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ON APPROXIMATION OF LAPLACIAN EIGENPROBLEM OVER A REGULAR HEXAGON WITH ZERO BOUNDARY CONDITIONS *1)

Jia-chang Sun
(*Parallel Computing Division, Institute of Software, Chinese Academy of Sciences Beijing 100080, China*)
(*E-mail: sunjc@mail.rdcps.ac.cn*)

Dedicated to Professor Zhong-ci Shi on the occasion of his 70th birthday

Abstract

In my earlier paper [4], an eigen-decompositions of the Laplacian operator is given on a unit regular hexagon with periodic boundary conditions. Since an exact decomposition with Dirichlet boundary conditions has not been explored in terms of any elementary form. In this paper, we investigate an approximate eigen-decomposition. The function space, corresponding all eigenfunction, have been decomposed into four orthogonal subspaces. Estimations of the first eight smallest eigenvalues and related orthogonal functions are given. In particulary we obtain an approximate value of the smallest eigenvalue $\lambda_1 \backsim \frac{29}{40}\pi^2 = 7.1555$, the absolute error is less than 0.0001.

Mathematics subject classification: 42B35, 65N25, 65T50
Key words: Laplacian eigen-decomposition, Regular hexagon domain, Dirichlet boundary conditions.

1. Introduction

Given an origin point O and two plane vectors e_1 and e_2, e.g. $e_1 = \{0, -1\}$, $e_2 = \{\frac{\sqrt{3}}{2}, \frac{1}{2}\}$, we form a 3-direction 2-D partition as drawn in Fig.1. To deal with symmetry along the three direction, we apply a 3-direction coordinates instead of the usual 2-D Cartesian coordinates and barycentry coordinates, which is a very useful within a triangle domain. Setting the origin point $O = (0, 0, 0)$, each partition line is represented by t_l=integer (l=1,2,3), and each 2-D point P is represented by

$$P = (t_1, t_2, t_3), \quad t_1 + t_2 + t_3 = 0, \tag{1.1}$$

and any function $f(P)$ defined on the plane can be written as $f(P) = f(t_1, t_2, t_3)$. In particulary, P_k is called an integer node if and only if $P_k = (k_1, k_2, k_3), \quad k_1 + k_2 + k_3 = 0$.

Let Ω be the unit regular hexagon domain

$$\Omega = \{P | P = (t_1, t_2, t_3) \qquad t_1 + t_2 + t_3 = 0, \quad -1 \le t_1, t_2, t_3 \le 1\} \tag{1.2}$$

we consider the following eigenvalue problem

$$-\Delta u = \lambda u, \tag{1.3}$$

with zero Dirichlet boundary

$$u|_{\partial\Omega} = 0 \tag{1.4}$$

In terms of the 3-direction partition form, the Laplacian operator can be written as

$$\mathcal{L} = -\frac{2}{3}\Delta = -(\frac{\partial}{\partial t_1} - \frac{\partial}{\partial t_2})^2 - (\frac{\partial}{\partial t_2} - \frac{\partial}{\partial t_3})^2 - (\frac{\partial}{\partial t_3} - \frac{\partial}{\partial t_1})^2. \tag{1.5}$$

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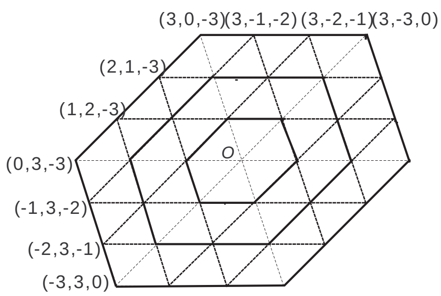

Figure 1.1: 3-direction partition

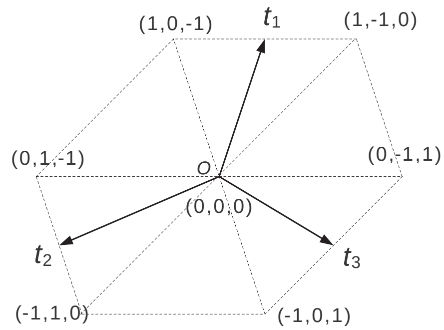

Figure 1.2: Parallel hexagon domain Ω

Definition 1.1. *A function $f(P)$, defined in the 3-direction coordinate, is called periodic, if for all $P = (t_1, t_2, t_3)$ the equality*

$$f(P + Q) = f(P)$$

holds for any integer vector $Q = (n_1, n_2, n_3)$ with $n_1^2 + n_2^2 + n_3^2 = 0 \pmod 6$

The following results have been proved in my earlier paper [4].

Theorem 1.1. *For all integer triple $j = (j_1, j_2, j_3)$, the function system*

$$g_j(P) = e^{i\frac{2\pi}{3}(j_1t_1+j_2t_2+j_3t_3)}$$

forms a complex eigen decomposition of the Laplacian operator (1.3) with three direction periodic boundary conditions, where $P = (t_1, t_2, t_3)$, $j_1 + j_2 + j_3 = 0$. The corresponding eigenvalues equal to

$$\lambda_{j_1,j_2,j_3} = (\frac{2\pi}{3})^2((j_1 - j_2)^2 + (j_2 - j_3)^2 + (j_3 - j_1)^2) \tag{1.6}$$

Since

$$\lambda_{j_1,j_2,j_3} = \lambda_{j_2,j_3,j_1} = \lambda_{j_3,j_1,j_1} = \lambda_{-j_1,-j_3,-j_2} = \lambda_{-j_3,-j_2,-j_1} = \lambda_{-j_2,-j_1,-j_3}$$

we have the following real eigen-decomposition.

