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DUAL BASES FOR A NEW FAMILY OF GENERALIZED BALL BASES *

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Abstract

This paper presents the dual bases for a new family of generalized Ball curves with a position parameter K, which includes the Bézier curve, generalized Said-Ball curve and some intermediate curves. Using the dual bases, the relative Marsden identity, conversion formulas of bases and control points of various curves are obtained.

Key words: Bézier curve, New generalized Ball curve, Dual basis, Marsden identity.

1. Introduction

The generalized Ball curve possesses many properties similar to the ones of the Bézier curve and an advantage that there are some more efficient recursive algorithms for computing the points on the curve [1~8]. For a given control polygon, the positions of both the generalized Ball curve and the Bézier curve are different. A new family of generalized Ball curves with a position parameter K has been presented recently in [9,10], which includes the Bézier curve, generalized Said-Ball curve and some other intermediate curves. When the position selecting of a curve is considered as important as the efficiency for evaluating, it is suggested to use the family of curves.

To converse various bases and various curves interactively, a powerful means is to find the corresponding dual basis. The dual basis for the generalized Said-Ball basis with its Marsden identity has been discussed in the papers [11~15]. In this paper, Section 2 gives the definition of new generalized Ball bases with a position parameter K and presents its dual bases. In Section 3, the Marsden identity about power basis expanded by new bases is obtained. Section 4 presents conversion formulas of the bases and the control points. In the end two examples of the proposed algorithm are provided.

2. Dual Basis

We adopt the notation of combination in [11]

$$\binom{\alpha}{k} = \frac{\alpha(\alpha-1)\cdots(\alpha-k+1)}{k!}$$

for real number α and positive integer k and

$$\binom{\alpha}{0} = 1, \quad \binom{\alpha}{k} = 0 \ (k<0).$$

It is easy to verify

$$(-1)^k\binom{\alpha}{k} = \binom{k-1-\alpha}{k}$$

and

$$\binom{n}{p} = \sum_{k=0}^{p}\binom{m}{p-k}\binom{n-m}{k} \quad \text{for } m\in R;\ n,p\in Z.$$

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In the paper [9] a new family of generalized Ball basis functions of degree n with a given integer K $(0 \le K \le m)$ has been introduced:

$$\alpha_i(u; 2m+1, K) = \begin{cases} \binom{m+K+i}{i} u^i (1-u)^{m+1+K}, & 0 \le i \le m-K; \\ \binom{2m+1}{i} u^i (1-u)^{2m+1-i}, & m-K+1 \le i \le m, \end{cases} \quad (0 \le u \le 1) \quad (1)$$

$$\alpha_{2m+1-i}(u; 2m+1, K) = \alpha_i(1-u; 2m+1, K), \quad 0 \le i \le m,$$

for odd degree $n = 2m+1$, and

$$\alpha_i(u; 2m, K) = \begin{cases} \binom{m+K+i}{i} u^i (1-u)^{m+1+K}, & 0 \le i \le m-K-1; \\ \binom{2m}{i} u^i (1-u)^{2m-i}, & m-K \le i \le m, \end{cases} \quad (0 \le u \le 1) \quad (2)$$

$$\alpha_{2m-i}(u; 2m, K) = \alpha_i(1-u; 2m, K), \quad 0 \le i \le m-1,$$

for even degree $n = 2m$.

The basis functions $\alpha_i(u; n, K)$ possess many properties similar to the ones of the Bernstein-Bézier basis and the generalized Ball basis, such as the nonnegativity, the partition of unity, and so on. When $K = 0$, it becomes the generalized Said-Ball basis, and $K = m$ gets the Bernstein- Bézier basis.

Let $\{\boldsymbol{p}\ \}_{i=0}^{n}$ be a set of control points in R^2 or R^3. Using bases $\{\alpha_i(u; n, K)\}$, a family of curves can be constructed

$$NB(u; n, K) = \sum_{i=0}^{n} \boldsymbol{p}\ \alpha_i(u; n, K), \qquad 0 \le u \le 1;\ K = 0, 1, \cdots, m. \quad (3)$$

In addition to the Bézier curve and generalized Said-Ball curve, the family of curves (3) includes some intermediate curves. The integer K can be regarded as a parameter, which determines the position of a curve. The figure 1 shows a family of curves $\{NB(u; n, K)\}_{K=0}^{3}$ of degree 7, where the Bézier curve with $K = 3$ is the nearest to the control polygon, the generalized Said-Ball curve with $K = 0$ is the farthest from the control polygon, and the other curves with $K = 1, 2$ are between the Bézier curve and Said-Ball curve.

