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REGULARITY OF BIRKHOFF INTERPOLATION*1)

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Abstract

A comparison theorem concerning the regularity of Birkhoff interpolation is given. As an application of this theorem the regularity of $(0, 1, \cdots, p-1, p+1, \cdots, M-1, q)$ interpolation $(0 < p < M \le q)$ is characterized.

1. Introduction

The following definitions and notations are taken from [1, pp. 2–3].

Let $G = \{g_0, g_1, \cdots, g_N\}$ be a system of linearly independent, m times continuously differentiable functions on $[-1, 1]$. A matrix

$$E = [e_{ik};\quad i = 1, 2, \cdots, n,\quad k = 0, 1, \cdots, m],\quad n \ge 1,\quad m \ge 0 \tag{1.1}$$

is called an interpolation matrix if its elements e_{ik} are 0 or 1 and if the number of 1's in E is equal to $N+1$, $|E| = \sum e_{ik} = N+1$. Let X denote a set of knots

$$1 \ge x_1 > x_2 > \cdots > x_n \ge -1. \tag{1.2}$$

A Birkhoff interpolation problem E, X (with respect to G) is, given a set of data c_{ik} (defined for $e_{ik} = 1$) to determine a polynomial $P = \sum_{j=0}^N a_j g_j$ (if any) such that

$$P^{(k)}(x_i) = c_{ik},\quad e_{ik} = 1,\quad e_{ik} \in E. \tag{1.3}$$

The pair E, X is called regular if the system of equations (1.3) has a unique solution for each given set of c_{ik}; otherwise the pair E, X is singular. The matrix E is called order regular if the pair E, X is regular for any ordered set of knots X. Since the system (1.3) consists of $N+1$ linear equations with $N+1$ unknowns a_j, a pair E, X is regular if and only if the determinant of the system

$$D(E,X) := D(E,X; g_0, \cdots, g_N) = \det\,[g_0^{(k)}(x_i), \cdots, g_N^{(k)}(x_i);\quad e_{ik} = 1,\quad e_{ik} \in E] \tag{1.4}$$

is nonzero; or equivalently, a pair E, X is singular if and only if some nontrivial polynomial P $\in$span G is annihilated by E, X, i.e., P satisfies the homogeneous equations

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$P^{(k)}(x_i) = 0$ for $e_{ik} = 1$. We order the pair E, X in (1.4) lexicographically [1, p.3]. By $A(E, X)$ we denote the $(N+1) \times (N+1)$ matrix that appears in (1.4).

A function $P_{ik} = \sum_{j=0}^{N} a_j g_j$ with $e_{ik} = 1$ and $e_{ik} \in E$ is said to be a fundamental function for the pair E, X if

$$P_{ik}^{(\mu)}(x_\nu) = \delta_{i\nu}\delta_{k\mu}, \quad e_{\nu\mu} = 1, \; e_{\nu\mu} \in E. \tag{1.5}$$

Clearly the determinant (1.4) is often very complicated; it is difficult to claim whether or not $D(E, X)$ vanishes. Thus simplification of $D(E, X)$ is of important interest.

One of the objects of this paper is to establish a comparison theorem, which makes it possible to decrease the order of $D(E, X)$ and to simplify $D(E, X)$ (Section 2). Then, in Section 3, we apply this theorem to $(0, 1, \cdots, p-1, p+1, \cdots, M-1, q)$ interpolation $(0 < p < M \leq q \leq N-n+1)$. (Here we agree that such a interpolation is $(0, 1, \cdots, M-2, q)$ interpolation when $p = M-1$.) That is the problem E, X, where E is the $n \times (N+1)$ matrix with

$$e_{ik} = \begin{cases} 1, & i = 1, 2, \cdots, n, \quad k = 0, 1, \cdots, p-1, p+1, \cdots, M-1, q, \\ 0, & \text{otherwise}. \end{cases} \tag{1.6}$$

In what follows we restrict ourselves to the case when span $G = \mathbf{P}_N$, the set of algebraic polynomials of degree at most N. In this case we can assume that $m \leq N$, and by adding zero columns if necessary, we can make $m = N$. Such a matrix we shall call normal.

In the following we have to apply a theorem several times proved by Atkinson and Sharma [1, Theorem 1.5, p. 10]. For the sake of convenience we shall state it here. To this end we need some further definitions from [1, pp. 7-9].