Corollary 1.1. *For all integer triple $j = (j_1, j_2, j_3)$, two function system*

$$\cos(\frac{2\pi}{3}(j_1t_1 + j_2t_2 + j_3t_3)) \quad \textit{and} \quad \sin(\frac{2\pi}{3}(j_1t_1 + j_2t_2 + j_3t_3))$$

form an eigen decomposition of the Laplacian operator (1.3) with three direction periodic boundary conditions. Moreover, except the smallest eigenvalue is single, all other eigenvalues are six multiple.

It is clear that the first eigenfunction is a trivial constant. Several figures of eigenfunctions, related from the second to the four without counting multiple, are drawn in Figure 1.3- 1.12.

Definition 1.2. $TSin_j(P) := \frac{1}{2i}\big[g_{j_1,j_2,j_3}(P) + g_{j_2,j_3,j_1}(P) + g_{j_3,j_1,j_2}(P)$

$$-g_{-j_1,-j_3,-j_2}(P) - g_{-j_2,-j_1,-j_3}(P) - g_{-j_3,-j_2,-j_1}(P)\big] \tag{1.7}$$

Definition 1.3. $TCos_j(P) := \frac{1}{2}\big[g_{j_1,j_2,j_3}(P) + g_{j_2,j_3,j_1}(P) + g_{j_3,j_1,j_2}(P)$

$$+g_{-j_1,-j_3,-j_2}(P) + g_{-j_2,-j_1,-j_3}(P) + g_{-j_3,-j_2,-j_1}(P)\big] \tag{1.8}$$

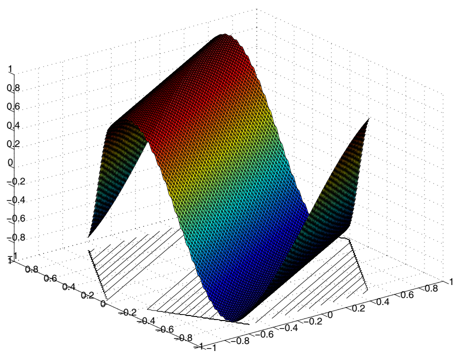

Figure 1.3: 2-nd eigen. function 1

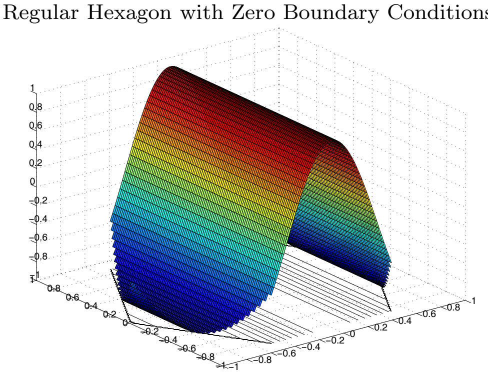

Figure 1.4: 2-nd eigen. function 2

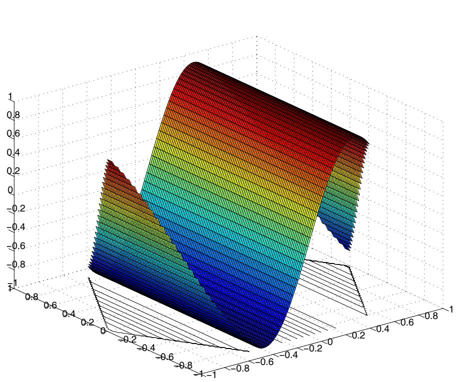

Figure 1.5: 2-nd eigen. function 3

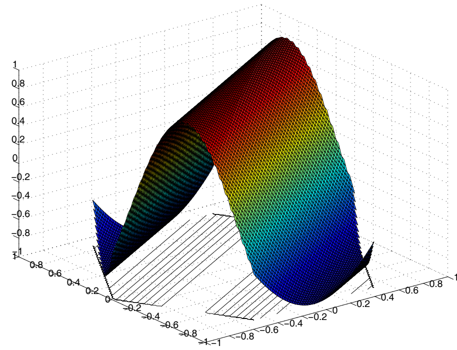

Figure 1.6: 2-nd eigen. function 4

Theorem 1.2. *$TSin_j(P)$ and $TCos_j(P)$ form an eigen-decoposition system of the Laplacian operator (1.3) on an equilateral triangle subdomain: $t_1 \leq 0$ and $t_2, t_3 \geq 0$ with zero Dirichlet and Neumann boundary conditions, respectively. The range of integer triple j in eigenvalue expression (1.6) corresponds*

$$j_1 > 0, \quad j_2, j_3 < 0$$

and

$$j_1 \geq 0, \quad j_2, j_3 \leq 0$$

respectively.