Definition 1. *Suppose that $\{b_i(u)\}_{i=0}^{n}$ is a basis of the space P_n of polynomials of degree non-exceeding n, if the linear functionals $\{\lambda_i(f)\}_{i=0}^{n}$ satisfy the conditions*

$$\lambda_i(b_j) = \delta_{ij}, \qquad i, j = 0, 1, \cdots, n,$$

where δ_{ij} is the Kronecker sign, then $\{\lambda_i(f)\}_{i=0}^{n}$ are called the dual bases or dual functionals of $\{b_i(u)\}_{i=0}^{n}$.

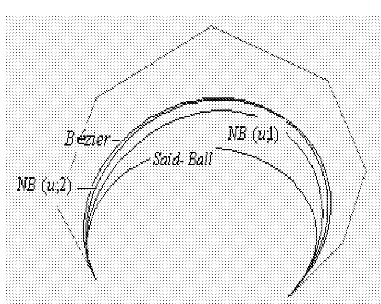

Figure 1 Bźier curve, Said-Ball curve and intermediate curves of degree 7

The following theorem is about the dual bases of the basis functions $\{\alpha_i(u; n, K)\}$.

Theorem 1. *The dual bases of the basis functions $\{\alpha_i(u;n,K)\}$ defined by (1) or (2) are*

$$\lambda_i^{(K)} = \begin{cases} \binom{m+K+i}{i}^{-1} \sum_{s=0}^{i} \binom{m+K+i-s}{i-s} \frac{1}{s!} f^{(s)}(0), & 0 \le i \le m-K; \\ \binom{2m+1}{i}^{-1} \sum_{s=0}^{i} \binom{2m+1-s}{i-s} \frac{1}{s!} f^{(s)}(0), & m-K+1 \le i \le m. \end{cases}$$

$$\lambda_{2m+1-i}^{(K)} = \begin{cases} \binom{m+K+i}{i}^{-1} \sum_{s=0}^{i} \binom{m+K+i-s}{i-s} (-1)^s \frac{1}{s!} f^{(s)}(1), & 0 \le i \le m-K; \\ \binom{2m+1}{i}^{-1} \sum_{s=0}^{i} \binom{2m+1-s}{i-s} (-1)^s \frac{1}{s!} f^{(s)}(1), & m-K+1 \le i \le m. \end{cases} \tag{4}$$

for odd degree $n = 2m+1$ and

$$\tilde{\lambda}_i^{(K)} = \begin{cases} \binom{m+K+i}{i}^{-1} \sum_{s=0}^{i} \binom{m+K+i-s}{i-s} \frac{1}{s!} f^{(s)}(0), & 0 \le i \le m-K-1; \\ \binom{2m}{i}^{-1} \sum_{s=0}^{i} \binom{2m-s}{i-s} \frac{1}{s!} f^{(s)}(0), & m-K \le i \le m. \end{cases}$$

$$\tilde{\lambda}_{2m-i}^{(K)} = \begin{cases} \binom{m+K+i}{i}^{-1} \sum_{s=0}^{i} \binom{m+K+i-s}{i-s} (-1)^s \frac{1}{s!} f^{(s)}(1), & 0 \le i \le m-K-1; \\ \binom{2m}{i}^{-1} \sum_{s=0}^{i} \binom{2m-s}{i-s} (-1)^s \frac{1}{s!} f^{(s)}(1), & m-K \le i \le m-1. \end{cases} \tag{5}$$

for even degree $n = 2m$ respectively.

Proof. Here we only consider the case of odd degree $n = 2m+1$, and the proof for $n = 2m$ is similar. From (1), it is obtained that

$$\alpha_j^{(s)}(0; 2m+1, K) = \begin{cases} 0, & s < j; \\ (-1)^{s-j} s! \binom{m+K+j}{j} \binom{m+1+K}{s-j}, & j \le s \le m; \end{cases}$$
$$\text{for } 0 \le j \le m-K, \tag{6}$$

$$\alpha_j^{(s)}(0; 2m+1, K) = \begin{cases} 0, & s < j; \\ (-1)^{s-j} s! \binom{2m+1}{j} \binom{2m+1-j}{s-j}, & j \le s \le m; \end{cases}$$
$$\text{for } m-K+1 \le j \le m, \tag{7}$$

and

$$\alpha_j^{(s)}(1; 2m+1, K) = 0 \qquad \text{for } 0 \le j,\ s \le m. \tag{8}$$

By the symmetry, we have

$$\begin{cases} \alpha_{2m+1-j}^{(s)}(0; 2m+1, K) = (-1)^s \alpha_j^{(s)}(1; 2m+1, K) = 0, \\ \alpha_{2m+1-j}^{(s)}(1; 2m+1, K) = (-1)^s \alpha_j^{(s)}(0; 2m+1, K). \end{cases} \quad \text{for } 0 \le j, s \le m. \tag{9}$$

First consider $0 \le i \le m$. We divide the values of i, j into following four cases:

(i) $i < j$. By (6) or (7), $\lambda_i(\alpha_j) = 0$.

