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Error Estimates and Superconvergence of Mixed Finite Element Methods for Optimal Control Problems with Low Regularity

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Abstract. In this paper, we investigate the error estimates and superconvergence property of mixed finite element methods for elliptic optimal control problems. The state and co-state are approximated by the lowest order Raviart-Thomas mixed finite element spaces and the control variable is approximated by piecewise constant functions. We derive L^2 and L^∞-error estimates for the control variable. Moreover, using a recovery operator, we also derive some superconvergence results for the control variable. Finally, a numerical example is given to demonstrate the theoretical results.

AMS subject classifications: 49J20, 65N30

Key words: Elliptic equations, optimal control problems, superconvergence, error estimates, mixed finite element methods.

1 Introduction

The finite element approximation of optimal control problems has been extensively studied in the literature. It is impossible to even give a very brief review here. For the studies about convergence and superconvergence of finite element approximations for optimal control problems, see, e.g., [5, 11, 13, 15, 17, 21–25, 28, 30–32]. A systematic introduction of finite element methods for PDEs and optimal control problems can be found in, e.g., [9, 19, 20, 29].

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Compared with standard finite element methods, the mixed finite methods have many numerical advantages. When the objective functional contains gradient of the state variable, we will firstly choose the mixed finite element methods. We have done some works on a priori error estimates and superconvergence properties of mixed finite elements for optimal control problems, see, e.g., [3,4,6,8,26]. In [4], we used the postprocessing projection operator, which was defined by Meyer and Rösch (see [21]) to prove a quadratic superconvergence of the control by mixed finite element methods. Recently, we derived error estimates and superconvergence of mixed methods for convex optimal control problems in [8]. However, in [8], the regularity assumption for the state and the co-state variables is a little strong.

The goal of this paper is to derive the error estimates and superconvergence of mixed finite element approximation for an elliptic control problem. Firstly, by use of the duality argument, we derive the superconvergence property between average L^2 projection and the approximation of the control variable, the convergence order is $h^{3/2}$ as that obtained in [8]. Then the error estimates of order h in the L^2-norm and in the L^∞-norm for the control variable are derived. Moreover, two global superconvergence results with the order $h^{3/2}$ for the control variable can be obtained by using a recovery operator. We can see that the regularity assumption for the state and the co-state variables is only $y,z \in H^2(\Omega)\cap W^{1,\infty}(\Omega)$. Finally, we present a numerical experiment to demonstrate the practical side of the theoretical results.

We consider the following linear optimal control problems for the state variables $\boldsymbol{p}$, y, and the control u with pointwise constraint:

$$\min_{u\in U_{ad}}\Big\{\frac{1}{2}\|\boldsymbol{p}-\boldsymbol{p}_d\|^2+\frac{1}{2}\|y-y_d\|^2+\frac{\nu}{2}\|u\|^2\Big\} \tag{1.1}$$

subject to the state equation

$$-\mathrm{div}(A(x)\nabla y)+a_0 y=u, \qquad x\in\Omega, \tag{1.2}$$

which can be written in the form of the first order system

$$\mathrm{div}\boldsymbol{p}+a_0 y=u, \quad \boldsymbol{p}=-A(x)\nabla y, \quad x\in\Omega \tag{1.3}$$

and the boundary condition

$$y=0, \qquad x\in\partial\Omega, \tag{1.4}$$

where Ω is a bounded domain in $\mathbb{R}^2$. U_{ad} denotes the admissible set of the control variable, defined by

$$U_{ad}=\big\{u\in L^\infty(\Omega):\ u\geq 0,\ \text{a.e. in}\ \Omega\big\}. \tag{1.5}$$