2. Two Kinds of Orthogonal Decomposition for a Function Defined on a Parallel Hexagon

Definition 2.1. *A function f, defined over a parallel hexagon domain Ω, is called partial symmetry with respect to direction t_1 if*

$$f(t_1, t_2, t_3) = f(-t_1, -t_3, -t_2)$$

or is called partial asymmetry with respect to direction t_1 if

$$f(t_1, t_2, t_3) = -f(-t_1, -t_3, -t_2)$$

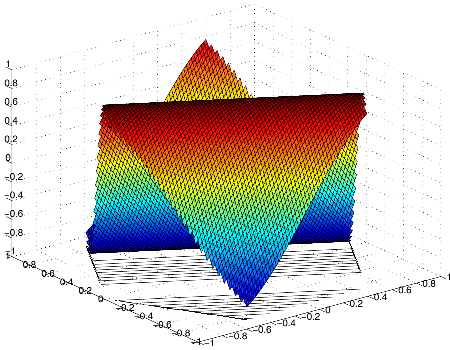

Figure 1.7: 2-nd eigen. function 5

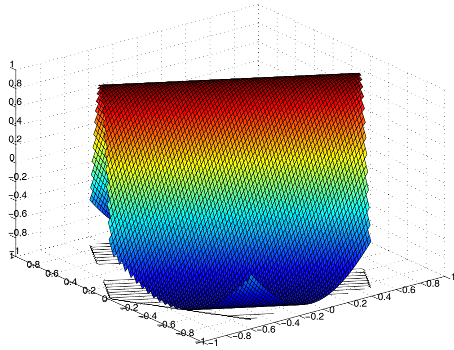

Figure 1.8: 2-nd eigen. function 6

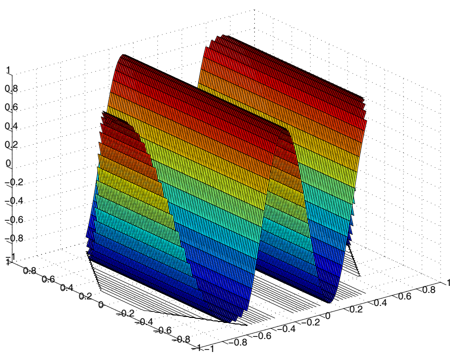

Figure 1.9: 3-nd eigen. function 1-3

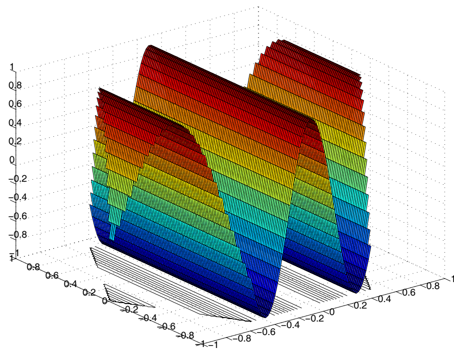

Figure 1.10: 2-nd eigen. function 4-6

Definition 2.2. *A function f, defined over a parallel hexagon domain Ω, is called partial symmetry with respect to axis $t_2 - t_3$ if*

$$f(t_1, t_2, t_3) = f(t_1, t_3, t_2)$$

or is called partial asymmetry with respect to axis $t_2 - t_3$ if

$$f(t_1, t_2, t_3) = -f(t_1, t_3, t_2)$$

Lemma 2.1. *If f is partial symmetry with respect to direction t_1 and g is partial asymmetry*

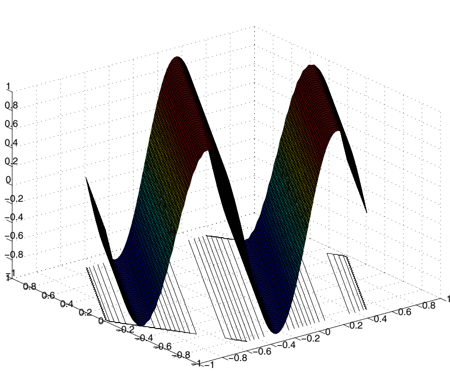

Figure 1.11: 4-th eigen. function 1-3

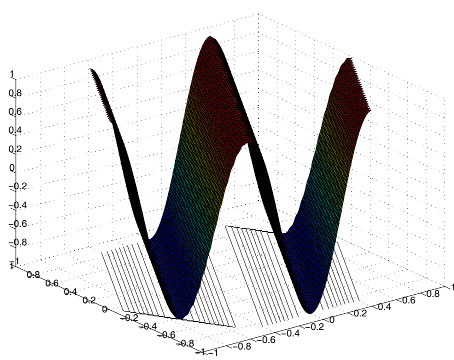

Figure 1.12: 4-th eigen. function 4-6

with respect to a direction t_1 on the same parallel hexagon, then the two functions are orthogonal

$$< f, g >:= \int_\Omega f(P)\bar{g}(P)dp = 0$$

Lemma 2.2. *If f is partial symmetry with respect to axis $t_2 - t_3$ and g is partial asymmetry with respect to axis $t_2 - t_3$ on the same parallel hexagon, then the two functions are orthogonal.*

There are similar definitions and lemmas of partial symmetry and asymmetry with respect to direction t_2, t_3 and axes $t_3 - t_1$ and $t_1 - t_2$, respectively.

By using the above lemmas, it leads to the following orthogonal decomposition theorem

Theorem 2.1. *Any function f, defined over a parallel hexagon domain Ω, can be decomposed into following four orthogonal sub-functions*

$$f(t_1, t_2, t_3) = f_1(t_1, t_2, t_3) + f_2(t_1, t_2, t_3) + f_3(t_1, t_2, t_3) + f_4(t_1, t_2, t_3) \tag{2.1}$$

where

$$f_1(t_1, t_2, t_3) = \frac{1}{4}\{f(t_1, t_2, t_3) + f(-t_1, -t_3, -t_2) + f(t_1, t_3, t_2) + f(-t_1, -t_2, -t_3)\}$$

is both partial symmetry with respect to direction t_1 and the axis $t_2 - t_3$,

$$f_2(t_1, t_2, t_3) = \frac{1}{4}\{f(t_1, t_2, t_3) - f(-t_1, -t_3, -t_2) + f(t_1, t_3, t_2) - f(-t_1, -t_2, -t_3)\}$$

is partial asymmetry with respect to direction t_1 and partial symmetry with the axis $t_2 - t_3$

$$f_3(t_1, t_2, t_3) = \frac{1}{4}\{f(t_1, t_2, t_3) + f(-t_1, -t_3, -t_2) - f(t_1, t_3, t_2) - f(-t_1, -t_2, -t_3)\}$$

is partial symmetry with respect to direction t_1 and partial asymmetry with the axis $t_2 - t_3$

$$f_4(t_1, t_2, t_3) = \frac{1}{4}\{f(t_1, t_2, t_3) - f(-t_1, -t_3, -t_2) - f(t_1, t_3, t_2) + f(-t_1, -t_2, -t_3)\}$$

is both partial asymmetry with respect to direction t_1 and the axis $t_2 - t_3$.