(ii) $j \le i \le m-K$. Substituting (6) into the first formula of (4), we get

$$\lambda_i(\alpha_j) = \binom{m+K+i}{i}^{-1} \sum_{s=j}^{i} \binom{m+K+i-s}{i-s} \frac{1}{s!}(-1)^{s-j} s! \binom{m+K+j}{j} \binom{m+1+K}{s-j}$$
$$= (-1)^{i-j} \binom{m+K+i}{i}^{-1} \binom{m+K+j}{j} \sum_{s=j}^{i} \binom{m+K+i-s}{i-s} (-1)^{i-s} \binom{m+1+K}{s-j}$$
$$= (-1)^{i-j} \binom{m+K+i}{i}^{-1} \binom{m+K+j}{j} \sum_{s=j}^{i} \binom{-m-K-1}{i-s} \binom{m+1+K}{s-j}$$
$$= (-1)^{i-j} \binom{m+K+i}{i}^{-1} \binom{m+K+j}{j} \binom{0}{i-j} = \delta_{ij}$$

(iii) $j \le m-K < i$. Substituting (6) into the second formula of (4), we get

$$\lambda_i(\alpha_j) = (-1)^{i-j} \binom{2m+1}{i}^{-1} \binom{m+K+j}{j} \sum_{s=j}^{i} \binom{2m+1-s}{i-s} (-1)^{i-s} \binom{m+1+K}{s-j}$$
$$= (-1)^{i-j} \binom{2m+1}{i}^{-1} \binom{m+K+j}{j} \sum_{s=j}^{i} \binom{i-2m-2}{i-s} \binom{m+1+K}{s-j}$$
$$= (-1)^{i-j} \binom{2m+1}{i}^{-1} \binom{m+K+j}{j} \binom{i-m+K-1}{i-j} = 0.$$

(iv) $m-K < j \le i$. Substituting (7) into the second formula of (4), we get

$$\lambda_i(\alpha_j) = (-1)^{i-j} \binom{2m+1}{i}^{-1} \binom{2m+1}{j} \sum_{s=j}^{i} \binom{2m+1-s}{i-s} (-1)^{i-s} \binom{2m+1-j}{s-j}$$
$$= (-1)^{i-j} \binom{2m+1}{i}^{-1} \binom{m+K+j}{j} \sum_{s=j}^{i} \binom{i-2m-2}{i-s} \binom{2m+1-j}{s-j}$$
$$= (-1)^{i-j} \binom{m+K+i}{i}^{-1} \binom{m+K+j}{j} \binom{i-j-1}{i-j} = \delta_{ij}$$

For $m+1 \le i \le 2m+1$, in the same way, it holds

$$\lambda_i(\alpha_j) = \delta_{ij}.$$

3. Marsden Identity

Theorem 2. *Assume $0 \le l \le m$. Then there exist identities*

$$\begin{aligned} u^l = & \sum_{i=0}^{m-K} \binom{m+i+K}{i}^{-1} \binom{m+i+K-l}{i-l} \alpha_i(u; 2m+1, K) \\ & + \sum_{i=m-K+1}^{m} \binom{2m+1}{i}^{-1} \binom{2m+1-l}{i-l} \alpha_i(u; 2m+1, K) \\ & + \sum_{i=0}^{m-K} \binom{m+i+K}{i}^{-1} \binom{m+i+K-l}{i} \alpha_{2m+1-i}(u; 2m+1, K) \\ & + \sum_{i=m-K+1}^{m} \binom{2m+1}{i}^{-1} \binom{2m+1-l}{i} \alpha_{2m+1-i}(u; 2m+1, K), \end{aligned} \tag{10}$$

$$\begin{aligned} u^{2m+1-l} = &\sum_{i=0}^{m-K}(-1)^i\binom{m+i+K}{i}^{-1}\binom{m-K-l}{i}\alpha_{2m+1-i}(u;2m+1,K) \\ &+\sum_{i=m-K+1}^{m}(-1)^i\binom{2m+1}{i}^{-1}\binom{i-1-l}{i}\alpha_{2m+1-i}(u;2m+1,K) \end{aligned} \tag{11}$$

Proof. Writing u^l as the linear combination of $\alpha_i(u;2m+1,K)$

$$u^l = \sum_{i=0}^{m}C_i\alpha_i(u;2m+1,K)+\sum_{i=0}^{m}C_{2m+1-i}\alpha_{2m+1-i}(u;2m+1,K).$$