Moreover, we assume that $0\leq a_0\in W^{1,\infty}(\Omega)$, $y_d\in H^1(\Omega)$ and $\boldsymbol{p}_d\in(H^1(\Omega))^2$. ν is a fixed positive number. The coefficient $A(x)=(a_{ij}(x))$ is a symmetric matrix function with $a_{ij}(x) \in W^{1,\infty}(\Omega)$, which satisfies the ellipticity condition

$$c_*|\xi|^2 \le \sum_{i,j=1}^{2} a_{ij}(x)\xi_i\xi_j, \quad \forall(\xi,x) \in \mathbb{R}^2 \times \bar{\Omega}, \quad c_* > 0.$$

The plan of this paper is as follows. In Section 2, we construct the mixed finite element approximation scheme for the optimal control problem (1.1)-(1.4) and give its equivalent optimality conditions. The main results of this paper are stated in Section 3. In Section 3, we derive the superconvergence properties between the average L^2 projection and the approximation of the control variable, as well as between the post-processing solution and the exact control solution. We also derive the L^2 and L^∞-error estimates for optimal control problem. In Section 4, we present a numerical example to demonstrate our theoretical results. In the last section, we briefly summarize the results obtained and some possible future extensions.

In this paper, we adopt the standard notation $W^{m,p}(\Omega)$ for Sobolev spaces on Ω with a norm $\|\cdot\|_{m,p}$ given by

$$\|v\|_{m,p}^p = \sum_{|\alpha|\le m} \|D^\alpha v\|_{L^p(\Omega)}^p,$$

a semi-norm $|\cdot|_{m,p}$ given by

$$|v|_{m,p}^p = \sum_{|\alpha|=m} \|D^\alpha v\|_{L^p(\Omega)}^p.$$

We set

$$W_0^{m,p}(\Omega) = \big\{v \in W^{m,p}(\Omega) : v|_{\partial\Omega} = 0\big\}.$$

For $p = 2$, we denote $H^m(\Omega) = W^{m,2}(\Omega)$, $H_0^m(\Omega) = W_0^{m,2}(\Omega)$ and $\|\cdot\|_m = \|\cdot\|_{m,2}$, $\|\cdot\| = \|\cdot\|_{0,2}$. In addition C denotes a general positive constant independent of h, where h is the spatial mesh-size for the control and state discretization.

2 Mixed methods for optimal control problems

In this section we shall construct mixed finite element approximation scheme of the control problem (1.1)-(1.4). For sake of simplicity, we assume that the domain Ω is a convex polygon. Now, we introduce the co-state elliptic equation

$$-\mathrm{div}(A(x)(\nabla z + \boldsymbol{p} - \boldsymbol{p}_d)) + a_0 z = y - y_d, \quad x \in \Omega, \tag{2.1}$$

which can be written in the form of the first order system

$$\mathrm{div}\boldsymbol{q} + a_0 z = y - y_d, \quad \boldsymbol{q} = -A(x)(\nabla z + \boldsymbol{p} - \boldsymbol{p}_d), \quad x \in \Omega, \tag{2.2}$$

and the boundary condition

$$z = 0, \quad x \in \partial\Omega. \tag{2.3}$$

Next, we recall a result from Grisvard [12].

Lemma 2.1. *(see [12]) For every p $(2 \leq p < p_\Omega)$ and every function $\psi \in L^p(\Omega)$, the solution ϕ of*

$$-\mathrm{div}(A\nabla\phi) + a_0\phi = \psi \quad \text{in } \Omega, \quad \phi|_{\partial\Omega} = 0 \tag{2.4}$$

belongs to $H_0^1(\Omega) \cap W^{2,p}(\Omega)$, where the constant $p_\Omega > 2$ depending on the biggest interior angle of Ω and A. Moreover, there exists a positive constant C, independent of a_0 such that

$$\|\phi\|_{W^{2,p}(\Omega)} \leq C\|\psi\|_{L^p(\Omega)}. \tag{2.5}$$

Let

$$\boldsymbol{V} = H(\mathrm{div};\Omega) = \{\boldsymbol{v} \in (L^2(\Omega))^2, \mathrm{div}\boldsymbol{v} \in L^2(\Omega)\}, \quad W = L^2(\Omega). \tag{2.6}$$