For normal matrices the condition

$$\sum_{k=0}^{s} \sum_{i=1}^{n} e_{ik} \geq s+1, \quad s = 0, 1, \cdots, N \tag{1.7}$$

is called the Pólya condition. A sequence of 1's of the ith row of E is supported if that (i, k) is the position of the first 1 of the sequence implies that there exist two 1's: $e_{i_1,k_1} = e_{i_2,k_2} = 1$ with $i_1 < i < i_2$, $k_1 < k$, and $k_2 < k$. Then we have

Theorem A. *A normal interpolation matrix is order regular for algebraic interpolation if it satisfies the Pólya condition and contains no odd supported sequences.*

2. A Comparison Theorem

Let E, E_1, and E_2 be $n \times (N+1)$ matrices, not necessarily normal, the elements in which take 1 or 0. We write $E = E_1 + E_2$ if it stands for the ordinary addition of matrices. The main result in this section is the following theorem, a special case of which can be found in [1, Theorem 8.1, p. 101].

Theorem 1. *Let $n \times (N+1)$ normal interpolation matrices E and E' have the forms*

$$E = E_1 + E_2, \quad E' = E_1 + E_2', \quad |E_2| = |E_2'| = s, \quad 1 \le s \le N. \tag{2.1}$$

Assume that the pair E', X is regular. Then the following statements are equivalent:

(a) *The pair E, X is regular;*

(b) *There exist s fundamental functions $r_{ik} \in \mathbf{P}_N$, $e_{ik} = 1$, $e_{ik} \in E_2$, for the pair E, X;*

(c) *There exist s linearly independent functions $h_i \in \mathbf{P}_N$, $i = 0, 1, \cdots, s-1$, annihilated by E_1, X, such that*

$$D(E_2, X; h_0, \cdots, h_{s-1}) := \det[h_0^{(k)}(x_i), \cdots, h_{s-1}^{(k)}(x_i); \quad e_{ik} = 1, \quad e_{ik} \in E_2] \tag{2.2}$$

is nonzero.

Proof.

(a)$\Longrightarrow$(b) Trivial.

(b)$\Longrightarrow$(c) We order the s functions r_{ik}, $e_{ik} = 1$, $e_{ik} \in E_2$ lexicographically and name them $h_0, \cdots, h_{s-1}$. If we order the pair in (2.2) in the same order then by (1.5) the matrix that appears in (2.2) is the unit matrix which implies that the determinant (2.2) is nonzero.

(c)$\Longrightarrow$(a) Let $f_{ik} \in \mathbf{P}_N$, $e_{ik} = 1$, $e_{ik} \in E'$ be the fundamental functions for the pair E', X. Then f_{ik}'s must exist uniquely, since the pair E', X is regular. Meanwhile f_{ik}'s span the set $\mathbf{P}_N$. We rewrite the functions f_{ik}, $e_{ik} = 1$, $e_{ik} \in E_2'$ as $g_0, \cdots, g_{s-1}$ in their lexicographic order and f_{ik}, $e_{ik} = 1$, $e_{ik} \in E_1$ as $g_s, \cdots, g_N$ in their lexicographic order, too. Thus the pair E, X is regular if and only if the determinant $D(E, X)$ is nonzero.

We rearrange the corresponding matrix $A(E, X)$ to have as the first s rows those corresponding to $e_{ik} = 1$ from E_2 in their lexicographic order, then rows corresponding to $e_{ik} = 1$ from E_1 in their lexicographic order also. This transforms $A(E, X)$ into

$$\begin{bmatrix} A(E_2, X; g_0, \cdots, g_{s-1}) & * \\ 0 & A(E_1, X; g_s, \cdots, g_N) \end{bmatrix}. \tag{2.3}$$

Hence

$$D(E, X) = D(E_2, X; g_0, \cdots, g_{s-1}) D(E_1, X; g_s, \cdots, g_N). \tag{2.4}$$

According to (1.5) we have that $A(E_1, X; g_s, \cdots, g_N)$ is the unit matrix and hence $D(E_1, X; g_s, \cdots, g_N) = 1$. Then

$$D(E, X) = D(E_2, X; g_0, \cdots, g_{s-1}). \tag{2.5}$$

It remains to show, with a nonzero constant C

$$D(E_2, X; h_0, \cdots, h_{s-1}) = C D(E_2, X; g_0, \cdots, g_{s-1}). \tag{2.6}$$