By the duality, these coefficients should be $C_i=\lambda_i(u^l)$. From the theorem 1,

$$\begin{aligned} C_i &= \binom{m+K+i}{i}^{-1}\sum_{s=0}^{i}\binom{m+K+i-s}{i-s}\frac{1}{s!}\frac{d^s}{du^s}(u^l)\bigg|_{u=0} \\ &= \binom{m+K+i}{i}^{-1}\binom{m+K+i-l}{i-l}, \qquad \text{for } 0\le i\le m-K; \end{aligned}$$

$$\begin{aligned} C_i &= \binom{2m+1}{i}^{-1}\sum_{s=0}^{i}\binom{2m+1-s}{i-s}\frac{1}{s!}\frac{d^s}{du^s}(u^l)\bigg|_{u=0} \\ &= \binom{2m+1}{i}^{-1}\binom{2m+1-l}{i-l}, \qquad \text{for } m-K+1\le i\le m; \end{aligned}$$

$$\begin{aligned} C_{2m+1-i} &= \binom{m+K+i}{i}^{-1}\sum_{s=0}^{i}\binom{m+K+i-s}{i-s}\frac{(-1)^s}{s!}\frac{d^s}{du^s}(u^l)\bigg|_{u=1} \\ &= (-1)^i\binom{m+K+i}{i}^{-1}\sum_{s=0}^{i}\binom{m+K+i-s}{i-s}(-1)^{i-s}\binom{l}{s} \\ &= (-1)^i\binom{m+K+i}{i}^{-1}\sum_{s=0}^{i}\binom{-m-K-1}{i-s}\binom{l}{s} \\ &= (-1)^i\binom{m+K+i}{i}^{-1}\binom{l-m-K-1}{i} \\ &= \binom{m+K+i}{i}^{-1}\binom{m+K+i-l}{i}, \qquad \text{for } 0\le i\le m-K; \end{aligned}$$

$$\begin{aligned} C_{2m+1-i} &= \binom{2m+1}{i}^{-1}\sum_{s=0}^{i}\binom{2m+1-s}{i-s}\frac{(-1)^s}{s!}\frac{d^s}{du^2}(u^l)\bigg|_{u=1} \\ &= \binom{2m+1}{i}^{-1}\binom{2m+1-l}{i}, \qquad \text{for } m-K+1\le i\le m. \end{aligned}$$

Therefore (10) is obtained.

Let

$$u^{2m+1-l} = \sum_{i=0}^{m}\widetilde{C}_i\alpha_i(u;2m+1,K)+\sum_{i=0}^{m}\widetilde{C}_{2m+1-i}\alpha_{2m+1-i}(u;2m+1,K),$$

then

$$\begin{aligned} \widetilde{C}_i &= \binom{m+K+i}{i}^{-1}\sum_{s=0}^{i}\binom{m+K+i-s}{i-s}\frac{1}{s!}\frac{d^s}{du^s}(u^{2m+1-l})\bigg|_{u=0} \\ &= 0, \qquad \text{for } 0\le i\le m-K; \end{aligned}$$

$$\begin{aligned}\widetilde{C}_i &= \binom{2m+1}{i}^{-1}\sum_{s=0}^{i}\binom{2m+1-s}{i-s}\frac{1}{s!}\frac{d^s}{du^s}(u^{2m+1-l})\Big|_{u=0}\\ &= 0, \qquad \text{for } m-K+1\le i\le m;\\ \widetilde{C}_{2m+1-i} &= \binom{m+K+i}{i}^{-1}\sum_{s=0}^{i}\binom{m+K+i-s}{i-s}\frac{(-1)^s}{s!}\frac{d^s}{du^s}(u^{2m+1-l})\Big|_{u=1}\\ &= (-1)^i\binom{m+K+i}{i}^{-1}\binom{m-K-l}{i}, \qquad \text{for } 0\le i\le m-K;\\ \widetilde{C}_{2m+1-i} &= \binom{2m+1}{i}^{-1}\sum_{s=0}^{i}\binom{2m+1-s}{i-s}\frac{(-1)^s}{s!}\frac{d^s}{du^s}(u^{2m+1-l})\Big|_{u=1}\\ &= (-1)^i\binom{2m+1}{i}^{-1}\binom{i-1-l}{i}, \qquad \text{for } m-K+1\le i\le m.\end{aligned}$$

Consequently (11) holds.

4. Basis Transformation

In this section we consider mutual transformation between the bases which have the same degree, $2m+1$, but different position parameters K and H.