We recast (1.1)-(1.4) as the following weak form: find $(\boldsymbol{p}, y, u) \in \boldsymbol{V} \times W \times U_{ad}$ such that

$$\min_{u\in U_{ad}} \Big\{\frac{1}{2}\|\boldsymbol{p} - \boldsymbol{p}_d\|^2 + \frac{1}{2}\|y - y_d\|^2 + \frac{\nu}{2}\|u\|^2\Big\}, \tag{2.7a}$$

$$(A^{-1}\boldsymbol{p}, \boldsymbol{v}) - (y, \mathrm{div}\boldsymbol{v}) = 0, \qquad \forall \boldsymbol{v} \in \boldsymbol{V}, \tag{2.7b}$$

$$(\mathrm{div}\boldsymbol{p}, w) + (a_0 y, w) = (u, w), \quad \forall w \in W. \tag{2.7c}$$

It follows from [19] that the optimal control problem (2.7a)-(2.7c) has a unique solution $(\boldsymbol{p}, y, u)$, and that a triplet $(\boldsymbol{p}, y, u)$ is the solution of (2.7a)-(2.7c) if and only if there is a co-state $(\boldsymbol{q}, z) \in \boldsymbol{V} \times W$ such that $(\boldsymbol{p}, y, \boldsymbol{q}, z, u)$ satisfies the following optimality conditions:

$$(A^{-1}\boldsymbol{p}, \boldsymbol{v}) - (y, \mathrm{div}\boldsymbol{v}) = 0, \qquad \forall \boldsymbol{v} \in \boldsymbol{V}, \tag{2.8a}$$

$$(\mathrm{div}\boldsymbol{p}, w) + (a_0 y, w) = (u, w), \qquad \forall w \in W, \tag{2.8b}$$

$$(A^{-1}\boldsymbol{q}, \boldsymbol{v}) - (z, \mathrm{div}\boldsymbol{v}) = -(\boldsymbol{p} - \boldsymbol{p}_d, \boldsymbol{v}), \qquad \forall \boldsymbol{v} \in \boldsymbol{V}, \tag{2.8c}$$

$$(\mathrm{div}\boldsymbol{q}, w) + (a_0 z, w) = (y - y_d, w), \qquad \forall w \in W, \tag{2.8d}$$

$$(\nu u + z, \tilde{u} - u) \geq 0, \qquad \forall \tilde{u} \in U_{ad}, \tag{2.8e}$$

where $(\cdot,\cdot)$ is the inner product of $L^2(\Omega)$.

The inequality (2.8e) can be expressed as

$$u = \max\{0, -z\}/\nu. \tag{2.9}$$

Let $\mathcal{T}_h$ denotes a regular triangulation of the polygonal domain Ω, h_T denotes the diameter of T and $h = \max h_T$. Let $\boldsymbol{V}_h \times W_h \subset \boldsymbol{V} \times W$ denotes the lowest order Raviart-Thomas mixed finite element space [10,27], namely,

$$\forall T \in \mathcal{T}_h, \quad \boldsymbol{V}(T) = \boldsymbol{P}_0(T) \oplus \mathrm{span}(xP_0(T)), \quad W(T) = P_0(T),$$

where $P_m(T)$ denotes polynomials of total degree at most m, $\boldsymbol{P}_0(T) = (P_0(T))^2$, $x = (x_1, x_2)$ which is treated as a vector, and

$$\boldsymbol{V}_h := \{\boldsymbol{v}_h \in \boldsymbol{V} : \forall T \in \mathcal{T}_h,\ \boldsymbol{v}_h|_T \in \boldsymbol{V}(T)\}, \tag{2.10a}$$

$$W_h := \{w_h \in W : \forall T \in \mathcal{T}_h, w_h|_T \in W(T)\}. \tag{2.10b}$$