Now we turn to construct another orthogonal decomposition.

Definition 2.3. *A function f, defined over a parallel hexagon domain Ω, is called centered symmetry if*

$$f(t_1, t_2, t_3) = f(-t_1, -t_2, -t_3)$$

or is called centered asymmetry if

$$f(t_1, t_2, t_3) = -f(-t_1, -t_2, -t_3)$$

Lemma 2.3. *If f is centered symmetry and g is centered asymmetry on the same parallel hexagon, then the two functions are orthogonal on the domain*

Definition 2.4. *A function f, defined over a parallel hexagon domain Ω, is called isotropic if f is centered symmetry and partial symmetry with respect to all three directions t_1,t_2 and t_3.*

Denote

$$H_I := \{f | f \quad isotropic\} \tag{2.2}$$

$H_{II} := \{f|f$ centered asymmetry & partial asymmetry with respect to a direction$\}$

$H_{III} := \{f|f$ centered symmetry & partial asymmetry with respect to a direction$\}$

$H_{IV} := \{f|f$ centered asymmetry & partial symmetry with respect to three directions$\}$

Lemma 2.4. *The above four function spaces H_I, H_{II},H_{III} and H_{IV} are orthogonal each other.*

Lemma 2.5. *For a regular hexagon domain, Laplacian operator preserves the above symmetry and asymmetry of the original function.*

Therefore, we lay down our foundation for find approximate eigenvalues and related orthogonal function in succeed sections.

Lemma 2.6. *Both of four spaces, f_1,f_2,f_3,f_4 defined in (2.1) and H_I,H_{II},H_{III},H_{IV} defined in (2.2), are orthogonal decomposition of any eigen-function of (1.3) with (1.4).*

3. Approximation to the Smallest Eight Eigenvalues and Related Functions

3.1. The smallest eigenvalue estimation

As is well know, the smallest eigenvalue of (1.3) with zero boundary conditions (1.4) is the minimum of the following quadratic functional

$$\lambda_1 = \inf_{u \in H_1^0(\Omega)} \frac{< \nabla u, \nabla u >}{< u, u >}$$

Denote Ω_S and Ω_B to be an interior circle with radius $R = \frac{\sqrt{3}}{2}$ and an exterior equilateral triangle with side length $H = 3$, respectively.

It is obvious $\Omega_S \subset \Omega \subset \Omega_B$, hence, by using comparison principle, we have the following inequalities

Lemma 3.1.

$$0.5926\pi^2 = \frac{16}{3H^2}\pi^2 = \lambda_{min}(\Omega_B) < \lambda_1 := \lambda_{min}(\Omega) < \lambda_{min}(\Omega_S) \leq \frac{(\pi^2+4)}{4(\pi^2-4)}\frac{\pi^2}{R^2} = 0.7814\pi^2 \tag{3.1}$$

The lower bound comes from eigenvalues expression of Laplacian in an equilateral triangle with side length 3, e.g. see [1] and [4]. To get the righthand side we rewrite the Laplacian eigen-problem in terms of polar coordinates

$$-\Delta u = -\frac{1}{r}\frac{\partial}{\partial r}(r\frac{\partial u}{\partial r}) - \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}, \qquad u(r,\theta)|_{r=R} = 0 \tag{3.2}$$

Hence, denote $\xi_{0,k}$ to be zero points of the Bessel function $J_0(r)$, then all isotropic eigenvalues of (3.2) equal to $\xi_{0,k}^2/R^2$. Since $J_0(\xi_{0,1})|_{\xi_{0,1}=2.405} = -0.000090558$, $J_0(\xi_{0,2})|_{\xi_{0,2}=5.52} = -0.0000265784$. Hence, the first two isotropic eigenvalues on the circle equal to $2.405^2 * 4/3 = 7.71203$ and $5.52^2 * 4/3 = 40.6272$ approximately, and the related Bessel functions $J_0(\xi_{0,k}r)$, $(k = 1, 2)$ are the strict eigenfunctions. In general, all zero points $\xi_{n,k}$ of Bessel functions $J_n(r)$ correspond eigenvalues of the Laplacian operator on the unit circle with zero boundary condition $\lambda_{n,k} = \xi_{n,k}^2$. The related double eigenfunctions are $J_n(\xi_{n,k}r)\cos n\theta$ and $J_n(\xi_{n,k}r)\sin n\theta$.