In fact, since $h_0, \cdots, h_{s-1}$ are annihilated by the pair E_1, X, by the definitions of $g_0, \cdots, g_{s-1}$ and $g_s, \cdots, g_N$ we obtain

$$h_\nu = \sum_{j=0}^{s-1} c_{\nu j} g_j, \quad \nu = 0, 1, \cdots, s-1, \tag{2.7}$$

which implies

$$h_\nu^{(k)}(x_i) = \sum_{j=0}^{s-1} c_{\nu j} g_j^{(k)}(x_i), \quad \nu = 0, 1, \cdots, s-1, \quad e_{ik} = 1, \ e_{ik} \in E_2.$$

Thus

$$A(E_2, X; h_0, \cdots, h_{s-1}) = [c_{\nu j}; \quad \nu, j = 0, 1, \cdots, s-1] \times A(E_2, X; g_0, \cdots, g_{s-1}).$$

This yields

$$D(E_2, X; h_0, \cdots, h_{s-1}) = \det [c_{\nu j}; \quad \nu, j = 0, 1, \cdots, s-1] \times D(E_2, X; g_0, \cdots, g_{s-1}).$$

Since $h_0, \cdots, h_{s-1}$ are linearly independent, $\det[c_{\nu j}; \quad \nu, j = 0, 1, \cdots, s-1] \neq 0$, which proves (2.6).

This completes the proof.

Remark. With this theorem Theorem 8.1 in [1, p. 101], which characterizes the regularity of an almost-Hermitian matrix, follows easily. In fact, the polynomials P_l's in the proof of that theorem are just the functions h_i's here. Then the determinant (8.22) in [1, p. 101] can be immediately obtained from (2.2). We omit the details.

3. $(0, 1, \cdots, p-1, p+1, \cdots, M-1, q)$ Interpolation

Let E be defined by (1.6). We set

$$\Phi = \{0, 1, \cdots, p-1, p+1, \cdots, M-1, q\}, \quad \Psi = \{1, 2, \cdots, n\}. \tag{3.1}$$

Then by definition a fundamental function $r_{ik} \in \mathbf{P}_N$, $i \in \Psi$, $k \in \Phi$, is defined by

$$r_{ik}^{(\mu)}(x_\nu) = \delta_{i\nu}\delta_{k\mu}, \quad \nu \in \Psi, \quad \mu \in \Phi. \tag{3.2}$$

In order to apply Theorem 1 it is suitable to take E' as an interpolation matrix corresponding to $(0, 1, \cdots, M-1)$ interpolation. Let $A_{ik} \in \mathbf{P}_N$ be defined by

$$A_{ik}^{(\mu)}(x_\nu) = \delta_{i\nu}\delta_{k\mu}, \quad i, \nu = 1, 2, \cdots, n, \quad k, \mu = 0, 1, \cdots, M-1. \tag{3.3}$$

Then each $P \in \mathbf{P}_N$ can be uniquely written as

$$P(x) = \sum_{i=1}^{n} \sum_{k=0}^{M-1} P^{(k)}(x_i) A_{ik}(x). \tag{3.4}$$

If we write

$$l_i(x) := \frac{\omega_n(x)}{(x-x_i)\omega_n'(x_i)}, \quad \omega_n(x) = (x-x_1)(x-x_2)\cdots(x-x_n), \tag{3.5}$$

we have the formulas [2]

$$\begin{aligned} A_{ik}(x) &= [a_{ik0}(x-x_i)^k + \cdots + a_{i,k,M-k-1}(x-x_i)^{M-1}]l_i^M(x), \\ &\quad i = 1,2,\cdots,n, \quad k = 0,1,\cdots,M-1, \end{aligned} \tag{3.6}$$

where

$$\begin{aligned} a_{ikj} = \frac{1}{k!j!}[l_i^{-M}(x)]^{(j)}_{x=x_i}, \quad & i = 1,2,\cdots,n, \quad k = 0,1,\cdots,M-1, \\ & j = 0,1,\cdots,M-k-1. \end{aligned} \tag{3.7}$$

Then we have

Theorem 2. *Let E correspond to the matrix of $(0,1,\cdots,p-1,p+1,\cdots,M-1,q)$ interpolation with $0 < p < M \le q \le N-n+1$. If $q = M$ and $M-p$ is even then the pair E,X is regular; if $q > M$ or $M-p$ is odd then the pair E,X is regular if and only if the matrix*

$$\mathbf{A}_n := [A_{ip}^{(q)}(x_j); \quad i,j = 1,2,\cdots,n] \tag{3.8}$$

is nonsingular.