Theorem 3. *Suppose that integers K, H satisfy $0 \le K, H \le m$ and $K \ne H$. Let*

$$\begin{aligned}A_i(u;K) &= \alpha_i(u;2m+1,K),\\ B_i(u;H) &= \alpha_i(u;2m+1,H),\end{aligned} \qquad i = 0,1,\cdots,2m+1 \tag{12}$$

be two bases and $\{B_i(u;H)\}$ be represented by the linear combination of $\{A_i(u;K)\}$:

$$B_i(u;H) = \sum_{j=0}^{2m+1} c_{ij}A_j(u;K). \tag{13}$$

(a) If $K < H$, then

$$c_{ij} = \begin{cases} 0, & j<i \text{ or } j>H-K+i;\\ (-1)^{j-i}\binom{m+H+i}{i}\binom{H-K}{j-i}\binom{m+K+j}{j}^{-1}, & i\le j\le H-K+i; \end{cases}$$

$$\textit{for } 0\le i\le m-H.$$

$$c_{ij} = \begin{cases} 0, & j<i \text{ or } j>m-K;\\ (-1)^{j-i}\binom{2m+1}{i}\binom{m-K-i}{j-i}\binom{m+K+j}{j}^{-1}, & i\le j\le m-K; \end{cases} \tag{14}$$

$$\textit{for } m-H+1\le i\le m-K.$$

$$c_{ij} = \delta_{ij} \qquad \textit{for } m-K+1\le i\le m,\ 0\le j\le 2m+1.$$

$$c_{2m+1-i,2m+1-j} = c_{ij} \qquad \textit{for } 0\le i\le m,\ 0\le j\le 2m+1.$$

(b) If $K > H$, then

$$c_{ij} = \begin{cases} 0, & j<i \text{ or } j>m;\\ \binom{m+H+i}{i}\binom{K-H-1+j-i}{j-i}\binom{m+K+j}{j}^{-1}, & i\le j\le m-K;\\ \binom{m+H+i}{i}\binom{m-H-i}{j-i}\binom{2m+1}{j}^{-1}, & m-K+1\le j\le m; \end{cases}$$

$$\textit{for } 0\le i\le m-K.$$

$$c_{ij}=\begin{cases}0, & j<i \textit{ or } j>m-H;\\ \binom{m+H+i}{i}\binom{m-H-i}{j-i}\binom{2m+1}{j}^{-1}, & i\le j\le m-K;\end{cases} \tag{15}$$

$$\textit{for } m-K+1\le i\le m-H.$$

$$c_{ij}=\delta_{ij} \qquad \textit{for } m-H+1\le i\le m,\ 0\le j\le 2m+1.$$

$$c_{2m+1-i,2m+1-j}=c_{ij} \qquad \textit{for } 0\le i\le m,\ 0\le j\le 2m+1.$$

Proof. (a) When $K<H$, we have $m-K>m-H$. By the definition

$$B_i(u;H)=A_i(u;K)=\binom{2m+1}{i}u^i(1-u)^{2m+1-i}, \qquad m-K+1\le i\le m.$$

Thus for $m-K+1\le i\le m$,

$$c_{ij}=\delta_{ij}.$$

For $0\le i\le m-H$,

$$\begin{aligned}B_i(u;H)&=\binom{m+H+i}{i}u^i(1-u)^{m+H+1}=\binom{m+H+i}{i}u^i(1-u)^{m+K+1}(1-u)^{H-K}\\&=\binom{m+H+i}{i}\sum_{j=0}^{H-K}(-1)^j\binom{H-K}{j}u^{i+j}(1-u)^{m+K+1}\\&=\binom{m+H+i}{i}\sum_{j=i}^{H-K+i}(-1)^{j-i}\binom{H-K}{j-i}\binom{m+K+j}{j}^{-1}A_j(u;K).\end{aligned}$$

For $m-H+1\le i\le m-K$,

$$\begin{aligned}B_i(u;H)&=\binom{2m+1}{i}u^i(1-u)^{2m+1-i}=\binom{2m+1}{i}u^i(1-u)^{m+K+1}(1-u)^{m-K-i}\\&=\binom{2m+1}{i}\sum_{j=0}^{m-K-i}(-1)^j\binom{m-K-i}{j}u^{i+j}(1-u)^{m+K+1}\\&=\binom{2m+1}{i}\sum_{j=i}^{m-K}(-1)^{j-i}\binom{m-K-i}{j-i}\binom{m+K+j}{j}^{-1}A_j(u;K).\end{aligned}$$

For $m+1\le i\le 2m+1$, consider the symmetry

$$A_i(u;K)=A_{2m+1-i}(1-u;K) \quad \text{and} \quad B_i(u;H)=B_{2m+1-i}(1-u;H).$$

This proves (a).

