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# ON THE GENERALIZED INVERSE NEVILLE-TYPE MATRIX-VALUED RATIONAL INTERPOLANTS*

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**Abstract**

A new kind of matrix-valued rational interpolants is recursively established by means of generalized Samelson inverse for matrices, with scalar numerator and matrix-valued denominator. In this respect, it is essentially different from that of the previous works [7, 9], where the matrix-valued rational interpolants is in Thiele-type continued fraction form with matrix-valued numerator and scalar denominator. For both univariate and bivariate cases, sufficient conditions for existence, characterisation and uniqueness in some sense are proved respectively, and an error formula for the univariate interpolating function is also given. The results obtained in this paper are illustrated with some numerical examples.

*Key words*: Generalized inverse for matrices; Neville-type; Rational interpolants.

## 1. Introduction

Many kinds of matrix-valued rational interpolation or approximation problems have appeared in recent years, which have been found to be useful in linear system theory, especially when the system is multi-input and multi-output. Padá interpolation and Padá approximation can be generalized to the matrical case to approximate a matrix-valued power series [3, 5]. By means of the reachability and the observability indices of defined pairs of matrices, Antoulas et al. [2] solved the minimal matrix rational interpolation problem. According to Loewner matrix, Anderson and Antoulas [1] discussed the problem of passing from interpolation data for a real rational transfer- function matrix to a minimal state-variable realization of the transfer-function matrix. Bose and Basu [4] discussed a matrix-valued approximant with matrix-valued numerator and denominator for the approximation of a bivariate matrix power series.

Motivated by Graves-Morris' Thiele-type vector-valued rational interpolants [6], Gu Chuanqing and Chen Zhibing [7] discussed the matrix-valued rational interpolants in Thiele-type continued fraction form, with matrix-valued numerator and scalar donominator. Given a set of distinct real points $\{x_i:\ i=0,1,\cdots,n\}$ and a corresponding set of matrical data $\{A_i:\ i=0,1,\cdots,n,\ A_i=A(x_i)\in C^{m\times m}\}$, [7] showed explicitly that

$$R(x)=\frac{N(x)}{D(x)}=A_0+\frac{x-x_0}{A_1}+\cdots+\frac{x-x_{n-1}}{A_n} \tag{1.1}$$

can serve to interpolate the given matrices. Gu Chuanqing also generalized (1.1) to the bivariate case [9]. The working tool of this kind of matrix-valued rational interpolants is closely ralated to the generalized Samelson inverse for matrices which was first introduced in [7] as following

$$\frac{1}{A}=A^{-1}=\frac{A^{\mathrm{H}}}{\|A\|^2},\qquad A\neq 0, \tag{1.2}$$

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where $A = (a_{ij}) \in C^{m\times m}$ and

$$\|A\| = \left(\mathrm{tr}A^{\mathrm{H}}A\right)^{\frac{1}{2}} = \Big(\sum_{i=1}^{m}\sum_{j=1}^{m}|a_{ij}|^2\Big)^{\frac{1}{2}}. \tag{1.3}$$

$A^{\mathrm{H}}$ is the conjugate transpose matrix of $A$. from (1.2) and (1.3), it can be derived that

$$(A^{-1})^{-1} = A, \tag{1.4}$$

which turns out to be an useful technique in this paper.

[10] shows that by using of the generalized Samelson inverse for matrices (1.2) in matrix-valued rational approximation problems, one need not have to define left and right approximation.

In this paper, we consider a new kind of matrix-valued rational interpolants based on (1.2), with scalar numerator and matrix-vlaued denominator, which is called Neville-type matrix-valued rational interpolants (NMRI). In this respect, it is essentially different from those of the authors' previous work [7, 9], where the matrix-valued rational interpolants is in Thiele-type continued fraction form with matrix-valued numerator and scalar denominator. Although NMRI is also based on the generalized Samelson inverse for matrices, compared with those obtained in [7] and [9], it has the following advantages: first, the total degrees of the numerator and denominator is low than that in [7] and [9](theorem 3.2, theorem 4.5); second, in the construction process, one need not compute each matrical inverse, by (1.5) one just "turn over" the matrix twice can make the computation easy(example 5.1-5.3); third, the interpolation is defined through recursive algorithm, hence, it is more suitable to calculate the value of a matrix-valued function for a given point (example 5.2).

