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THE MECHANICAL QUADRATURE METHODS AND THEIR EXTRAPOLATION FOR SOLVING BIE OF STEKLOV EIGENVALUE PROBLEMS *1)

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Abstract

By means of the potential theory Steklov eigenvalue problems are transformed into general eigenvalue problems of boundary integral equations (BIE) with the logarithmic singularity.Using the quadrature rules[1], the paper presents quadrature methods for BIE of Steklov eigenvalue problem, which possess high accuracies $O(h^3)$ and low computing complexities. Moreover, an asymptotic expansion of the errors with odd powers is shown. Using h^3- Richardson extrapolation, we can not only improve the accuracy order of approximations, but also derive a posterior estimate as adaptive algorithms. The efficiency of the algorithm is illustrated by some examples.

Mathematics subject classification: 65N25.
Key words: Steklov eigenvalue problem, Boundary integral equation, Quadrature method, Richardson extrapolation.

1. Introduction

Consider the following Steklov eigenvalue problem:

$$\begin{cases} \Delta\tilde{u} - \alpha^2\tilde{u} = 0, \text{ in } \Omega, \\ \frac{\partial\tilde{u}}{\partial n} = \lambda\tilde{u}, \text{ on } \Gamma, \end{cases} \tag{1.1}$$

where α is a constant. Ω is a bounded domain with the boundary Γ, and $\frac{\partial}{\partial n}$ is the outward normal derivative on Γ. This problem arises from many applications, e.g., the free membrane and heat flow problems.

Courant and Hilbert[5] studied the problem (1.1). Bramble and Osborn[3] gave finite element method and its error estimate for solving the equation (1.1). Liu and Ortiz[9] gave finite difference methods and Tao-Method. Obviously, the problem (1.1) is easily converted into the eigenvalue problem of the boundary integral equation, so the boundary element method(BEM) solved (1.1) is more advantageous. Using a new variational formula of BIE, Han, Guan and He[6,7], and Tang, Guan and Han[13] derived a new BIE of the problem (1.1) and obtained its approximate BEM, which can keep the self adjoint property of the original problem. Although their approximate methods are very efficient, however, there exist two disadvantages: (1) each element of the discrete matrix as full has to calculate a double improper integral; (2) the order of accuracy only is $O(h^2)$. In [10,11], the mechanical quadrature methods for solving the boundary integral equations are constructed, where the convergence and asymptotic expansion with h^3 are proved by the collectively compact and asymptotically compact theory[1,2,4].

In this paper Steklov eigenvalue problem will be transformed into a general eigenvalue problem of BIE with the logarithmic singularity. Applying Side's quadrature rules[12], we present the mechanical quadrature methods of BIE for solving Steklov eigenvalue problem,

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in which the generation of the discrete matrix is without any calculations of integrals. Since the asymptotic expansions of the errors with the power h^3 are shown, our methods imply that using h^3-Richardson extrapolation not only improves the accuracy order of approximations, but also a posteriori estimate as adaptive algorithm is got. The numerical examples show that our methods have the accuracy order $O(h^3)$. The extrapolation and a posteriori estimate are very effective.

By the potential theory, the problem (1.1) is easily transformed into the following eigenvalue problem of BIE

$$\frac{1}{2}\tilde{u}\,(x)-\int_{\Gamma}\tilde{u}(y)\frac{\partial}{\partial n_y}\Phi(x,y)ds_y=-\lambda\int_{\Gamma}\tilde{u}(y)\Phi(x,y)ds_y, x\in\Gamma, \tag{1.2}$$

where $x=(x_1,x_2)$, $y=(y_1,y_2)$, $|x-y|^2=(x_1-y_1)^2+(x_2-y_2)^2$,

$$\Phi(x,y)=\begin{cases} -K_0(\alpha|x-y|)/(2\pi), \text{ for } \alpha>0\\ (2\pi)^{-1}\log|x-y|, \text{ for } \alpha=0\end{cases}$$

is the fundamental solution of equation (1.1); $K_0\,(z)$ is the modified Bessel function with

$$K_0\,(z)\approx-\log z+\log 2-\gamma,\ z\to 0;$$

and $\gamma\approx 0.5772$ is Euler constant.