Because three areas of the circle, the regular hexagon and the equilateral triangle equal to

$$S_{\Omega_S} = 3\pi/4, \quad S_\Omega = \frac{3\sqrt{3}}{2}, \quad S_{\Omega_B} = \frac{9\sqrt{3}}{4}$$

or

$$S_\Omega = S_{\Omega_S}^{1-t} S_{\Omega_B}^t, \quad t = \frac{\log(\frac{S_\Omega}{S_{\Omega_S}})}{\log(\frac{S_{\Omega_B}}{S_{\Omega_S}})} = 0.1942$$

Corollary 3.1. *We may take an approximation for the first eigenvalue as a generalized geometry means between the upper bound and the lower bound of (3.1)*

$$\lambda_1 \backsimeq \lambda_{min}(\Omega_S)^{(1-t)} \lambda_{min}(\Omega_B)^t \pi^2 = 0.7499\pi^2, \quad where \quad t = 0.1942. \tag{3.3}$$

To get an elementary approximation for the first eigenfunction, we substitute $u(r) = \cos\frac{r\pi}{2R}$ into the following expression

$$\lambda_1 = \inf_{u(R)=0,u(0) bounded} \frac{\int_0^R ru'^2 dr}{\int_0^R ru^2 dr} \tag{3.4}$$

then

$$\lambda_1 \leq \frac{(4+\pi^2)/(16R^2)}{(\pi^2-4)/(4\pi^2)} = \frac{(4+\pi^2)}{4R^2(\pi^2-4)}\pi^2 = 7.7739.$$

Hence, in this sense the function is a good approximate of the first eigenfunction. To obtain a further approximation for an approximation of the first eigenfunction, we denote

$$F_{1,1} = 4\cos\frac{t_1\pi}{2}\cos\frac{t_2\pi}{2}\cos\frac{t_3\pi}{2} = \cos t_1\pi + \cos t_2\pi + \cos t_3\pi + 1; \tag{3.5}$$

$$F_{1,2} = (1-t_1^2)(1-t_2^2)(1-t_3^2). \tag{3.6}$$

These two functions are isotropic and belong to space H_I. The graphics of the function $F_{1,1}$ is shown in Figure 3.1.

A straight computation leads to

$$\frac{<\nabla F_{1,1}, \nabla F_{1,1}>}{<F_{1,1}, F_{1,1}>} = \frac{8\sqrt{3}+3\sqrt{3}\pi^2}{\frac{3}{4}\sqrt{3}(5\pi^2+16)/\pi^2} = \frac{4(8+3\pi^2)\pi^2}{3(5\pi^2+16)} = 0.7674\pi^2$$

$$\frac{<\nabla F_{1,2}, \nabla F_{1,2}>}{<F_{1,2}, F_{1,2}>} = \frac{\frac{24641}{9450}\sqrt{3}}{\frac{3167}{8820}\sqrt{3}} = 0.7358\pi^2$$

Further, if test is done in the space spanned of $F_{1,1}$ and $F_{1,2}$.

Therefore function $F1$ can be taken as the first eigen-function.

Next, the computation can be done in the space spanned of $F21$ and $F22$. In fact, set a stiffness matrix

$$K1 = \begin{bmatrix} <\nabla F_{1,1}, \nabla F_{1,1}> & <\nabla F_{1,1}, \nabla F_{1,2}> \\ <\nabla F_{1,2}, \nabla F_{1,1}> & <\nabla F_{1,2}, \nabla F_{1,2}> \end{bmatrix} = \begin{bmatrix} 65.1404 & 16.9864 \\ 16.9864 & 4.5163 \end{bmatrix}$$

and mass matrix

$$M1 = \begin{bmatrix} <F_{1,1}, F_{1,1}> & <F_{1,1}, F_{1,2}> \\ <F_{1,2}, F_{1,1}> & <F_{1,2}, F_{1,2}> \end{bmatrix} = \begin{bmatrix} 8.6011 & 2.3084 \\ 2.3084 & 0.6219 \end{bmatrix}$$

The smallest eigenvalue of the related generalized matrix eigenvalue problem equals to $7.1847 = 0.7280\pi^2$.

The estimation can be improve further if we take the minimum in more wide space

$$F1(t_1,t_2,t_3) = Span_{\nu=1,\ldots,6}\{F_{1,\nu}\}$$

where

$$F_{13} = W_1 + W_2 + W_3; \qquad W_1 = 4\cos\frac{\pi t_1}{2}\sin\pi t_2 \sin\pi t_3;$$

$$W_2 = 4\sin\pi t_1\cos\frac{\pi t_2}{2}\sin\pi t_3; \qquad W_3 = 4\sin\pi t_1\sin\pi t_2\cos\frac{\pi t_3}{2};$$

$$F_{14} = 4\cos\frac{3\pi t_1}{2}\cos\frac{\pi t_2}{2}\cos\frac{\pi t_1 3}{2}; \quad F_{15} = 4\cos\frac{\pi t_1}{2}\cos\frac{3\pi t_2}{2}\cos\frac{\pi t_3}{2};$$

$$F_{16} = 4\cos\frac{\pi t_1}{2}\cos\frac{\pi t_1}{2}\cos\frac{3\pi t_3}{2}.$$

The minimum value

$$\lambda_1 \backsimeq 7.1650 = 0.7260\pi^2$$

reaches when

$$F_1 = -0.8946F_{1,1} + 3.9425F_{1,2} - 0.1105F_{13} + 0.0713(F_{14}+F_{15}+F_{16})$$

Finally, if we add the following extra five high frequency functions

$$F_{17}=4\cos\frac{\pi t_1}{2}\cos\frac{3\pi t_2}{2}\cos\frac{3\pi t_3}{2};\qquad F_{18}=4\cos\frac{3\pi t_1}{2}\cos\frac{\pi t_2}{2}\cos\frac{3\pi t_3}{2};$$

$$F_{19}=4\cos\frac{3\pi t_1}{2}\cos\frac{3\pi t_2}{2}\cos\frac{\pi t_3}{2};\qquad F_{1,10}=4\cos\frac{3\pi t_1}{2}\cos\frac{3\pi t_2}{2}\cos\frac{3\pi t_3}{2};$$

and

$$F_{1,11}=F_{1,2}^2$$

into the above space, we may obtain more accuracy upper bound for the smallest eigenvalue

$$\lambda_1\leq 7.1567=0.72512\pi^2.$$

As is well known that the numerical results of first smallest eigen-values are less than their true values. Therefore, we obtain a very close pair of upper bound and lower bound for the first smallest eigenvalue of (1.3) with zero boundary conditions (1.4).