In section 2, we iteratively construct NMRI. In section 3, some important conclusions such as characterisation and uniqueness in some sense are proven respectfully , and an error formula for NMRI is also given and proven. In section 4, most results obtained in section 3 are extended to the bivariate case (BNMRI). In last section, some numerical examples are given to illustrate the results in this paper.

## 2. NMRI

Given a set of distinct real points $\{x_i : i = s, s+1, \cdots, s+v, x_i \in R\}$ and a corresponding set of matrix data $\{A_i : i = s, s+1, \cdots, s+v,\ A_i = A(x_i) \in C^{m\times m}\}$, we will construct NMRI as

$$M_s^v(x) = \frac{N_s^v(x)}{D_s^v(x)}, \tag{2.1}$$

where $N_s^v(x)$ is a real polynomial and $D_s^v(x)$ is a real or complex polynomial matrix, such that

$$M_s^v(x_i) = \frac{N_s^v(x_i)}{D_s^v(x_i)} = \frac{1}{A_i}, \qquad i = s, s+1, \cdots, s+v. \tag{2.2}$$

By (1.2), it is easy to prove

**Lemma 2.1.** *For $A, C \in R^{m\times m}$ and $b \in R, b \neq 0$, then*

$$\frac{b}{A} = \frac{1}{C} \Longleftrightarrow A = bC. \tag{2.3}$$

For simplicity, we define

**Definition 2.2.**

$$U_s^v(x,A) = D_s^v(x) - N_s^v(x)A, \tag{2.4}$$
$$\alpha_i = x - x_i, \qquad i = s, s+1, \cdots, s+v. \tag{2.5}$$

from (2.3) and (2.4), it is obvious that (2.2) equals

$$U_s^v(x_i, A_i) = D_s^v(x_i) - N_s^v(x_i)A_i = 0, \quad i = s, s+1, \cdots, s+v. \tag{2.6}$$

The following theorem shows that $N_s^v(x)$ and $D_s^v(x)$ can be recursively defined,
**Theorem 2.3.** *Let* $N_s^0(x) = 1, \quad D_s^0(x) = A_s$, *then*

$$N_s^v(x) = \alpha_s N_s^{v-1}(x) N_{s+1}^{v-1}(x) - \alpha_{s+v} N_{s+1}^{v-1}(x) N_s^{v-1}(x), \tag{2.7}$$

$$D_s^v(x) = \alpha_s N_s^{v-1}(x) D_{s+1}^{v-1}(x) - \alpha_{s+v} N_{s+1}^{v-1}(x) D_s^{v-1}(x). \tag{2.8}$$

*Proof.* We only need to prove (2.6). It is clear that

$$U_s^0(x_s, A_s) = 0.$$

Suppose $M_s^{v-1}(x)$ and $M_{s+1}^{v-1}(x)$ satisfy

$$U_s^{v-1}(x_i, A_i) = 0, \qquad i = s, s+1, \cdots, s+v-1, \tag{2.9}$$

$$U_{s+1}^{v-1}(x_i, A_i) = 0, \qquad i = s+1, s+2, \cdots, s+v, \tag{2.10}$$

noticing

$$U_s^v(x,A) = \alpha_s N_s^{v-1}(x) U_{s+1}^{v-1}(x,A) - \alpha_{s+v} N_{s+1}^{v-1}(x) U_s^{v-1}(x,A) \tag{2.11}$$

and the definition of $\alpha_s$ and $\alpha_{s+v}$, we finally get

$$U_s^v(x_i, A_i) = 0, \qquad i = s, s+1, \cdots, s+v. \tag{2.12}$$

by induction, the proof is finished.

The coefficients $N_s^{v-1}(x)$, $N_{s+1}^{v-1}(x)$ in (2.7) and (2.8) can be eliminated by the following theorem.
**Theorem 2.4** *Let* $M_s^0(x) = A_s^{-1}$, *if* $N_s^{v-1}(x) N_{s+1}^{v-1}(x) \neq 0$, *then*

$$M_s^v(x) = \frac{\alpha_s - \alpha_{s+v}}{\dfrac{\alpha_s}{M_{s+1}^{v-1}(x)} - \dfrac{\alpha_{s+v}}{M_s^{v-1}(x)}}, \tag{2.13}$$

where $\alpha_s, \alpha_{s+v}$ are defined as in (2.5).