And the approximated space of control is given by

$$U_h := \{\tilde{u}_h \in U_{ad} : \forall T \in \mathcal{T}_h,\ \tilde{u}_h|_T = \text{constant}\}. \tag{2.11}$$

Before the mixed finite element scheme is given, we introduce two operators. Firstly, we define the standard $L^2(\Omega)$-projection [10] $P_h : W \to W_h$, which satisfies: for any $\phi \in W$

$$(P_h\phi - \phi, w_h) = 0, \qquad \forall\, w_h \in W_h, \tag{2.12a}$$

$$\|\phi - P_h\phi\|_{-s,\rho} \le Ch^{1+s}\|\phi\|_{1,\rho}, \qquad s = 0, 1,\ \ 2 \le \rho \le \infty,\ \ \forall\phi \in W^{1,\rho}(\Omega). \tag{2.12b}$$

Next, recall the Fortin projection (see [2] and [10]) $\Pi_h : \boldsymbol{V} \to \boldsymbol{V}_h$, which satisfies: for any $\boldsymbol{q} \in \boldsymbol{V}$

$$(\mathrm{div}(\Pi_h\boldsymbol{q} - \boldsymbol{q}), w_h) = 0, \qquad \forall w_h \in W_h, \tag{2.13a}$$

$$\|\boldsymbol{q} - \Pi_h\boldsymbol{q}\|_{0,\rho} \le Ch\|\boldsymbol{q}\|_{1,\rho}, \qquad 2 \le \rho \le \infty,\ \ \forall\boldsymbol{q} \in (W^{1,\rho}(\Omega))^2, \tag{2.13b}$$

$$\|\mathrm{div}(\boldsymbol{q} - \Pi_h\boldsymbol{q})\| \le Ch\|\mathrm{div}\boldsymbol{q}\|_1, \qquad \forall\mathrm{div}\boldsymbol{q} \in H^1(\Omega). \tag{2.13c}$$

We have the commuting diagram property

$$\mathrm{div} \circ \Pi_h = P_h \circ \mathrm{div} : \boldsymbol{V} \to W_h \quad \text{and} \quad \mathrm{div}(I - \Pi_h)\boldsymbol{V} \perp W_h, \tag{2.14}$$

where and after, I denote identity operator.

Then the mixed finite element discretization of (2.7a)-(2.7c) is as follows: find $(\boldsymbol{p}_h, y_h, u_h) \in \boldsymbol{V}_h \times W_h \times U_h$ such that

$$\min_{u_h \in U_h} \Big\{\frac{1}{2}\|\boldsymbol{p}_h - \boldsymbol{p}_d\|^2 + \frac{1}{2}\|y_h - y_d\|^2 + \frac{\nu}{2}\|u_h\|^2\Big\}, \tag{2.15a}$$

$$(A^{-1}\boldsymbol{p}_h, \boldsymbol{v}_h) - (y_h, \mathrm{div}\boldsymbol{v}_h) = 0, \qquad \forall\boldsymbol{v}_h \in \boldsymbol{V}_h, \tag{2.15b}$$

$$(\mathrm{div}\boldsymbol{p}_h, w_h) + (a_0 y_h, w_h) = (u_h, w_h), \qquad \forall w_h \in W_h. \tag{2.15c}$$

The optimal control problem (2.15a)-(2.15c) again has a unique solution $(\boldsymbol{p}_h, y_h, u_h)$, and that a triplet $(\boldsymbol{p}_h, y_h, u_h)$ is the solution of (2.15a)-(2.15c) if and only if there is a co-state $(\boldsymbol{q}_h, z_h) \in \boldsymbol{V}_h \times W_h$ such that $(\boldsymbol{p}_h, y_h, \boldsymbol{q}_h, z_h, u_h)$ satisfies the following optimality conditions:

$$(A^{-1}\boldsymbol{p}_h, \boldsymbol{v}_h) - (y_h, \mathrm{div}\boldsymbol{v}_h) = 0, \qquad \forall\boldsymbol{v}_h \in \boldsymbol{V}_h, \tag{2.16a}$$