2. Mechanical Quadrature Methods

Assume that Γ is a smooth closed curve described by the parameter mapping $x(t)=(x_1(t),x_2(t))$ with $(x_1^{'}(t))^2+(x_2^{'}(t))^2>0$. Let $C^m[0,2\pi]$ denote the set of m times differentiable periodic functions with periodic 2π. Define the following integral operators on $C^m[0,2\pi]$

$$(Ku)(s)=\int_0^{2\pi}k\,(t,s)\,u(t)dt, \tag{2.1}$$

and

$$(Bu)(s)=\int_0^{2\pi}b(t,s)u\,(t)\,dt, \tag{2.2}$$

where $u(t)=\tilde{u}(x_1(t),x_2(t))$. $k\,(t,s)=\frac{\partial}{\partial n_y}(\Phi(x(t),x(s)))|x^{'}(t)|/\pi$ is smooth function; however, $b(t,s)=\Phi(x(t),x(s))|x^{'}(t)|/\pi$ is with the logarithmic singularity. The equation (1.2) is equivalent to the operator eigenvalue problem

$$\begin{cases} (I-K)u=-\lambda Bu,\\ ||u||_{0,\Gamma}^2=\int_0^{2\pi}u^2(s)ds=1.\end{cases} \tag{2.3}$$

Take a mesh width $h=2\pi/n$, and $t_j=s_j=jh$, $j=0,\cdots,n-1$. By the trapezoidal and quadrature rule[12] we construct the approximate operators of K and B

$$\begin{cases} (K_hu)(s)=h\sum_{j=0}^{n-1}k(t_j,s)u(t_j),\\ (B_hu)(s)=h\sum_{j=0}^{n-1}b_n(t_j,s)u(t_j),\end{cases} \tag{2.4}$$

where

$$b_n(t,s)=\begin{cases} b(t,s), \text{ for } |t-s|\geq h;\\ h[\log\frac{h}{2\pi}+\log|x^{'}\,(s)\,|+\varepsilon_\alpha]|x'\,(s)\,|, \text{ for } |t-s|<h\end{cases}$$

is a continuous approximation of $b(t,s)$ and

$$\varepsilon_\alpha=\begin{cases} -\log{(2\alpha)}-\gamma,\ \text{for } \alpha>0;\\ 0, \text{for } \alpha=0.\end{cases}$$

From (2.4) we derive an approximate eigenvalue problem of (2.3): find λ_h, and $u_h \in C[0,2\pi]$ satisfying

$$\begin{cases} u_h - K_h u_h = -\lambda_h B_h u_h, \\ h\sum_{j=0}^{n-1}(u_h(t_j))^2 = 1, \end{cases} \tag{2.5}$$

which can be solved as follows: first solve an algebraic approximate eigenvalue problem

$$\begin{cases} u_h(t_i) - h\sum_{j=0}^{n-1} k(t_j,t_i)u_h(t_j) = -\lambda_h\{h\sum_{j=0,j\neq i}^{n-1} b(t_j,t_i)u_h(t_j) \\ \quad +h[\log|hx'(t_i)/(2\pi)| + \varepsilon_\alpha]|x'(t_i)|u_h(t_i)\}, i = 0,\cdots,n-1 \\ \quad h\sum_{j=0}^{n-1}(u_h(t_j))^2 = 1, \end{cases} \tag{2.6}$$

then use Nystrom iterate

$$u_h(s) = h\sum_{j=0}^{n-1} k(t_j,s)u_h(t_j) - \lambda_h h\sum_{j=0}^{n-1} b_n(t_j,s)u_h(t_j).$$

3. Error Analysis of Quadrature Methods

Suppose that the eigenvalues of both K and K_h are without 1, then the problems (2.3) and (2.5) can be rewritten as follows: find μ and $u \in C[0,2\pi]$ satisfying

$$Au = (I-K)^{-1}Bu = \mu u, \text{ and } ||u||_{0,\Gamma} = 1; \tag{3.1}$$

and find μ_h and $u_h \in C[0,2\pi]$ satisfying

$$A_h u_h = (I-K_h)^{-1} B_h u_h = \mu_h u_h, \text{ and } h\sum_{j=0}^{n-1} u_h^2(t_j) = 1, \tag{3.2}$$

where $\mu = -1/\lambda, \mu_h = -1/\lambda_h$. In order to prove that (μ_h, u_h) converges to (μ, u), we first prove the following lemmas.