Proof. It is easy to see that E satisfies the Pólya condition. If $q = M$ and $M-p$ is even then the matrix E has no odd supported sequences. Applying Theorem A we conclude that the pair E,X is always regular. If $q > M$ or $M-p$ is odd then taking E' as the matrix of $(0,1,\cdots,M-1)$ interpolation and applying Theorem 1 we conclude that the pair E,X is regular if and only if the matrix $\mathbf{A}_n$ is nonsingular.

This completes the proof.

As immediate consequences of this theorem we state some corollaries.

Corollary 1. Let E correspond to the matrix of $(0,1,\cdots,p-1,p+1,\cdots,M-1,M)$ interpolation with $0 < p < M$. If $M-p$ is even then the matrix $\mathbf{A}_n$ is always nonsingular.

Proof. Apply Theorem A and Theorem 1.

Corollary 2. Let E correspond to the matrix of $(0,1,\cdots,p-1,p+1,\cdots,M-1,q)$ interpolation with $0 < p < M \le q \le N-n+1$. If both $q-p$ and n are odd and if X satisfies

$$x_i = -x_{n-i+1}, \quad i = 1,2,\cdots,n \tag{3.9}$$

then the pair E,X is singular.

Proof. By the definition of A_{ip} it follows from (3.9) that

$$A_{ip}(x) = (-1)^p A_{n-i+1,p}(-x), \quad i = 1,2,\cdots,n.$$

Hence

$$A_{ip}^{(q)}(x_j) = (-1)^{q-p} A_{n-i+1,p}^{(q)}(-x_j) = -A_{n-i+1,p}^{(q)}(x_{n-j+1}), \quad i,j = 1,2,\cdots,n.$$

Then

$$\begin{aligned} \det \mathbf{A}_n &= \det [A_{ip}^{(q)}(x_j); \;\; i,j = 1,2,\cdots,n] \\ &= (-1)^n \det [A_{n-i+1,p}^{(q)}(x_{n-j+1}); \;\; i,j = 1,2,\cdots,n] \\ &= (-1)^n \det \mathbf{A}_n = -\det \mathbf{A}_n. \end{aligned}$$

This means that $\det\mathbf{A}_n = 0$ and the pair E, X is singular.

Theorem 3. *Let E correspond to the matrix of $(0,1,\cdots,p-1,p+1,\cdots,M-1,q)$ interpolation with $0 < p < M \le q \le N-n+1$. If a fundamental function r_{ik}, $i \in \Psi$, $k \in \Phi$, exists then*

$$r_{ik}(x) = A_{ik}(x) + \sum_{j=1}^{n} r_{ik}^{(p)}(x_j) A_{jp}(x) \tag{3.10}$$

(The functions A_{iq} are assumed to be 0), and if the pair E, X is regular then

$$r_{ik}(x) = A_{ik}(x) - \sum_{j=1}^{n} A_{ik}^{(q)}(x_j) r_{jq}(x), \quad i \in \Psi, \quad k \in \Phi. \tag{3.11}$$

Proof. (3.10) follows from (3.2)-(3.4) and (3.11) follows from (3.2) and (3.4).

This completes the proof.

In general $\det\mathbf{A}_n$ is still complicated. But for the case when $q = M$ and $p = M-1$ it is of a simpler form.

Theorem 4. *Let E correspond to the matrix of $(0,1,\cdots,M-2,M)$ interpolation. Then*

$$\det \mathbf{A}_n = M^n \begin{vmatrix} Ml_1'(x_1) & l_2'(x_1) & \cdots & l_n'(x_1) \\ l_1'(x_2) & Ml_2'(x_2) & \cdots & l_n'(x_2) \\ \cdots & \cdots & \cdots & \cdots \\ l_1'(x_n) & l_2'(x_n) & \cdots & Ml_n'(x_n) \end{vmatrix} \tag{3.12}$$

and

$$\det \mathbf{A}_n = M^n \begin{vmatrix} \sum_{j\ne 1} \frac{M}{x_1-x_j} & \frac{1}{x_1-x_2} & \cdots & \frac{1}{x_1-x_n} \\ \frac{1}{x_2-x_1} & \sum_{j\ne 2} \frac{M}{x_2-x_j} & \cdots & \frac{1}{x_2-x_n} \\ \cdots & \cdots & \cdots & \cdots \\ \frac{1}{x_n-x_1} & \frac{1}{x_n-x_2} & \cdots & \sum_{j\ne n} \frac{M}{x_n-x_j} \end{vmatrix}. \tag{3.13}$$