$$(\mathrm{div}\boldsymbol{p}_h, w_h) + (a_0 y_h, w_h) = (u_h, w_h), \qquad \forall w_h \in W_h, \tag{2.16b}$$

$$(A^{-1}\boldsymbol{q}_h, \boldsymbol{v}_h) - (z_h, \mathrm{div}\boldsymbol{v}_h) = -(\boldsymbol{p}_h - \boldsymbol{p}_d, \boldsymbol{v}_h), \qquad \forall\boldsymbol{v}_h \in \boldsymbol{V}_h, \tag{2.16c}$$

$$(\mathrm{div}\boldsymbol{q}_h, w_h) + (a_0 z_h, w_h) = (y_h - y_d, w_h), \qquad \forall w_h \in W_h, \tag{2.16d}$$

$$(\nu u_h + z_h, \tilde{u}_h - u_h) \ge 0, \qquad \forall\tilde{u}_h \in U_h. \tag{2.16e}$$

Similar to (2.9), the control inequality (2.16e) can be expressed as

$$u_h = \max\{0, -z_h\}/\nu. \tag{2.17}$$

In the rest of the paper, we shall use some intermediate variables. For any control function $\tilde{u} \in U_{ad}$, we first define the state solution $(\boldsymbol{p}(\tilde{u}), y(\tilde{u}), \boldsymbol{q}(\tilde{u}), z(\tilde{u})) \in (\boldsymbol{V} \times W)^2$ associated with $\tilde{u}$ that satisfies

$$(A^{-1}\boldsymbol{p}(\tilde{u}), \boldsymbol{v}) - (y(\tilde{u}), \mathrm{div}\boldsymbol{v}) = 0, \qquad \forall \boldsymbol{v} \in \boldsymbol{V}, \tag{2.18a}$$
$$(\mathrm{div}\boldsymbol{p}(\tilde{u}), w) + (a_0 y(\tilde{u}), w) = (\tilde{u}, w), \qquad \forall w \in W, \tag{2.18b}$$
$$(A^{-1}\boldsymbol{q}(\tilde{u}), \boldsymbol{v}) - (z(\tilde{u}), \mathrm{div}\boldsymbol{v}) = -(\boldsymbol{p}(\tilde{u}) - \boldsymbol{p}_d, \boldsymbol{v}), \qquad \forall \boldsymbol{v} \in \boldsymbol{V}, \tag{2.18c}$$
$$(\mathrm{div}\boldsymbol{q}(\tilde{u}), w) + (a_0 z(\tilde{u}), w) = (y(\tilde{u}) - y_d, w), \qquad \forall w \in W. \tag{2.18d}$$

Then, we define the discrete state solution $(\boldsymbol{p}_h(\tilde{u}), y_h(\tilde{u}), \boldsymbol{q}_h(\tilde{u}), z_h(\tilde{u})) \in (\boldsymbol{V}_h \times W_h)^2$ associated with $\tilde{u}$ that satisfies

$$(A^{-1}\boldsymbol{p}_h(\tilde{u}), \boldsymbol{v}_h) - (y_h(\tilde{u}), \mathrm{div}\boldsymbol{v}_h) = 0, \qquad \forall \boldsymbol{v}_h \in \boldsymbol{V}_h, \tag{2.19a}$$
$$(\mathrm{div}\boldsymbol{p}_h(\tilde{u}), w_h) + (a_0 y_h(\tilde{u}), w_h) = (\tilde{u}, w_h), \qquad \forall w_h \in W_h, \tag{2.19b}$$
$$(A^{-1}\boldsymbol{q}_h(\tilde{u}), \boldsymbol{v}_h) - (z_h(\tilde{u}), \mathrm{div}\boldsymbol{v}_h) = -(\boldsymbol{p}_h(\tilde{u}) - \boldsymbol{p}_d, \boldsymbol{v}_h), \qquad \forall \boldsymbol{v}_h \in \boldsymbol{V}_h, \tag{2.19c}$$
$$(\mathrm{div}\boldsymbol{q}_h(\tilde{u}), w_h) + (a_0 z_h(\tilde{u}), w_h) = (y_h(\tilde{u}) - y_d, w_h), \qquad \forall w_h \in W_h. \tag{2.19d}$$