(b) When $K>H$, we have $m-K<m-H$. By the definition

$$B_i(u;H)=A_i(u;K)=\binom{2m+1}{i}u^i(1-u)^{2m+1-i}, \quad m-H+1\le i\le m.$$

Thus for $m-H+1\le i\le m$

$$c_{ij}=\delta_{ij}.$$

For $0\le i\le m-K$, let

$$B_i(u;H)=\binom{m+H+i}{i}u^i(1-u)^{m+H+1}=\sum_{j=0}^{2m+1}c_{ij}A_j(u;K),$$

then by operating the dual basis λ_j on both the sides, we obtain

$$c_{ij}=\lambda_jB_i(u;H).$$

The index j can be divided into following four cases:

(i) $0 \le j < i$. Since $B_i^{(s)}(0;H) = 0$ for $s < i$, then

$$c_{ij} = \binom{m+K+j}{j}^{-1} \sum_{s=i}^{j} \binom{m+K+j-s}{j-s} \frac{1}{s!} B_i^{(s)}(0;H) = 0.$$

(ii) $m+1 \le j \le 2m+1$. From $B_i^{(s)}(1;H) = 0$ for $s \le m$, it is obvious that

$$c_{ij} = 0.$$

(iii) $i \le j \le m-K$. By simple calculation, one can verify

$$\begin{aligned}
c_{ij} &= \binom{m+H+i}{i} \binom{m+K+j}{j}^{-1} \sum_{s=i}^{j} \binom{m+K+j-s}{j-s} (-1)^{s-i} \binom{m+H+1}{s-i} \\
&= (-1)^{i-j} \binom{m+H+i}{i} \binom{m+K+j}{j}^{-1} \sum_{s=i}^{j} \binom{m+K+j-s}{j-s} (-1)^{j-s} \binom{m+H+1}{s-i} \\
&= (-1)^{i-j} \binom{m+H+i}{i} \binom{m+K+j}{j}^{-1} \sum_{s=i}^{j} \binom{-m-K-1}{j-s} \binom{m+H+1}{s-i} \\
&\underline{\underline{s'=s-i}}\ (-1)^{i-j} \binom{m+H+i}{i} \binom{m+K+j}{j}^{-1} \sum_{s'=0}^{j-i} \binom{-m-K-1}{j-s'-i} \binom{m+H+1}{s'} \\
&= (-1)^{i-j} \binom{m+H+i}{i} \binom{m+K+j}{j}^{-1} \binom{H-K}{j-i} \\
&= \binom{m+H+i}{i} \binom{m+K+j}{j}^{-1} \binom{K-H+j-i-1}{j-i}.
\end{aligned}$$

(iv) $m-K+1 \le j \le m$. Similarly, it can be obtained

$$\begin{aligned}
c_{ij} &= \binom{m+H+i}{i} \binom{2m+1}{j}^{-1} \sum_{s=i}^{j} \binom{2m+1-s}{j-s} (-1)^{s-i} \binom{m+H+1}{s-i} \\
&= \binom{m+H+i}{i} \binom{2m+1}{j}^{-1} \binom{m-H-i}{j-i}.
\end{aligned}$$

For $m-K+1 \le i \le m-H$,

$$\begin{aligned}
B_i(u;H) &= \binom{m+H+i}{i} u^i (1-u)^{m+H+1} \\
&= \binom{m+H+i}{i} u^i (1-u)^{m+H+1} [u + (1-u)]^{m-H-i} \\
&= \binom{m+H+i}{i} \sum_{j=i}^{m-H} \binom{m-H-i}{j-i} \binom{2m+1}{j}^{-1} A_j(u;K).
\end{aligned}$$

The proof of (b) is completed.

Let $C = (c_{ij})$ be the transformation matrix from a basis $\{A_i(u;K)\}$ to another basis $\{B_i(u;H)\}$, i.e.

$$\begin{pmatrix} B_0(u;H) \\ B_1(u;H) \\ \cdots \\ B_{2m+1}(u;H) \end{pmatrix} = \begin{pmatrix} c_{00} & c_{01} & \cdots & c_{0,2m+1} \\ c_{10} & c_{11} & \cdots & c_{1,2m+1} \\ \cdots & \cdots & \cdots & \cdots \\ c_{2m+1,0} & c_{2m+1,1} & \cdots & c_{2m+1,2m+1} \end{pmatrix} \begin{pmatrix} A_0(u;K) \\ A_1(u;K) \\ \cdots \\ A_{2m+1}(u;K) \end{pmatrix}. \tag{16}$$