*Proof.* From (2.1), (2.7), (2.8) and by using of (1.5), one can easily derive

$$\begin{aligned} M_s^v(x) &= \frac{\alpha_s N_s^{v-1}(x) N_{s+1}^{v-1}(x) - \alpha_{s+v} N_{s+1}^{v-1}(x) N_s^{v-1}(x)}{\alpha_s N_s^{v-1}(x) D_{s+1}^{v-1}(x) - \alpha_{s+v} N_{s+1}^{v-1}(x) D_s^{v-1}(x)} \\ &= \frac{\alpha_s - \alpha_{s+v}}{\dfrac{\alpha_s D_{s+1}^{v-1}(x)}{N_{s+1}^{v-1}(x)} - \dfrac{\alpha_{s+v} D_s^{v-1}(x)}{N_s^{v-1}(x)}} \\ &= \frac{\alpha_s - \alpha_{s+v}}{\dfrac{\alpha_s}{\dfrac{N_{s+1}^{v-1}(x)}{D_{s+1}^{v-1}(x)}} - \dfrac{\alpha_{s+v}}{\dfrac{N_s^{v-1}(x)}{D_s^{v-1}(x)}}} \\ &= \frac{\alpha_s - \alpha_{s+v}}{\dfrac{\alpha_s}{M_{s+1}^{v-1}(x)} - \dfrac{\alpha_{s+v}}{M_s^{v-1}(x)}}. \end{aligned}$$

Theorem 2.4 shows that $M_s^v(x)$ can be recursively defined and computed .

## 3. Characterisation ,uniqueness and error formula

**Definition 3.1.** *A matrix-valued polynomial*

$$A(x) = \Big(a_{ij}(x)\Big)_{m\times m}$$

*is said to be of degree $n$ and denoted by $d\{A(x)\} = n$, if $d\{a_{ij}(x)\} \le n$, for $i, j = 1, 2, \cdots, m$ and $d\{a_{ij}(x)\} = n$ for some $i, j$ $(1 \le i, j \le m)$.*

**Theorem 3.2.** *Let $N_s^0(x) = 1,\ D_s^0(x) = A_s$, then*

$$d\{N_s^v(x)\} = 0, \tag{3.1}$$

$$d\{D_s^v(x)\} \le v. \tag{3.2}$$

*Proof.* From (2.7), (2.8) and by induction, it is easy to derive (3.1) and (3.2).

It is proved in [7] that in (1.1)

$$\mathrm{d}\{N(x)\} = n, \qquad \mathrm{d}\{D(x)\} = 2[\frac{n}{2}]. \tag{3.3}$$

here $[x]$ represents the integer function. (3.1) and (3.2) shows that compared to $R(x)$, $M_s^v(x)$ turns out to be easy and concise. The following theorem shows that in some sense, $M_s^v(x)$ is unique.

**Theorem 3.3.** *Let $M_s^v(x) = N_s^v(x)/D_s^v(x)$ and $\overline{M}_s^v(x) = \overline{N}_s^v(x)/\overline{D}_s^v(x)$ be arbitrary two NMRI which satisfy*

$$M_s^v(x_i) = \overline{M}_s^v(x_i) = \frac{1}{A_i}, \qquad i = s, s+1, \cdots, s+v; \tag{3.4}$$

*and*

$$d\{N_s^v(x)\} = d\{\overline{N}_s^v(x)\} = 0, \qquad d\{D_s^v(x)\} \le v, \qquad d\{\overline{D}_s^v(x)\} \le v. \tag{3.5}$$

*then*

$$\overline{N}_s^v(x) D_s^v(x) \equiv N_s^v(x) \overline{D}_s^v(x).$$

*Proof.* From (3.4) and Lemma 2.1, it is esay to derive

$$\frac{N_s^v(x_i)}{D_s^v(x_i)} = \frac{\overline{N}_s^v(x_i)}{\overline{D}_s^v(x_i)},$$

$$\overline{N}_s^v(x_i)D_s^v(x_i) = N_s^v(x_i)\overline{D}_s^v(x_i),$$

$$\overline{N}_s^v(x_i)D_s^v(x_i) - N_s^v(x_i)\overline{D}_s^v(x_i) = 0, \qquad i = s, s+1, \ldots, s+v.$$