Thus, as we defined, the exact solution and its approximation can be written in the following way:

$$(\boldsymbol{p}, y, \boldsymbol{q}, z) = (\boldsymbol{p}(u), y(u), \boldsymbol{q}(u), z(u)),$$
$$(\boldsymbol{p}_h, y_h, \boldsymbol{q}_h, z_h) = (\boldsymbol{p}_h(u_h), y_h(u_h), \boldsymbol{q}_h(u_h), z_h(u_h)).$$

3 Error estimates and superconvergence analysis

Superconvergence has been one of the important features for the finite element methods, see, e.g., [7, 8, 14, 31]. In this section, we will derive the error estimates and some superconvergent results for the control variable.

Now, we are in the position of deriving the estimate for $\|P_h z(u_h) - z_h\|$.

Lemma 3.1. *Let* $(\boldsymbol{p}(u_h), y(u_h), \boldsymbol{q}(u_h), z(u_h)) \in (\boldsymbol{V} \times W)^2$ *and* $(\boldsymbol{p}_h, y_h, \boldsymbol{q}_h, z_h) \in (\boldsymbol{V}_h \times W_h)^2$ *be the solutions of* (2.18a)-(2.18d) *and* (2.19a)-(2.19d) *with* $\tilde{u} = u_h$ *respectively. If the solution satisfies*

$$\boldsymbol{p}(u_h),\ \boldsymbol{q}(u_h) \in (H^1(\Omega))^2 \ \ \text{and} \ \ y(u_h),\ z(u_h) \in W^{1,\infty}(\Omega),$$

then we have

$$\|P_h y(u_h) - y_h\| \le Ch^2, \tag{3.1a}$$
$$\|P_h z(u_h) - z_h\| \le Ch^2. \tag{3.1b}$$

Proof. From Eqs. (2.18a)-(2.18d) and (2.19a)-(2.19d), we can easily obtain the following error equations

$$(A^{-1}(\boldsymbol{p}(u_h)-\boldsymbol{p}_h),\boldsymbol{v}_h)-(y(u_h)-y_h,\mathrm{div}\boldsymbol{v}_h)=0, \tag{3.2a}$$
$$(\mathrm{div}(\boldsymbol{p}(u_h)-\boldsymbol{p}_h),w_h)+(a_0(y(u_h)-y_h),w_h)=0, \tag{3.2b}$$
$$(A^{-1}(\boldsymbol{q}(u_h)-\boldsymbol{q}_h),\boldsymbol{v}_h)-(z(u_h)-z_h,\mathrm{div}\boldsymbol{v}_h)=-(\boldsymbol{p}(u_h)-\boldsymbol{p}_h,\boldsymbol{v}_h), \tag{3.2c}$$
$$(\mathrm{div}(\boldsymbol{q}(u_h)-\boldsymbol{q}_h),w_h)+(a_0(z(u_h)-z_h),w_h)=(y(u_h)-y_h,w_h), \tag{3.2d}$$

for any $\boldsymbol{v}_h\in\boldsymbol{V}_h$ and $w_h\in W_h$.