Proposition 3.1.

$$0.72490\pi^2<\lambda_1<0.72512\pi^2 \tag{3.7}$$

3.2. The 2nd to 5th eigenvalue approximation

According to the eigenvalue max-min property, we know

$$\lambda_2=\sup_{u_1}\inf_{u\in H_1^0(\Omega)\bigcap<u,u1>=0}\frac{<\nabla u,\nabla u>}{<u,u>}$$

Since the first eigen-function is invariant under rotation 60^o with non zero within the hexagon domain, the second eigen function must be partial asymmetry along a bisection line, e.g. $t_3=0$ or $t_2=t_1$, and the line in the only zero line within the domain. Therefore, the second eigenvalue is double multiple.

Denote

$$G_1=4\sin\pi t_1\cos\frac{\pi t_2}{2}\cos\frac{\pi t_3}{2};\quad G_2=4\cos\frac{\pi t_1}{2}\sin\pi t_2\cos\frac{\pi t_3}{2};\quad G_3=4\cos\frac{\pi t_1}{2}\cos\frac{\pi t_2}{2}\sin\pi t_3;$$

The three functions are partial asymmetry along direction t_1, t_2 and t_3, respectively. They have no zero lines except on their asymmetry axis.

$$F_{21}=G_2+G_3-2G_1;\qquad F_{22}=\sqrt{3}(G_2-G_3);$$

It is obvious that function F_{21} and F_{22} are partial asymmetry with respect to direction t_1 and t_2-t_3, respectively. Hence, both of them belong to H_{II}, orthogonal to $F1$ in H_I, and they are orthogonal each other.

By symmetry and asymmetry, it is obvious

$$\begin{cases}<G_1,F_1>=<G_2,F_1>=<G_3,F_1>=0,\\ <G_1,G_2>=<G_2,G_3>=<G_3,G_1>,\\ <G_1,G_1>=<G_2,G_2>=<G_3,G_3>.\end{cases} \tag{3.8}$$

Hence

$$<F_{21},F_{22}>=<F_{21},F_1>=<F_{22},F_1>=0$$

Moreover

$$<F_{21},F_{21}>=<F_{22},F_{22}>=\frac{2\sqrt{3}}{75\pi^2}(4048+720\pi+675\pi^2)=60.7064$$

$$<\Delta F_{21},F_{21}>=<\Delta F_{22},F_{22}>=(\frac{40192}{225}+\frac{592}{15}\pi+36\pi^2)\sqrt{3}=1139.6$$

$$\lambda_2=\lambda_3\leq=\frac{1139.6}{60.7064}=8.7717=1.9020\pi^2$$

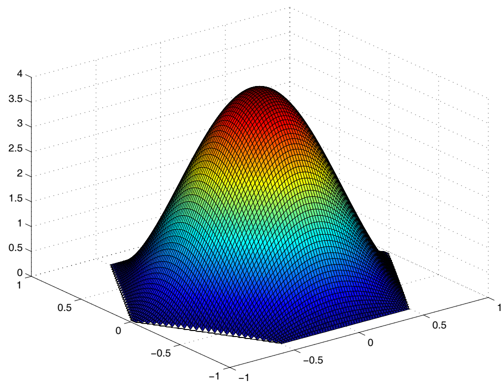

Figure 3.13: Minimum surface approximation $F_{1,1}$

Similarly, the $4th$ and $5th$ eigenvalue are also double multiple. Their corresponding functions have two across bisection zero lines.

Denote

$$W_1 = 4\cos\frac{\pi t_1}{2}\sin\pi t_2\sin\pi t_3; W_2 = 4\sin\pi t_1\cos\frac{\pi t_2}{2}\sin\pi t_3; W_3 = 4\sin\pi t_1\sin\pi t2\cos\frac{\pi t_3}{2};$$

$$F_{31} = \sqrt{3}(W_2 - W_3); \qquad F_{32} = W_2 + W_3 - 2W_1;$$

The two functions are orthogonal each other $< F_{31}, F_{32} >= 0$, because of permutation symmetry,

$$\begin{cases} < W_1, W_2 >=< W_2, W_3 >=< W_3, W_1 >, \\ < W_1, W_1 >=< W_2, W_2 >=< W_3, W_3 > . \end{cases} \tag{3.9}$$