but from theorem 3.2

$$\mathrm{d}\{\overline{N}_s^v(x)D_s^v(x) - N_s^v(x)\overline{D}_s^v(x)\} \leq v,$$

it follows that

$$\overline{N}_s^v(x)D_s^v(x) \equiv N_s^v(x)\overline{D}_s^v(x).$$

**Theorem 3.4.** *Suppose $[a,b]$ is the minimal closed interval in which the points $x_s, x_{s+1}, \cdots$, $x_{s+v}$ are included, $A(x) = \big(a_{ij}(x)\big)_{m\times m}$ is of order $v+1$ continuous derivatives, and $N_s^v(x) \neq 0$, then For each $\overline{x} \in [a,b]$, there exists $\Xi \in R^{m\times m}$, such that*

$$A(\overline{x}) - \Big(M_s^v(\overline{x})\Big)^{-1} = \frac{\omega(\overline{x})}{(v+1)!}\Xi, \tag{3.6}$$

*here*

$$\omega(x) = (x - x_s)(x - x_{s+1})\cdots(x - x_{s+v}), \tag{3.7}$$

$$\Xi = \Big(a_{ij}^{(v+1)}(\xi_{ij})\Big)_{m\times m}, \tag{3.8}$$

$$\xi_{ij} \in (a,b), \qquad i,j = 1,2,\cdots,m. \tag{3.9}$$

*Proof.* Without losing generality, we suppose

$$\overline{x} \neq x_i, \qquad i = s, s+1, \cdots, s+v.$$

since

$$\Big(M_s^v(\overline{x})\Big)^{-1} = \Big(\frac{N_s^v(\overline{x})}{D_s^v(\overline{x})}\Big)^{-1} = \frac{D_s^v(\overline{x})}{N_s^v(\overline{x})} = \frac{\Big(d_{s(ij)}^v(\overline{x})\Big)_{m\times m}}{N_s^v(\overline{x})},$$

we only need to prove that for $i,j = 1,2,\cdots,m$

$$a_{ij}(\overline{x}) - \frac{d_{s(ij)}^v(\overline{x})}{N_s^v(\overline{x})} = \frac{\omega(\overline{x})}{(v+1)!}a_{ij}^{(v+1)}(\xi_{ij}), \tag{3.10}$$

for $\overline{x} \in [a,b]$, we can find $k_{ij} \in R$, such that

$$F_{ij}(x) := a_{ij}(x) - \frac{d_{s(ij)}^v(x)}{N_s^v(x)} - k_{ij}\omega(x) \tag{3.11}$$

serves

$$F_{ij}(\overline{x}) = 0. \tag{3.12}$$

and obviously, $F_{ij}(x)$ has $v+2$ roots in $[a,b]$

$$x_s, x_{s+1}, \cdots, x_{s+v}, \overline{x},$$

by using Rolle's theorem, there exists $\xi_{ij} \in (a,b)$, such that

$$F_{ij}^{(v+1)}(\xi_{ij}) = a_{ij}^{(v+1)}(\xi_{ij}) - \frac{1}{N_s^v(x)}\Big(d_{s(ij)}^v(x)\Big)^{(v+1)}(\xi_{ij}) - k_{ij}(v+1)! = 0 \tag{3.13}$$

from

$$\mathrm{d}\{D_s^v(x)\} \le v,$$

it is derived that

$$a_{ij}^{(v+1)}(\xi_{ij}) - k_{ij}(v+1)! = 0,$$

that is

$$k_{ij} = \frac{a_{ij}^{(v+1)}(\xi_{ij})}{(v+1)!}.$$

thus by (3.11) and (3.12)

$$a_{ij}(\overline{x}) - \frac{d_{s(ij)}^v(\overline{x})}{N_s^v(\overline{x})} = \frac{a_{ij}^{(v+1)}(\xi_{ij})}{(v+1)!}\omega(\overline{x}),$$

hence

$$A(\overline{x}) - \Big(M_s^v(\overline{x})\Big)^{-1} = \frac{\omega(\overline{x})}{(v+1)!}\Xi.$$

where $\Xi$ is given by (3.8).