A same parametric curve $\boldsymbol{r}(\mathrm{u})$ of degree $2m+1$ can be represented in two bases

$$\boldsymbol{r}(u) = \sum_{i=0}^{2m+1} \boldsymbol{p}_i A_i(u;K) \quad \text{and} \quad \boldsymbol{r}(u) = \sum_{i=0}^{2m+1} \boldsymbol{q}_i B_i(u;H), \tag{17}$$

where $\{\boldsymbol{p}_i\}$ and $\{\boldsymbol{q}_i\}$ are control points corresponding to the bases $\{A_i(u;K)\}$ and the $\{B_i(u;H)\}$ respectively. Using the matrix notation, one can obtain

$$\boldsymbol{r}(u) = (\boldsymbol{p}_0 \quad \boldsymbol{p}_1 \quad \cdots \quad \boldsymbol{p}_{2m+1}) \begin{pmatrix} A_0(u;K) \\ A_1(u;K) \\ \cdots \\ A_{2m+1}(u;K) \end{pmatrix} = (\boldsymbol{q}_0 \quad \boldsymbol{q}_1 \quad \cdots \quad \boldsymbol{q}_{2m+1}) \begin{pmatrix} B_0(u;K) \\ B_1(u;K) \\ \cdots \\ B_{2m+1}(u;K) \end{pmatrix}$$

$$= (\boldsymbol{q}_0 \quad \boldsymbol{q}_1 \quad \cdots \quad \boldsymbol{q}_{2m+1}) \begin{pmatrix} c_{00} & c_{01} & \cdots & c_{0,2m+1} \\ c_{10} & c_{11} & \cdots & c_{1,2m+1} \\ \cdots & \cdots & \cdots & \cdots \\ c_{2m+1,0} & c_{2m+1,1} & \cdots & c_{2m+1,2m+1} \end{pmatrix} \begin{pmatrix} A_0(u;K) \\ A_1(u;K) \\ \cdots \\ A_{2m+1}(u;K) \end{pmatrix}.$$

Substituting the c_{ij}'s in Theorem 3 into the above equation, we get the following theorem:

Theorem 4. *Suppose that $\{\boldsymbol{p}_i\}$ and $\{\boldsymbol{q}\}$ are the control points of the same curve $\boldsymbol{r}(u)$ on the bases $\{A_i(u;K)\}$ and $\{B_i(u;H)\}$ respectively.*

(a) If $K < H$, then

$$\begin{cases} \boldsymbol{p}_i = \binom{m+K+i}{i}^{-1} \sum\limits_{s=0}^{i} \binom{m+H+s}{s} (-1)^{i-s} \binom{H-K}{i-s} \boldsymbol{q}_s, \\ \boldsymbol{p}_{2m+1-i} = \binom{m+K+i}{i}^{-1} \sum\limits_{s=0}^{i} \binom{m+H+s}{s} (-1)^{i-s} \binom{H-K}{i-s} \boldsymbol{q}_{2m+1-s}, \end{cases} \quad \text{for } 0 \le i \le m-H. \tag{18}$$

$$\begin{cases} \boldsymbol{p}_i = \binom{m+K+i}{i}^{-1} \sum\limits_{s=0}^{m-H} \binom{m+H+s}{s} (-1)^{i-s} \binom{H-K}{i-s} \boldsymbol{q}_s, \\ \qquad + \binom{m+K+i}{i}^{-1} \sum\limits_{s=m-H+1}^{i} \binom{2m+1}{s} (-1)^{i-s} \binom{m-K-s}{i-s} \boldsymbol{q}_s, \\ \boldsymbol{p}_{2m+1-i} = \binom{m+K+i}{i}^{-1} \sum\limits_{s=0}^{m-H} \binom{m+H+s}{s} (-1)^{i-s} \binom{H-K}{i-s} \boldsymbol{q}_{2m+1-s}, \\ \qquad + \binom{m+K+i}{i}^{-1} \sum\limits_{s=m-H+1}^{i} \binom{2m+1}{s} (-1)^{i-s} \binom{m-K-s}{i-s} \boldsymbol{q}_{2m+1-s} \end{cases} \quad \text{for } m-H+1 \le i \le m-K. \tag{19}$$

$$\boldsymbol{p}_i = \boldsymbol{q}_i, \quad \boldsymbol{p}_{2m+1-i} = \boldsymbol{q}_{2m+1-i} \quad \text{for } m-K+1 \le i \le m.$$