Proof. In this case using (3.6) and (3.7) the functions $A_{i,M-1}$ have the simpler forms

$$A_{i,M-1}(x) = \frac{1}{(M-1)!}(x-x_i)^{M-1} l_i^M(x), \quad i = 1,2,\cdots,n.$$

This yields

$$A_{i,M-1}^{(M)}(x_i) = M[l_i^M(x)]'_{x=x_i} = M^2 l_i'(x_i) \tag{3.14}$$

and for $j \neq i$

$$A_{i,M-1}^{(M)}(x_j) = \frac{(x_j - x_i)^{M-1}}{(M-1)!}[l_i^M(x)]_{x=x_j}^{(M)} = M(x_j - x_i)^{M-1} l_i'(x_j)^M. \tag{3.15}$$

Thus

$$\det \mathbf{A}_n = M^n \begin{vmatrix} Ml_1'(x_1) & (x_1 - x_2)^{M-1} l_2'(x_1)^M & \cdots & (x_1 - x_n)^{M-1} l_n'(x_1)^M \\ (x_2 - x_1)^{M-1} l_1'(x_2)^M & Ml_2'(x_2) & \cdots & (x_2 - x_n)^{M-1} l_n'(x_2)^M \\ \cdots & \cdots & \cdots & \cdots \\ (x_n - x_1)^{M-1} l_1'(x_n)^M & (x_n - x_2)^{M-1} l_2'(x_n)^M & \cdots & Ml_n'(x_n) \end{vmatrix}. \tag{3.16}$$

It is easy to check that

$$l_i'(x_i) = \sum_{j \neq i} \frac{1}{x_i - x_j}$$

and for $j \neq i$

$$l_i'(x_j) = \frac{\omega_n'(x_j)}{(x_j - x_i)\omega_n'(x_i)}.$$

By means of these formulas the jth row of the determinant (3.16) times $\omega_n'(x_j)^{-k}$ and the jth column times $\omega_n'(x_j)^k$ yields (3.12) if $k = M - 1$ and (3.13) if $k = M$.

This completes the proof.

It is natural to ask the existence of which subsets of the set $\{r_{ik} : e_{ik} = 1, e_{ik} \in E\}$ implies the regularity of the pair E, X. According to Theorem 1 it is enough to transform E into E' by shifting 1's in some positions into new positions so that the pair E', X is regular. Then we can conclude that the existence of the fundamental functions in the original positions implies the regularity of the pair E, X. The following theorems give answers to this question for $(0, 1, \cdots, p-1, p+1, \cdots, M-1, q)$ interpolation. Since by Theorem 2 the pair E, X is always regular if $q = M$ and $M - p$ is even, we assume that $q > M$ or $M - p$ is odd. We treat the case when $q = M$ and the case when $q > M$, separately. First for the case when $q = M$ we have

Theorem 5. *Let E correspond to the matrix of $(0, 1, \cdots, p-1, p+1, \cdots, M)$ interpolation with $0 < p < M$. Assume that $M - p$ is odd. Then the pair E, X is regular if one of the following conditions is valid:*

(a) *For some index $k = M - 2\nu > p$ with a nonnegative integer ν there exist at least $s = [\frac{n+1}{2}]$ functions from the set $\{r_{ik} : \ i = 1, 2, \cdots, n\}$;*

(b) *For some index $k = M - 2\nu < p$ with a positive integer ν or $k = 0$ with $p > 1$ all the functions r_{ik}, $i = 1, 2, \cdots, n$, exist;*

(c) *For $k = 0$ and $p = 1$ there exist at least $n - s$ functions r_{i0}, $i = 1, 2, \cdots, t$ and $i = t + s + 1, \cdots, n$, where $0 \leq t \leq n - s$.*

Proof. Let the statement (a) be true and let such functions be r_{ik}, $i = i_1, \cdots, i_s$. We replace E by a new matrix E', obtained from E by removing the 1's in positions $(i_1, k), \cdots, (i_s, k)$ into the new positions (i, p), $i \in \Psi \setminus \{i_1, \cdots, i_s\}$ if n is even and (i, p), $i \in \Psi \setminus \{i_2, \cdots, i_s\}$ if n is odd. This transforms E into E' so that E' has no odd supported sequences. Thus the pair E', X is regular. Hence by Theorem 1 the statement (a) implies the regularity of the pair E, X.

For the statement (b) we distinguish two cases when $k > 0$ and when $k = 0$ with $p > 1$.