## 4. Bivariate Neville type matrix valued rational interpolants (BNMRI)

In this section, most results obtained in section 2 and section 3 are extended to the bivariate case. Given a set of distinct real points in $\mathbf{R^2}$

$$\{(x_i,y_j) : i = s, s+1, \cdots, s+v; j = t, t+1, \cdots, t+w\}$$

and a correspoding set of matrices

$$\{A_{ij} : i = s, s+1, \cdots, s+v; j = t, t+1, \cdots, t+w, A_{ij} = A(x_{ij}) \in C^{m\times m}\}$$

we will construct BNMRI as

$$M_{s,t}^{v,w}(x,y) = \frac{N_{s,t}^{v,w}(x,y)}{D_{s,t}^{v,w}(x,y)}, \tag{4.1}$$

where $N_{s,t}^{v,w}(x,y)$ is a bivariate real polynomial and $D_{s,t}^{v,w}(x,y)$ is a bivariate real or complex polynomial matrix, such that

$$M_{s,t}^{v,w}(x_i,y_j) = \frac{N_{s,t}^{v,w}(x_i,y_j)}{D_{s,t}^{v,w}(x_i,y_j)} = \frac{1}{A_{ij}}, \qquad i = s, s+1, \cdots, s+v; j = t, t+1, \cdots, t+w. \tag{4.2}$$

**Definition 4.1.**

$$U_{s,t}^{v,w}(x,y;A) = D_{s,t}^{v,w}(x,y) - N_{s,t}^{v,w}(x,y)A, \tag{4.3}$$
$$\alpha_i = x - x_i, \qquad i = s, s+1, \cdots, s+v, \tag{4.4}$$
$$\beta_j = y - y_j, \qquad j = t, t+1, \cdots, t+w. \tag{4.5}$$

By using of the same approaches as in the proof of theorem 2.3, theorem 2.4, theorem 3.2 and theorem 3.3, we can derive the following three theorems.

**Theorem 4.2.** *Let* $N_{s,t}^{0,0}(x,y)=1, \quad D_{s,t}^{0,0}(x,y)=A_{s,t}$ *then*
(i) when $\quad v\geq 1, \quad w=0$

$$N_{s,t}^{v,0}=\alpha_s N_{s,t}^{v-1,0}N_{s+1,t}^{v-1,0}-\alpha_{s+v}N_{s+1,t}^{v-1,0}N_{s,t}^{v-1,0}, \tag{4.6}$$

$$D_{s,t}^{v,0}=\alpha_s N_{s,t}^{v-1,0}D_{s+1,t}^{v-1,0}-\alpha_{s+v}N_{s+1,t}^{v-1,0}D_{s,t}^{v-1,0}. \tag{4.7}$$

(ii) when $\quad v=0, \quad w\geq 1$

$$N_{s,t}^{0,w}=\beta_t N_{s,t}^{0,w-1}N_{s,t+1}^{0,w-1}-\beta_{t+w}N_{s,t+1}^{0,w-1}N_{s,t}^{0,w-1}, \tag{4.8}$$

$$D_{s,t}^{0,w}=\beta_t N_{s,t}^{0,w-1}D_{s,t+1}^{0,w-1}-\beta_{t+w}N_{s,t+1}^{0,w-1}D_{s,t}^{0,w-1}. \tag{4.9}$$

(iii) when $\quad v\geq 1, \quad w\geq 1$

$$N_{s,t}^{v,w}=N_{s,t}^{v-1,w-1}N_{s,t+1}^{v-1,w-1}N_{s+1,t}^{v-1,w-1}N_{s+1,t+1}^{v-1,w-1}(\alpha_{s+v}\beta_{t+w}-\alpha_{s+v}\beta_t-\alpha_s\beta_{t+w}+\alpha_s\beta_t) \tag{4.10}$$

$$\begin{aligned}D_{s,t}^{v,w}&=\alpha_{s+v}\beta_{t+w}N_{s,t+1}^{v-1,w-1}N_{s+1,t}^{v-1,w-1}N_{s+1,t+1}^{v-1,w-1}D_{s,t}^{v-1,w-1}\\&-\alpha_{s+v}\beta_t N_{s,t}^{v-1,w-1}N_{s+1,t}^{v-1,w-1}N_{s+1,t+1}^{v-1,w-1}D_{s,t+1}^{v-1,w-1}\\&-\alpha_s\beta_{t+w}N_{s,t}^{v-1,w-1}N_{s,t+1}^{v-1,w-1}N_{s+1,t+1}^{v-1,w-1}D_{s+1,t}^{v-1,w-1}\\&+\alpha_s\beta_t N_{s,t}^{v-1,w-1}N_{s,t+1}^{v-1,w-1}N_{s+1,t}^{v-1,w-1}D_{s+1,t+1}^{v-1,w-1}\end{aligned} \tag{4.11}$$