Lemma 1. *The approximate operator sequence $\{A_h : C[0,2\pi] \to C[0,2\pi]\}$ is the asymptotically compact convergence to A, i,e.*

$$A_h \overset{a.c}{\to} A.$$

Proof. Since $k(t,s)$ is a continuous kernel, it means that K_h is collectively compact convergence to K, i,e. $K_h \overset{c.c}{\to} K$, as $n \to \infty$ (see [1,4]). On the other hand, because $b_n(t,s)$ is a continuous approximation of $b(t,s)$, from [1,2], the approximate operator B_h is the asymptotically compact convergence to B, i,e. $B_h \overset{a.c}{\to} B$, as $n \to \infty$. It implies that for any boundary sequence $\{y_m\}$ there exists a convergent subsequence in $\{B_h y_m\}$. Without loss of generality we assume $B_h y_m \to z$, $(m \to \infty)$. By the properties of collectively compact convergence, we have

$$\begin{aligned} ||A_h y_m - (I-K)^{-1}z|| &\le ||(I-K_h)^{-1}B_h y_m - (I-K)^{-1}z|| \\ &\le ||(I-K_h)^{-1}||||B_h y_m - z|| \\ +||(I-K_h)^{-1}(K_h - K)(I-K)^{-1}z|| &\to 0, \text{ as } m \to \infty, \text{ and } h \to 0, \end{aligned} \tag{3.3}$$

where $\|\cdot\|$ is the norm of $\mathfrak{L}(C[0,2\pi], C[0,2\pi])$. It shows that $\{A_h : C[0,2\pi] \to C[0,2\pi]\}$ is an asymptotically compact operator sequence. Moreover we shall show that $A_h \overset{a.c}{\to} A$, as $n \to \infty$. In fact, since $B_h \overset{a.c}{\to} B$, for $\forall y \in C[0,2\pi]$ we get

$$||B_h y - By|| \to 0. \tag{3.4}$$

From (3.4) we derive

$$||A_h y - Ay|| \le ||(I-K_h)^{-1}||||B_h y - By||$$

$$+||(I-K_h)^{-1}(K_h-K)(I-K)^{-1}By|| \to 0, \text{as } h \to 0, \tag{3.5}$$

which get the proof of Lemma 1.

Corollary 1. *Under the assumption of Lemma 1, it holds that*

$$\|(A_h - A)\, A\| \to 0 \text{ and } \|(A_h - A)\, A_h\| \to 0, \text{as } h \to 0, \tag{3.6}$$

where $\|\cdot\|$ *is the norm in* $\mathfrak{L}(C[0,2\pi], C[0,2\pi])$.

Lemma 2. *If* $\varphi(s) \in C^{2m}[0,2\pi]$ *and* $k(t,s) \in C^{2m}[0,2\pi]$, *then it holds the asymptotic expansion*

$$(A_h - A)\varphi(s) = \sum_{j=1}^{m-1} \psi_j(s)\, h^{2j+1} + O\left(h^{2m}\right), \tag{3.7}$$

where $\psi_j(s) \in C^{2m-j}[0,2\pi]$, $j = 1, ..., m-1$, *are functions independent of* h.