Note that both of F_{31} and F_{32} belong to H_{III}, they are orthogonal to $F_1 \in H_I$ and $F_{21}, F_{22} \in H_{II}$. Moreover the two Rayleigh quotations are the same, because

$$< F_{31}, F_{31} >=< F_{32}, F_{32} >= \frac{2\sqrt{3}}{3675\pi^2}(70208 + 6720\pi + 33075\pi^2) = 39.8985$$

$$< \Delta F_{31}, F_{31} >=< \Delta F_{32}, F_{32} >= (\frac{1301248}{11025} + \frac{3904}{105}\pi + 54\pi^2)\sqrt{3} = 1329.9$$

$$\lambda_4 = \lambda_5 \leq= \frac{1329.9}{39.8985} = 33.3310 = 3.3771\pi^2$$

3.3. The 6th to 8th eigenvalue approximation

As another isotropic function in H_I we may take

$$F6 = W_1 + W_2 + W_3;$$

By using (3.9) and isotropic property of $F6 \in H_I$, it is orthogonal to the above four functions $F_{21}, F_{22} \in H_{II}$ and $F_{31}, F_{32} \in H_{III}$.

$$< F6, F_{21} >=< F6, F_{22} >=< F6, F_{31} >=< F6, F_{32} >= 0.$$

To get an approximation to the second isotropic eigenfunction, we find a linear combination of two isotropic functions $F_{1,1}$ in (3.5) and the above $F6$ such that

$$F_6 = 0.7822F1 + 0.6230F6$$

It corresponding the larger generalized eigenvalues of the sub-space spanned of two functions $F1$ and $F6$. And

$$< F_6, F_6 >= 2.4580, \quad < -\Delta F_6, F_6 >= 94.1353$$

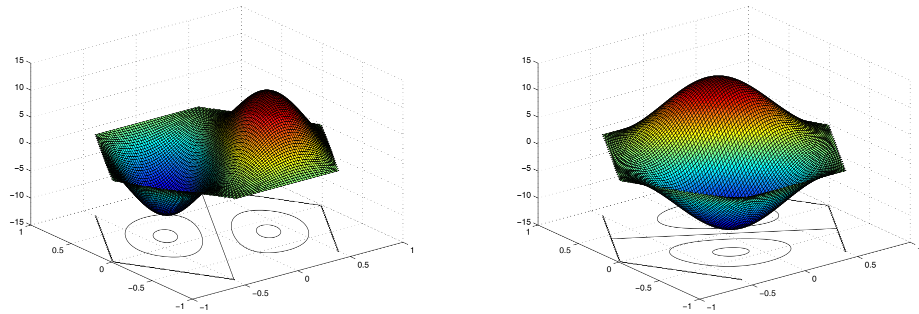

Figure 3.14: Two orthogonal surfaces F_{21} andF_{22} related to $\lambda_2 = \lambda_3$

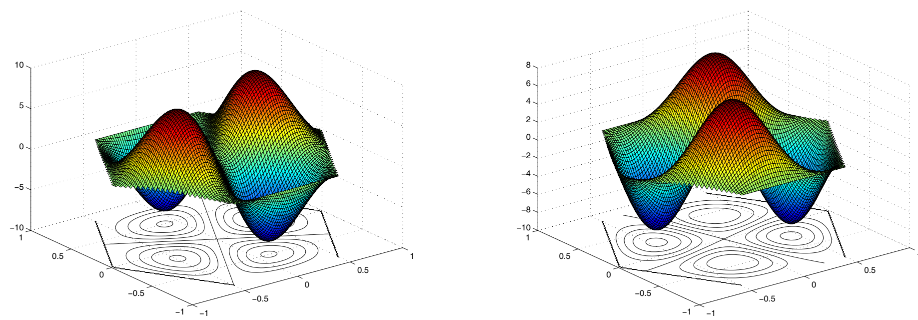

Figure 3.15: Two orthogonal surfaces F_{31} and F_{32} related to $\lambda_4 = \lambda_5$

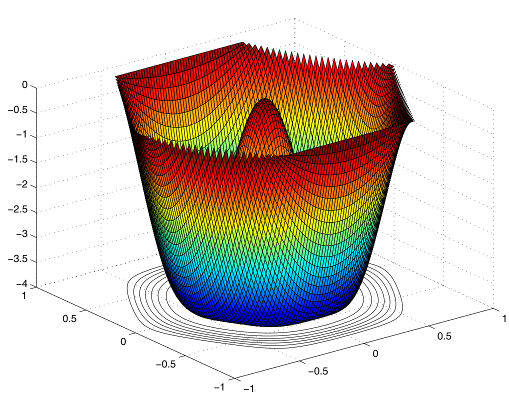

Figure 3.16: Isotropic surface F_6 related to λ_6

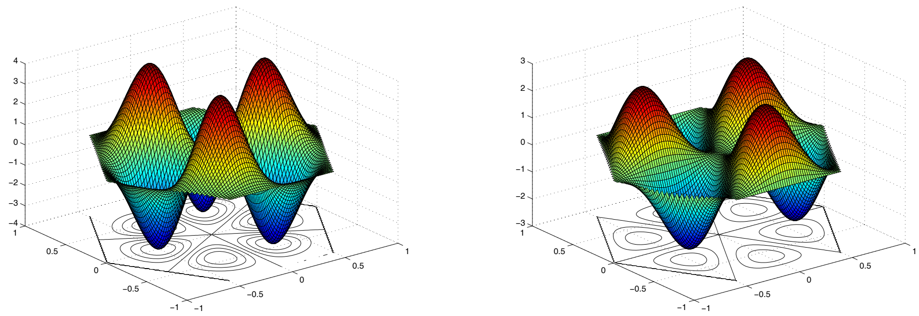

Figure 3.17: Two orthogonal surfaces F_7 and F_8 related to λ_7 and $\lambda_8(\lambda_7 < \lambda_8)$

the Rayleigh quotation

$$\lambda_6 \leq \frac{94.1353}{2.4580} = 38.2969 = 3.8803\pi^2$$

The functions corresponding the 7th and 8th belong to space H_{IV}. They are centered asymmetry and have zero lines on three direction $t_\nu = integer$ or on the three bisection $t_\mu - t_\nu = integer, \quad \mu \neq \nu$, respectively.