**Theorem 4.3.** *Let* $M_{s,t}^{0,0}(x,y)=A_{s,t}^{-1}$*, then corresponding to theorem 4.2*
*(i) if* $N_{s,t}^{v-1,0}N_{s+1,t}^{v-1,0}\neq 0$*, then for* $v\geq 1$*,*

$$M_{s,t}^{v,0}=\frac{\alpha_s-\alpha_{s+v}}{\dfrac{\alpha_s}{M_{s+1,t}^{v-1,0}}-\dfrac{\alpha_{s+v}}{M_{s,t}^{v-1,0}}}. \tag{4.12}$$

*(ii) if* $N_{s,t}^{0,v-1}N_{s,t+1}^{0,w-1}\neq 0$*, then for* $w\geq 1$*,*

$$M_{s,t}^{0,w}=\frac{\beta_t-\beta_{t+w}}{\dfrac{\beta_t}{M_{s,t+1}^{0,w-1}}-\dfrac{\beta_{t+w}}{M_{s,t}^{0,w-1}}}. \tag{4.13}$$

*(iii) if* $N_{s,t}^{v-1,w-1}N_{s,t+1}^{v-1,w-1}N_{s+1,t}^{v-1,w-1}N_{s+1,t+1}^{v-1,w-1}\neq 0$*, then for* $v\geq 1, w\geq 1$

$$M_{s,t}^{v,w}=\frac{(\alpha_{s+v}-\alpha_s)(\beta_{t+w}-\beta_t)}{\dfrac{\alpha_{s+v}\beta_{t+w}}{M_{s,t}^{v-1,w-1}}-\dfrac{\alpha_{s+v}\beta_t}{M_{s,t+1}^{v-1,w-1}}-\dfrac{\alpha_s\beta_{t+w}}{M_{s+1,t}^{v-1,w-1}}+\dfrac{\alpha_s\beta_t}{M_{s+1,t+1}^{v-1,w-1}}}. \tag{4.14}$$

**Definition 4.4.** *It is called*

$$\lim_{(x,y)\to(x_0,y_0)}M(x,y)=(b_{ij})_{m\times m},$$

*if*

$$\lim_{(x,y)\to(x_0,y_0)}M_{ij}(x,y)=b_{ij}, \quad i,j=1,2,\cdots,m, \quad M(x,y)=\Big(M_{ij}(x,y)\Big)_{m\times m}.$$

*and if*

$$\lim_{\Delta x\to 0}\frac{M(x+\Delta x,y)-M(x,y)}{\Delta x}$$

*exists, the limit is denoted by* $\partial M(x,y)/\partial x$, *by induction, if*

$$\lim_{\Delta x\to 0}\frac{\dfrac{\partial^{n-1}}{\partial x^{n-1}}M(x+\Delta x,y)-\dfrac{\partial^{n-1}}{\partial x^{n-1}}M(x,y)}{\Delta x}$$

*exists, the limit is denoted by* $\partial^n M(x,y)/\partial x^n$, *the meaning of* $\partial^n M(x,y)/\partial y^n$ *is similar.*

**Theorem 4.5.** *Let* $N_{s,t}^{0,0}(x,y)=1, D_{s,t}^{0,0}(x,y)=A_{s,t}$, *then*

$$d\{N_{s,t}^{v,w}(x,y)\}=0, \tag{4.15}$$

$$d\{D_{s,t}^{v,w}(x,y)\}\le v+w, \tag{4.16}$$

$$\frac{\partial^{v+1}}{\partial x^{v+1}}D_{s,t}^{v,w}=0, \tag{4.17}$$

$$\frac{\partial^{w+1}}{\partial y^{w+1}}D_{s,t}^{v,w}=0, \tag{4.18}$$

*here* $v,w\ge 0$ .

The following theorem shows that in some sense, $M_{s,t}^{v,w}$ is unique.