Proof. First we substitute the asymptotic expansion of the error of the quadrature rule[12]

$$(B - B_h)\,\varphi = \sum_{j=1}^{m-1} a_j(s)\, h^{2j+1} + O\left(h^{2m}\right) \tag{3.8}$$

with $a_j(s) = \frac{d^{2j}}{ds^{2j}}(\varphi(s)\,|x'(s)\,|)$ into the identity

$$\begin{gathered}(A_h - A)\varphi = \\ (I-K)^{-1}(B_h - B)\,\varphi + (I-K_h)^{-1}(K_h - K)\,(I-K)^{-1}(B_h - B)\,\varphi \\ +(I-K_h)^{-1}(K_h - K)\,(I-K)^{-1}B\varphi,\end{gathered} \tag{3.9}$$

then let $\psi_j(s) = (I-K)^{-1} a_j(s)$. Thus using the remainder estimate of the trapezoidal of the periodic function

$$\max_{0\le s\le 2\pi} |\,(K-K_h)\,\phi(s)| = ||\,(K-K_h)\,\phi|| = O(h^k), \forall \phi \in C^k[0,2\pi], \tag{3.10}$$

we get the proof of (3.7).

We all know that if μ is an isolated eigenvalue of (3.1), then the dimension of its eigenspace is finite[4] and the conjugate complex number $\bar{\mu}$ of μ is also an eigenvalue of conjugate operator A^*. Let $\overset{*}{V}_\mu$ =span$\left\{\overset{*}{u}_1, ..., \overset{*}{u}_\chi\right\}$ and V_μ =span$\{u_1, ..., u_\chi\}$ be the eigenspace of A^* and A, respectively, which construct a biorthogonal system

$$\left\langle u_i, \overset{*}{u}_j \right\rangle = \delta_{ij}, \quad i,j = 1, \cdots, \chi, \tag{3.12}$$

with $\|u_i\|_0 = 1,\ i = 1, ..., \chi$. Let μ_h and $V_{\mu h}$ be the eigenvalue and the eigenspace of A_h, which correspond to μ and V_μ, respectively. By [4] , there exists dim $V_{\mu h} = \chi_1 \le$ dim $V_\mu \le \chi$. Assume that $\{u_{hi}\}$ and $\{\overset{*}{u}_{hi}\}$ are such the approximate eigenfunctions of u_i and $\overset{*}{u}_i$ $(i = 1, ...\chi_1)$, that satisfy the following normalized conditions

$$\begin{cases} \left\langle u_{hi}, \overset{*}{u}_{hj} \right\rangle = \delta_{ij}, \quad i,j = 1, \cdots, \chi_1, \\ \left\langle u_{hi}, \overset{*}{u}_i \right\rangle = 1, i = 1, ..., \chi_1. \end{cases} \tag{3.13}$$

Theorem 1. *Under the hypotheses of Lemma 1,* (3.13) *and* (3.12)*, then*

$$\begin{cases} |\mu_h - \mu| = O(||A\,(A - A_h)\,||_0), \\ ||u_i - u_{hi}||_0 = O\,(||A\,(A - A_h)\,||_0)\,. \end{cases} \tag{3.14}$$

Proof. Taking the inner product by u_i^* on the both sides of the identity

$$(\mu I - A)\,(u_i - u_{hi}) = -\mu u_{hi} + A u_{hi}$$

$$= (\mu_h - \mu)\, u_{hi} + (Au_{hi} - A_h u_{hi}), \tag{3.15}$$

and using (3.13) and (3.12), we get

$$0 = \left\langle (\mu I - A)(u_i - u_{hi}), \overset{*}{u}_i \right\rangle$$

$$= (\mu_h - \mu) \left\langle u_{hi}, \overset{*}{u}_i \right\rangle + \left\langle Au_{hi} - A_h u_{hi}, \overset{*}{u}_i \right\rangle$$

$$= \mu_h - \mu + \left\langle A (A - A_h) u_{hi}, \overset{*}{u}_i \right\rangle / \mu,$$

or

$$|\mu_h - \mu| = O(||A(A - A_h)||_0). \tag{3.16}$$

Define the subspace

$$V_\mu^\perp = \left\{ v : \left\langle v, \overset{*}{u}_j \right\rangle = 0,\ j = 1, ..., \chi \right\}.$$

Obviously, under the restriction of the subspace $V_\mu^\perp$, $(\mu I - A)^{-1}$ is existent. Since $u_i - u_{hi} \in V_\mu^\perp$, from (3.15) we deduce that there exists a constant $c > 0$ satisfying

$$c||u_i - u_{hi}||_0 \leq ||(\mu I - A)(u_i - u_{hi})||_0$$

$$\leq |\mu_h - \mu| ||u_{hi}||_0 + ||(A - A_h) A_h||_0 ||u_{hi}||_0 / |\mu_h| \tag{3.17}$$

Using (3.17), (3.16) and $||u_{hi}||_0 \leq ||u_i - u_{hi}||_0 + ||u_i||_0$, we complete the proof of (3.14).