It is particularly simple to treat the case when $k > 0$. Let E' correspond to the matrix of $(0, 1, \cdots, k-1, k+1, \cdots, M)$ interpolation. Obviously the pair E', X is regular, because $M - k = 2\nu$ is even. It remains to apply Theorem 1.

In case $k = 0$ with $p > 1$ we shift the 1's in positions $(1, 0), \cdots, (n-1, 0)$ in E into the new positions $(1, p), \cdots, (n-1, p)$. The obtained matrix E' still satisfies the conditions of Theorem A and the pair E', X is regular. Thus by Theorem 1 we obtain even that the existence of r_{io}, $i = 1, 2, \cdots, n-1$, implies the regularity of the pair E, X.

For the statement (c) we obtain E' by moving the 1's in positions $(i, 0)$, $i = 1, 2, \cdots, t$ and $i = t+s+1, \cdots, n$ in E into the new positions $(i, 1)$, $i = t+1, \cdots, t+n-s$. This matrix E' is still order regular. Then applying Theorem 1 gives the required conclusion.

This completes the proof.

Theorem 6. *Let E correspond to the matrix of $(0, 1, \cdots, p-1, p+1, \cdots, M-1, q)$ interpolation with $0 < p < M < q \leq N - n + 1$. Then the pair E, X is regular if all the functions r_{iq}, $i = 1, 2, \cdots, n$, exist or all the others exist.*

Proof. Let E' correspond to the matrix of $(0, 1, \cdots, M-1)$ interpolation. Then we apply Theorem 1 to obtain the first conclusion.

In order to prove the second conclusion by (3.4) we write x^s, $s = q, q+1, \cdots, q+n-1$, as

$$x^s = \sum_{i=1}^{n} \sum_{k=0}^{M-1} \frac{s!}{(s-k)!} x_i^{s-k} A_{ik}(x).$$

Differentiating q times and putting $x = x_j$ for $j = 1, 2, \cdots, n$ yields

$$x_j^{s-q} = \sum_{i=1}^{n} \sum_{k=0}^{M-1} \frac{(s-q)!}{(s-k)!} x_i^{s-k} A_{ik}^{(q)}(x_j). \tag{3.17}$$

On the other hand, from (3.10) we obtain for $k \neq p$ and $k \neq q$

$$0 = r_{ik}^{(q)}(x_j) = A_{ik}^{(q)}(x_j) + \sum_{\nu=1}^{n} r_{ik}^{(p)}(x_\nu) A_{\nu p}^{(q)}(x_j)$$

or

$$A_{ik}^{(q)}(x_j) = -\sum_{\nu=1}^{n} r_{ik}^{(p)}(x_\nu) A_{\nu p}^{(q)}(x_j), \quad i, j = 1, 2, \cdots, n.$$

Substituting these values into (3.17) we get for $j = 1, 2, \cdots, n$

$$x_j^{s-q} = \sum_{i=1}^{n} \frac{(s-q)!}{(s-p)!} x_i^{s-p} A_{ip}^{(q)}(x_j) - \sum_{i=1}^{n} \sum_{\substack{0 \le k \le M-1 \\ k \ne p}} \frac{(s-q)!}{(s-k)!} x_i^{s-k} \sum_{\nu=1}^{n} r_{ik}^{(p)}(x_\nu) A_{\nu p}^{(q)}(x_j)$$

$$= \sum_{i=1}^{n} \left[\frac{(s-q)!}{(s-p)!} x_i^{s-p} - \sum_{\nu=1}^{n} \sum_{\substack{0 \le k \le M-1 \\ k \ne p}} \frac{(s-q)!}{(s-k)!} x_\nu^{s-k} r_{\nu k}^{(p)}(x_i) \right] A_{ip}^{(q)}(x_j).$$

This shows that the vector $[x_1^{s-q}, \cdots, x_n^{s-q}]$ belongs to the linear span of the set

$$\{[A_{ip}^{(q)}(x_1), \cdots, A_{ip}^{(q)}(x_n)] : \quad i = 1, 2, \cdots, n\}. \tag{3.18}$$

Remembering that $s-q = 0, 1, \cdots, n-1$ and the n vectors $[x_1^k, \cdots, x_n^k]$, $k = 0, 1, \cdots, n-1$, are linearly independent we conclude that the vectors in the set (3.18) are also linearly independent. Therefore the matrix $\mathbf{A}_n$ must be nonsingular. Namely the pair E, X is regular.

This completes the proof.

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