**Theorem 4.6.** *Let* $M_{s,t}^{v,w}(x,y)=N_{s,t}^{v,w}(x,y)/D_{s,t}^{v,w}(x,y)$ *and* $\overline{M}_{s,t}^{v,w}(x,y)=\overline{N}_{s,t}^{v,w}(x,y)\ /\ \overline{D}_{s,t}^{v,w}(x,y)$ *be arbitrary two BNMRI which satisfy*

$$M_{s,t}^{v,w}(x_i,y_j)=\overline{M}_{s,t}^{v,w}(x_i,y_j)=\frac{1}{A_{ij}},\qquad i=s,s+1,\cdots,s+v; j=t,t+1,\cdots,t+w. \tag{4.19}$$

*and*

$$d\{N_{s,t}^{v,w}(x,y)\}=d\{\overline{N}_{s,t}^{v,w}(x,y)\}=0,\qquad d\{D_{s,t}^{v,w}(x,y)\}\le v+w,\qquad d\{\overline{D}_{s,t}^{v,w}(x,y)\}\le v+w, \tag{4.20}$$

*then*

$$\overline{N}_{s,t}^{v,w}(x,y)D_{s,t}^{v,w}(x,y)\equiv N_{s,t}^{v,w}(x,y)\overline{D}_{s,t}^{v,w}(x,y).$$

*Proof.* From (4.19) and Lemma 2.1,

$$\frac{N_{s,t}^{v,w}(x_i,y_j)}{D_{s,t}^{v,w}(x_i,y_j)}=\frac{\overline{N}_{s,t}^{v,w}(x_i,y_j)}{\overline{D}_{s,t}^{v,w}(x_i,y_j)},$$

$$\overline{N}_{s,t}^{v,w}(x_i,y_j)D_{s,t}^{v,w}(x_i,y_j)-N_{s,t}^{v,w}(x_i,y_j)\overline{D}_{s,t}^{v,w}(x_i,y_j)=0, \tag{4.21}$$

for fixed $j$, by (4.17)

$$d\{\overline{N}_{s,t}^{v,w}(x,y_j)D_{s,t}^{v,w}(x,y_j)-N_{s,t}^{v,w}(x,y_j)\overline{D}_{s,t}^{v,w}(x,y_j)\}\le v,$$

thus from (4.21), it is derived that

$$\overline{N}_{s,t}^{v,w}(x,y_j)D_{s,t}^{v,w}(x,y_j)-N_{s,t}^{v,w}(x,y_j)\overline{D}_{s,t}^{v,w}(x,y_j)\equiv 0 \tag{4.22}$$

now fix x, by (4.18)

$$d\{\overline{N}_{s,t}^{v,w}(x,y)D_{s,t}^{v,w}(x,y)-N_{s,t}^{v,w}(x,y)\overline{D}_{s,t}^{v,w}(x,y)\}\le w,$$

hence from (4.22), we finally get

$$\overline{N}_{s,t}^{v,w}(x,y)D_{s,t}^{v,w}(x,y)-N_{s,t}^{v,w}(x,y)\overline{D}_{s,t}^{v,w}(x,y)\equiv 0,$$

that is

$$\overline{N}_{s,t}^{v,w}(x,y)D_{s,t}^{v,w}(x,y) \equiv N_{s,t}^{v,w}(x,y)\overline{D}_{s,t}^{v,w}(x,y).$$

**Example 4.7.** Suppose $s = t = 0, v = 1, w = 2$, and

$$(x_0, y_0) = (0,0) \qquad A_{0,0} = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix},$$

$$(x_0, y_1) = (0,1) \qquad A_{0,1} = \begin{pmatrix} 3 & 1 \\ 1 & 0 \end{pmatrix},$$

$$(x_0, y_2) = (0,2) \qquad A_{0,2} = \begin{pmatrix} 3 & 0 \\ 1 & 2 \end{pmatrix},$$

$$(x_1, y_0) = (1,0) \qquad A_{1,0} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix},$$

$$(x_1, y_1) = (1,1) \qquad A_{1,1} = \begin{pmatrix} 0 & 2 \\ 2 & 3 \end{pmatrix},$$

$$(x_1, y_2) = (1,2) \qquad A_{1,2} = \begin{pmatrix} 4 & 3 \\ 2 & 1 \end{pmatrix},$$

find $M_{0,0}^{1,2}(x,y)$.