Corollary 2. *If $u_i, u_i^* \in C^2[0, 2\pi]$, $j = 1, ..., \chi$, then*

$$|\mu_h - \mu| = O(h^2), \tag{3.18}$$

$$||u_i - u_{hi}||_0 = O\left(h^2\right), \tag{3.19}$$

and

$$||u_i^* - u_{hi}^*||_0 = O\left(h^2\right). \tag{3.20}$$

Proof. Since (3.18) and (3.19) are true (see [4,8,11]), we only give the proof of (3.20).

Define the subspace

$$V_{\mu h}^{*\perp} = \left\{ v \in L^2[0, 2\pi] : \langle v, u_{hj} \rangle = 0,\ j = 1, ..., \chi_1 \right\}.$$

Note that under the restriction of the subspace $V_{\mu h}^{*\perp}$, $(\bar{\mu} I - A_h^*)^{-1}$ is existent and uniformly bounded. Since $u_i^* - u_{hi}^* \in V_{\mu h}^{*\perp}$ by (3.13), from

$$(\bar{\mu}_h I - A_h^*)(u_i^* - u_{hi}^*) = \bar{\mu}_h u_i^* - A_h^* u_i$$

$$= \frac{\bar{\mu}_h}{\bar{\mu}} A^* u_i^* - A_h^* u_i^* = \frac{1}{\bar{\mu}} (\frac{\bar{\mu}_h}{\bar{\mu}} A^* - A_h^*) A^* u_i^*,$$

$$= \frac{1}{\bar{\mu}} (A^* - A_h^*) A^* u_i^* + O(h^2) = O(h^2)$$

we get the proof of (3.20).

4. h^3-Richardson Extrapolation and a Posterior Error Estimate

Theorem 2. *Under the hypotheses of Lemma 2 and Theorem 1, if $\{\mu, u_i\}$ and $\{\mu_h, u_{hi}\}$ are the eigenvalue and eigenfunctions of* (3.1) *and* (3.2), *respectively, then there exist a constants c_1 and functions $v_{i1} \in C^4[0, 2\pi]$, $i = 1, ..., \chi_1$, independent of h, such that*

$$\mu_h - \mu = c_1 h^3 + O\left(h^4\right), \tag{4.1a}$$

$$u_{hi} - u_i = v_{i1}h^3 + O\left(h^4\right). \tag{4.1b}$$

Proof. From Lemma 2 we obtain

$$\begin{aligned} & A_h\left(u_i + v_{i1}h^3\right) - \left(\mu + c_1h^3\right)\left(u_i + v_{i1}h^3\right) \\ = & \left(A_h - A\right)u_i + h^3\left(A_hv_{i1} - c_1u_i - \mu v_{i1}\right) - c_1v_{i1}h^6 \\ = & h^3\left(Av_{i1} - \mu v_{i1} - c_1u_i + \psi_1\right) + O\left(h^4\right). \end{aligned} \tag{4.2}$$

Choose the constant c_1 and the function v_{i1} satisfying the following operator equations:

$$Av_{i1} - \mu v_{i1} = c_1u_i - \psi_1, \tag{4.3a}$$

$$\langle c_1u_i - \psi_1, \phi\rangle = 0, \ \forall\phi \in V_\mu^\perp \tag{4.3b}$$

Obviously, under the restriction of $(4.3b)$, there exists a unique solution v_{i1} in (4.3). Taking $\phi = u_i^*$, we get

$$c_1 = \langle\psi_1, u_i^*\rangle.$$

Thus (4.2) becomes

$$A_h\left(u_i + v_{i1}h^3\right) - \left(\mu + c_1h^3\right)\left(u_i + v_{i1}h^3\right) = O\left(h^4\right). \tag{4.4}$$