$$F_7 = 4\sin(t_1 - t_2)\pi \sin(t_2 - t_3)\pi \sin(t_3 - t_1)\pi F_1;$$
$$F_8 = 4\sin\pi t_1 \sin\pi t_2 \sin\pi t_3; \tag{3.10}$$

Since F_7 is partial symmetry to axis $t_2 - t_3$ and F_8 is partial asymmetry to $t_2 - t_3$, hence they are orthogonal both in ordinary sense and in energy sense.

$$< F_7, F_8 >=< -\Delta F_7, F_8 >= 0$$

As F_7 and F_8 belong to space H_{IV}, by Lemma 2.4, they are orthogonal to $F_1, F_6 \in H_I$, $F_{21}, F_{22} \in H_{II}$ and $F_{31}, F_{32} \in H_{III}$.

Moreover, a straight computation leads to

$$< -\Delta F_7, F_7 >= (-\frac{295408}{11025} - \frac{1844}{105}\pi + \frac{39}{4}\pi^2)\sqrt{3} = 24.7024$$

$$< F_7, F_7 >= \frac{\sqrt{3}}{2450\pi^2}(-33016 - 21840\pi + 11025\pi^2) = 0.5146$$

The Rayleigh quotation

$$\frac{< -\Delta F_7, F_7 >}{< F_7, F_7 >} = \frac{24.7024}{0.5146} = 48.0031 = 4.8637\pi^2$$

$$\frac{< -\Delta F_8, F_8 >}{< F_8, F_8 >} = \frac{205.135}{3.8971} = 52.6379 = 5.3333\pi^2$$

4. Lower Bound of Eigenvalues Via Difference Scheme

To get a lower bound of eigenvalues, we may use difference scheme of (1.3) with zero boundary conditions (1.4). Let mesh size $h = 1/(N+1)$, using two variable Taylor expanding follows

$$L^h u_j = \frac{2}{3h^2}\{6u(j_1, j_2, j_3) - u(j_1, j_2 + h, j_3 - h) - u(j_1, j_2 - h, j_3 + h)$$
$$-u(j_1 + h, j_2, j_3 - h) - u(j_1 - h, j_2, j_3 + h) - u(j_1 - h, j_2 + h, j_3) - u(j_1 - h, j_2, j_3 + h)\}$$
$$= -\Delta u_j - \frac{h^2}{16}\Delta^2 u_j - \frac{h^4}{5760}\{11u^{(6,0)} + 15u^{(4,2)} + 45u^{(2,4)} + 9u^{(0,6)}\}|_{u=u_j} + O(h^6) \tag{4.1}$$

Since the biharmonic operator is positive, hence, for sufficient smooth function and suitable small mesh size h, we can take eigenvalues of the partial difference equation as a lower bound of the related Laplacian eigenvalues.

The number of total inner knots equals to $3N(N+1)+1$. The related stiffness matrix has tri-diagonal block form

$$A = Tridiag\{A_{11}, A_{12}; A_{21}, A_{22}, A_{23}; \cdots ; A_{N,N-1}, A_{N,N}, A_{N,N+1}; \cdots ; A_{2N-1,2N}, A_{2N,2N}\}$$

As an example, for h=1/2, the related stiffness matrix becomes

$$A = \frac{2}{3h^2}\begin{bmatrix} 6 & -1 & -1 & -1 & & & \\ -1 & 6 & & -1 & -1 & & \\ -1 & & 6 & -1 & & -1 & \\ -1 & -1 & -1 & 6 & -1 & -1 & -1 \\ & -1 & & -1 & 6 & & -1 \\ & & -1 & -1 & & 6 & -1 \\ & & & -1 & -1 & -1 & 6 \end{bmatrix} \tag{4.2}$$

For $N = 48$, the first eight approximate smallest eigenvalues are equal to

$$\{0.7249, 1.8363, 1.8363, 3.2853, 3.2853, 3.7950, 4.8199, 5.3260\}\pi^2 \tag{4.3}$$

respectively.

5. Conclusion

$TSin_j(P)$ in Theorem 1.2, as eigenfunctions of Laplacian operator over an equilateral triangle with zero boundary conditions, still are a part of eigenfunctions in the regular hexagon case. The function space, corresponding all eigenfunction, have been decomposed into four orthogonal subspaces. For the smallest eight eigenvalues of (1.3) over the regular hexagon with zero boundary condition (1.4), we get an upper bound

$$\{0.7251, 1.9020, 1.9020, 3.3771, 3.3771, 3.8803, 4.8637, 5.3333\}\pi^2$$

and a lower bound

$$\{0.7249, 1.8363, 1.8363, 3.2853, 3.2853, 3.7950, 4.8199, 5.3260\}\pi^2$$

Their maximum relative gaps are less than 4%. The pictures of eigenfunction, corresponding from the smallest to $8th$, are shown in Fig. 3.1 to Fig. 3.17, where $\lambda_2 = \lambda_3$ and $\lambda_4 = \lambda_5$ are two double roots, other four eigenvalues are single roots. Moreover, so far only λ_8 and related eigenfunction (3.10) are exact. Approximations to other eigenvalues and related functions can be constructed in a similar way.

How to use these approximation of partial eigen-decomposition for constructing a fast solver of the related discrete system is our future work.

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