solution: By (4.14)

$$M_{0,0}^{1,2} = \frac{(0-1)(0-2)}{\dfrac{(x-1)(y-2)}{M_{0,0}^{0,1}} - \dfrac{(x-1)y}{M_{0,1}^{0,1}} - \dfrac{x(y-2)}{M_{1,0}^{0,1}} + \dfrac{xy}{M_{1,1}^{0,1}}}. \tag{4.23}$$

here by (4.13)

$$M_{0,0}^{0,1} = \frac{0-1}{\dfrac{y-1}{M_{0,0}^{0,0}} - \dfrac{y}{M_{0,1}^{0,0}}} = \frac{1}{\begin{pmatrix} y+2 & y \\ y & -2y+2 \end{pmatrix}},$$

$$M_{0,1}^{0,1} = \frac{1-2}{\dfrac{y-2}{M_{0,1}^{0,0}} - \dfrac{y-1}{M_{0,2}^{0,0}}} = \frac{1}{\begin{pmatrix} 3 & -y+2 \\ 1 & 2y-2 \end{pmatrix}},$$

$$M_{1,0}^{0,1} = \frac{0-1}{\dfrac{y-1}{M_{1,0}^{0,0}} - \dfrac{y}{M_{1,1}^{0,0}}} = \frac{1}{\begin{pmatrix} -y+1 & y+1 \\ y+1 & 3y \end{pmatrix}},$$

$$M_{1,1}^{0,1} = \frac{1-2}{\dfrac{y-2}{M_{1,1}^{0,0}} - \dfrac{y-1}{M_{1,2}^{0,0}}} = \frac{1}{\begin{pmatrix} 4y-4 & y+1 \\ 2 & -2y+5 \end{pmatrix}},$$

by (4.23). it is easy to derive that

$$M_{0,0}^{1,2} = \frac{2}{\begin{pmatrix} 6xy^2 - 10xy - y^2 - 2x + 3y + 4 & 2xy^2 - 2xy - 2y^2 + 2x + 4y \\ -y^2 + 2x + 3y & -9xy^2 + 19xy + 4y^2 - 4x - 8y + 4 \end{pmatrix}}.$$

## References


[1] B. D. O. Anderson, A. C. Antoulas, Rational interpolation and state- variable realizations, *Linear Algebra Appl.* **137/138**(1990), 479-509.

[2] A. C. Antoulas, J. A. Ball, J. Kang, J. C. Willems, On the solution of the minimal rational interpolation problem, *Linear Algebra Appl.* **137/138**(1990), 511-573.

[3] B. Beckermann, G. Labahn, Uniform approach for Hermite Padá and Simultaneous Padá approximants and their matrix-type generalization, *Numer. Algorithms* **3**(1992), 45-54.

[4] N. K. Bose, S. Basu, Multidimensional systems theory: matrix Padá approximants, *Pro. Conf. on Decision and control,* IEEE Control System Society, 1978.

[5] N. K. Bose, S. Basu, Two-dimensional matrix Padá approximants: existence, nonuniqueness and recursive computation, *IEEE Trans. Automat. Control* AC-25 (1980), 509-514.

[6] P. R. Graves-Morris, Vector valued rational interpolants I, *Numer. Math.* **42**(1983), 331-348.

[7] Gu Chuanqing, Chen Zhibing, Matrix valued rational interpolants and its error formula , *Math. Nummer. Sinica* **17**(1995), 73-77.

[8] Gu Chuanqing, Zhu Gongqin, Matrix valued rational approximation, *J. Math. Res. Exposition* **16**(1996), 301-306.

[9] Gu Chuanqing, Bivariate Thiele-type matrix valued rational interpolants, *J. Comput. Appl. Math.* **80**(1997), 71-82.

[10] Gu Chuanqing, Generalized inverse matrix valued Padá approximants, *Math. Numer. Sinica* **19**(1997), 19-28.

[11] Gu Chuanqing, Thiele-type and Lagrange-type generalized inverse rational interpolation for rectangular complex matrices, *Linear Algebra Appl.* **295**(1999), 7-30.

[12] J. Stoer, R. Bulirsch, *Introduction to Numerical Analysis,* (Springer-Verlag, New York, 1993.)