Since $\{\mu_h, u_{hi}\}$ satisfies

$$A_hu_{hi} - \mu_hu_{hi} = 0, \tag{4.5}$$

by (4.4) and (4.5) we get

$$\begin{aligned} & A_h\left(u_{hi} - u_i - v_{i1}h^3\right) - \mu_h\left(u_{hi} - u_i - v_{i1}h^3\right) - \\ & \left(\mu_h - \mu - c_1h^3\right)\left(u_i + v_{i1}h^3\right) = O\left(h^4\right). \end{aligned} \tag{4.6}$$

Taking the inner product on the both sides of (4.6) by u_{hi}^* and using (3.12)- (3.14) and

$$\begin{aligned} \langle u_i, u_{hi}^*\rangle &= \langle u_i, u_i^*\rangle + \langle u_i, u_{hi}^* - u_i^*\rangle \\ &= 1 + \langle u_i - u_{hi}, u_{hi}^* - u_i^*\rangle = 1 + O(h^4), \end{aligned}$$

we get

$$\mu_h - \mu - c_1h^3 = O\left(h^4\right), \tag{4.7}$$

Substituting (4.7) into (4.6) we have

$$\left(A_h - \mu_hI\right)\left(u_{hi} - u_i - h^3v_{i1}\right) = O\left(h^4\right) \tag{4.8}$$

Obviously, under the restriction in the invariant subspace

$$V_{\mu h}^\perp = \left\{v : \left\langle v, \overset{*}{u}_{hi}\right\rangle \ = 0, i = 1, ..., \chi_1\right\}$$

the operator $(A_h - \mu_hI)$ is invertible and $(A_h - \mu_hI)^{-1}$ is uniformly bounded. Generally speaking $u_{hi} - u_i - h^3v_{i1} = g \notin V_{\mu h}^\perp$, but $g - P_hg \in V_{\mu h}^\perp$, where

$$P_hg = \sum_{i=1}^{\chi_1}\left\langle g, \overset{*}{u}_{hi}\right\rangle u_{hi} \tag{4.9}$$

is a projection of g on $V_{\mu h}$.By means of (3.13) we get the estimate

$$\begin{aligned} & \left|\left\langle g, \overset{*}{u}_{hi}\right\rangle\right| = \left|\left\langle g, \overset{*}{u}_{hi} - \overset{*}{u}_i\right\rangle\right| \leq \\ & \left|\left\langle u_{hi} - u_i, \overset{*}{u}_{hi} - \overset{*}{u}_i\right\rangle\right| + \left|\left\langle v_{i1}, \overset{*}{u}_{hi} - \overset{*}{u}_i\right\rangle\right| h^3 = O\left(h^4\right), \end{aligned}$$

which means $\|P_h g\|_0 = O(h^4)$. However, from (4.8) we have

$$\begin{aligned} O(h^4) &= \|(A_h - \mu_h I)\, g\|_0 \ge \|(A_h - \mu_h I)\,(g - P_h g)\|_0 - O(h^4) \\ &\ge C\|g - P_h g)\|_0 - O(h^4), \end{aligned}$$

i.e., $\|g - P_h g)\|_0 = O(h^4)$. Therefore we get $g = u_{hi} - u_i - h^3 v_{i1} = O\left(h^4\right)$, that is, (4.1) is shown.

Let $(\mu_h,\, u_h)$ and $(\mu_{h/2},\, u_{h/2})$ be the solutions of (2.6) according to mesh widths h, $h/2$, respectively. Then from (4.1) h^3-Richardson extrapolations

$$\bar{\mu}_h = (8\mu_{h/2} - \mu_h)/7, \tag{4.10}$$

and

$$\bar{u}_h\left(t_i\right) = (8u_{h/2}\left(t_i\right) - u_h\left(t_i\right))/7, t_i = ih, i = 0, ..., n-1 \tag{4.11}$$

have the error estimates: $|\bar{\mu}_h - \mu| = O(h^4)$ and $|\bar{u}_h(t_i) - u(t_i)| = O(h^4)$.

