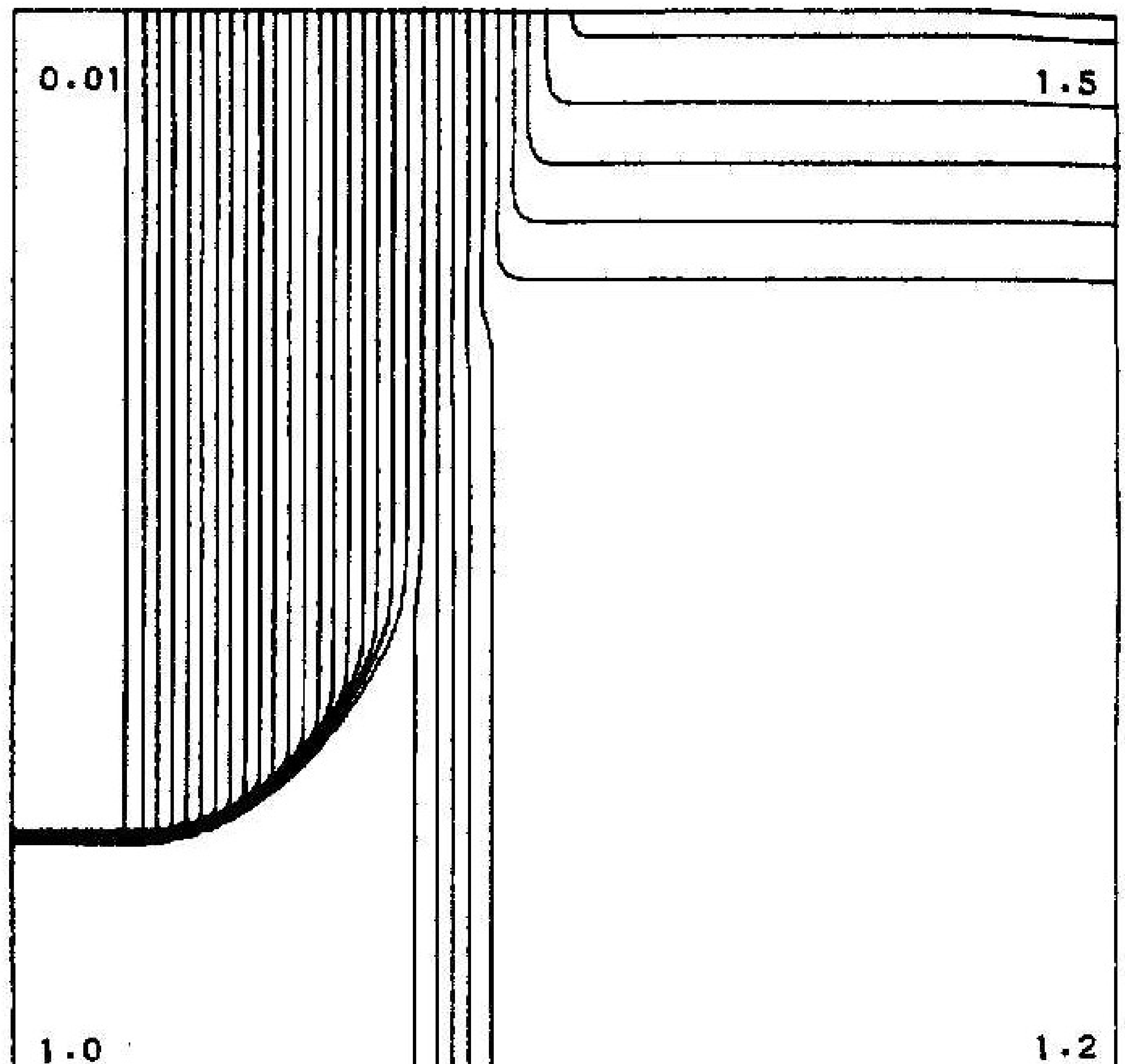

Fig.8b R-(1-2), S-(2-3), R-(3-4), S-(4-1)

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