(b) If $K > H$, then

$$\begin{cases} \boldsymbol{p}_i = \binom{m+K+i}{i}^{-1} \sum\limits_{s=0}^{i} \binom{K-H+i-s-1}{i-s} \binom{m+H+s}{s} \boldsymbol{q}_s, \\ \boldsymbol{p}_{2m+1-i} = \binom{m+K+i}{i}^{-1} \sum\limits_{s=0}^{i} \binom{K-H+i-s-1}{i-s} \binom{m+H+s}{i-s} \boldsymbol{q}_{2m+1-s}, \end{cases} \quad \text{for } 0 \le i \le m-K. \tag{20}$$

$$\begin{cases} \boldsymbol{p}_i = \binom{2m+1}{i}^{-1} \sum\limits_{s=0}^{i} \binom{m+H+s}{s} \binom{m-H-s}{i-s} \boldsymbol{q}_s, \\ \boldsymbol{p}_{2m+1-i} = \binom{2m+1}{i}^{-1} \sum\limits_{s=0}^{i} \binom{m+H+s}{s} \binom{m-H-s}{i-s} \boldsymbol{q}_{2m+1-s}, \end{cases} \quad \text{for } m-K+1 \le i \le m-H. \tag{21}$$

$$\boldsymbol{p}_i = \boldsymbol{q}_i, \quad \boldsymbol{p}_{2m+1-i} = \boldsymbol{q}_{2m+1-i} \quad \text{for } m-H+1 \le i \le m.$$

The results of Theorem 4 include all transformation formulae of the control points among the Bézier curve, Said-Ball curve and other intermediate curves.

Example 1. *For $m = 3$, $H = m$, $K = 0, 1, 2$, evaluate the control points $\{\boldsymbol{p}_i\}$ by $\{\boldsymbol{q}_i\}$. (here only the first m control points are listed, and the last m control points can be written in symmetric form)*

$$\boldsymbol{p}_0 = \boldsymbol{q}_0, \quad \boldsymbol{p}_1 = \frac{1}{4}(-3\boldsymbol{q}_0 + 7\boldsymbol{q}_1), \quad \boldsymbol{p}_2 = \frac{1}{10}(3\boldsymbol{q}_0 - 14\boldsymbol{q}_1 + 21\boldsymbol{q}_2),$$

$$\boldsymbol{p}_3 = \frac{1}{20}(-\boldsymbol{q}_0 + 7\boldsymbol{q}_1 - 21\boldsymbol{q}_2 + 35\boldsymbol{q}_3), \qquad \textit{for } K = 0.$$

$$\boldsymbol{p}_0 = \boldsymbol{q}_0, \quad \boldsymbol{p}_1 = \frac{1}{5}(-2\boldsymbol{q}_0 + 7\boldsymbol{q}_1), \quad \boldsymbol{p}_2 = \frac{1}{15}(\boldsymbol{q}_0 - 7\boldsymbol{q}_1 + 21\boldsymbol{q}_2), \quad \boldsymbol{p}_3 = \boldsymbol{q}_3, \qquad \textit{for } K = 1.$$

$$\boldsymbol{p}_0 = \boldsymbol{q}_0, \quad \boldsymbol{p}_1 = \frac{1}{6}(-\boldsymbol{q}_0 + 7\boldsymbol{q}_1), \quad \boldsymbol{p}_2 = \boldsymbol{q}_2, \quad \boldsymbol{p}_3 = \boldsymbol{q}_3, \qquad \textit{for } K = 2.$$

Example 2. *For* $m = 3,\ K = m,\ H = 0, 1, 2$, *evaluate the control points* $\{\boldsymbol{p}_i\}$ *by* $\{\boldsymbol{q}_i\}$.

$$\boldsymbol{p}_0 = \boldsymbol{q}_0, \quad \boldsymbol{p}_1 = \frac{1}{7}(3\boldsymbol{q}_0 + 4\boldsymbol{q}_1), \quad \boldsymbol{p}_2 = \frac{1}{21}(3\boldsymbol{q}_0 + 8\boldsymbol{q}_1 + 10\boldsymbol{q}_2),$$

$$\boldsymbol{p}_3 = \frac{1}{35}(\boldsymbol{q}_0 + 4\boldsymbol{q}_1 + 10\boldsymbol{q}_2 + 20\boldsymbol{q}_3), \qquad \textit{for } H = 0.$$

$$\boldsymbol{p}_0 = \boldsymbol{q}_0, \quad \boldsymbol{p}_1 = \frac{1}{7}(2\boldsymbol{q}_0 + 5\boldsymbol{q}_1), \quad \boldsymbol{p}_2 = \frac{1}{21}(\boldsymbol{q}_0 + 5\boldsymbol{q}_1 + 15\boldsymbol{q}_2), \quad \boldsymbol{p}_3 = \boldsymbol{q}_3, \qquad \textit{for } H = 1.$$

$$\boldsymbol{p}_0 = \boldsymbol{q}_0, \quad \boldsymbol{p}_1 = \frac{1}{7}(\boldsymbol{q}_0 + 6\boldsymbol{q}_1), \quad \boldsymbol{p}_2 = \boldsymbol{q}_2, \quad \boldsymbol{p}_3 = \boldsymbol{q}_3, \qquad \textit{for } H = 2.$$

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