Moreover from the asymptotic expansions (4.1), we shall derive the following a posterior error estimates

$$\begin{aligned} |\mu_{h/2} - \mu| &\le |8/7\mu_{h/2} - 1/7\mu_h - \mu_{h/2}| + O\left(h^4\right) \\ &\le |\mu_{h/2} - \mu_h|/7 + O\left(h^4\right) \end{aligned} \tag{4.12}$$

and

$$\begin{aligned} |u_{h/2}(t_i) - u(t_i)| &\le |8/7u_{h/2}\left(t_i\right) - 1/7u_h\left(t_i\right) - u_{h/2}\left(t_i\right)| + O\left(h^4\right) \\ &\le 1/7|u_{h/2}\left(t_i\right) - u_h\left(t_i\right)| + O\left(h^4\right), \end{aligned} \tag{4.13}$$

which can be used to construct adaptive algorithms.

5. Numerical Example

Assume that the boundary Γ of the problem (1.1) is the unit circle and $\alpha = 1$, then its eigenvalues are as follows[13]

$$\lambda_0 = \frac{\sum_{k=0}^{\infty} 2^{-2k-1}/[k!(k+1)!]}{\sum_{k=0}^{\infty} 2^{-2k-1}/[k!k!]} \text{ and } \lambda_n = \frac{\sum_{k=0}^{\infty} 2^{-2k}(2k+n)/[k!(k+n)!]}{\sum_{k=0}^{\infty} 2^{-2k}/[k!(k+n)!]}$$

where λ_0 is simple and λ_n $(n > 0)$ is double.

Table 1 and Table 2 show the errors of the approximate eigenvalues by using (2.6) and the methods in [13], when n=4, 8 and n=16, 32, respectively. It is easy to see that the error of the first eigenvalue in [13] is smaller than our's, the others bigger than our's. Moreover the order of accuracy of our methods is $O(h^3)$; the extrapolations and a posteriori estimate are very effective.

Table 1. the errors $E_i = |\lambda_i - \lambda_{hi}|(n = 4$ and $n = 8)$

n	4	4	8	8	8	8
method	[13]	(2.6)	[13]	(2.6)	extra.	pos err
E_0	4.7e-6	9.13e-3	8.0e-7	5.61e-4	6.92e-5	1.13e-3
E_1	8.39e-2	5.9e-2	4.09e-3	7.7e-3	3.74e-4	7.33e-3
E_2	9.83e-1	5.77e-1	9.69e-2	8.54e-2	1.15e-2	7.02e-2
E_3			7.17e-1	3.91e-1		
E_4			1.6352	1.11		

Table 2. the errors $E_i = |\lambda_i - \lambda_{hi}|(n = 16 \text{ and } n = 32)$

n	16	16	16	16	32	32	32
method	[13]	(2.6)	extra.	pos err	(2.6)	extra.	pos err
E_0	3.0e-7	1.42e-4	1.9e-6	1.5e-4	1.91e-5	2.0e-7	1.89e-5
E_1	3.26e-4	9.66e-4	3.8e-6	9.6e-4	1.07e-4	2.0e-7	1.2e-4
E_2	5.71e-3	1.07e-2	3.14e-3	1.06e-2	1.39e-3	3.0e-6	1.34e-3
E_3	4.03e-2	5.08e-2	2.24e-3	4.86e-2	6.32e-3	3.0e-5	6.36e-3
E_4	1.76e-1	1.56e-1	1.96e-2	1.37e-1	1.95e-2	5.77e-5	1.96e-2
E_5	9.05e-1	3.76e-1			4.72e-2	3.25e-4	4.69e-2
E_6	1.6645	7.62e-1			9.76e-2	2.70e-3	9.49e-2
E_7	2.5396	1.3773			1.80e-1	1.10e-2	1.69e-1
E_8	3.3717	2.2270			3.07e-1	3.32e-2	2.74e-1

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