As a result of (2.12a), we can rewrite (3.2a)-(3.2d) as

$$(A^{-1}(\boldsymbol{p}(u_h)-\boldsymbol{p}_h),\boldsymbol{v}_h)-(P_hy(u_h)-y_h,\mathrm{div}\boldsymbol{v}_h)=0, \tag{3.3a}$$
$$(\mathrm{div}(\boldsymbol{p}(u_h)-\boldsymbol{p}_h),w_h)+(a_0(y(u_h)-y_h),w_h)=0, \tag{3.3b}$$
$$(A^{-1}(\boldsymbol{q}(u_h)-\boldsymbol{q}_h),\boldsymbol{v}_h)-(P_hz(u_h)-z_h,\mathrm{div}\boldsymbol{v}_h)=-(\boldsymbol{p}(u_h)-\boldsymbol{p}_h,\boldsymbol{v}_h), \tag{3.3c}$$
$$(\mathrm{div}(\boldsymbol{q}(u_h)-\boldsymbol{q}_h),w_h)+(a_0(z(u_h)-z_h),w_h)=(P_hy(u_h)-y_h,w_h), \tag{3.3d}$$

for any $\boldsymbol{v}_h\in\boldsymbol{V}_h$ and $w_h\in W_h$.

For sake of simplicity, we now denote

$$\tau=P_hy(u_h)-y_h,\quad e=P_hz(u_h)-z_h. \tag{3.4}$$

Then, we estimate (3.1a) and (3.1b) in Part I and Part II, respectively.

Part I. As we can see,

$$\|\tau\|=\sup_{\psi\in L^2(\Omega),\ \psi\neq 0}\frac{(\tau,\psi)}{\|\psi\|}, \tag{3.5}$$

we then need to bound (τ,ψ) for $\psi\in L^2(\Omega)$. Let $\phi\in H^2(\Omega)\cap H_0^1(\Omega)$ be the solution of (2.4). We can see from (2.13a) and (3.3a)

$$\begin{aligned}(\tau,\psi)=&(\tau,-\mathrm{div}(A\nabla\phi))+(\tau,a_0\phi)\\ =&-(\tau,\mathrm{div}(\Pi_h(A\nabla\phi)))+(\tau,a_0\phi)\\ =&-(A^{-1}(\boldsymbol{p}(u_h)-\boldsymbol{p}_h),\Pi_h(A\nabla\phi))+(\tau,a_0\phi).\end{aligned} \tag{3.6}$$

Note that

$$(\mathrm{div}(\boldsymbol{p}(u_h)-\boldsymbol{p}_h),\phi)+(A^{-1}(\boldsymbol{p}(u_h)-\boldsymbol{p}_h),A\nabla\phi)=0. \tag{3.7}$$

Thus, from (3.3b), (3.6) and (3.7), we derive

$$\begin{aligned}(\tau,\psi)=&(A^{-1}(\boldsymbol{p}(u_h)-\boldsymbol{p}_h),A\nabla\phi-\Pi_h(A\nabla\phi))\\ &+(\mathrm{div}(\boldsymbol{p}(u_h)-\boldsymbol{p}_h),\phi-P_h\phi)+(a_0\tau,\phi-P_h\phi)\\ &+(a_0(y(u_h)-P_h(y(u_h))),\phi-P_h\phi)-(a_0(y(u_h)-P_h(y(u_h))),\phi).\end{aligned} \tag{3.8}$$

From (2.13b), we have

$$(A^{-1}(\boldsymbol{p}(u_h)-\boldsymbol{p}_h), A\nabla\phi-\Pi_h(A\nabla\phi)) \leq Ch\|\boldsymbol{p}(u_h)-\boldsymbol{p}_h\|\cdot\|\phi\|_2. \tag{3.9}$$

Let $\tilde{u}=u_h$ and $w=\mathrm{div}\boldsymbol{p}(u_h)+a_0y(u_h)-u_h$ in (2.18b), we can find that

$$\mathrm{div}\boldsymbol{p}(u_h)+a_0y(u_h)-u_h=0. \tag{3.10}$$

Similarly, by (2.12a) and (2.16b), it is easy to see that

$$\mathrm{div}\boldsymbol{p}_h=u_h-P_ha_0y_h. \tag{3.11}$$

By (3.10), (3.11), (2.12a) and (2.12b), we have

$$\begin{aligned}(\mathrm{div}(\boldsymbol{p}(u_h)-\boldsymbol{p}_h),\phi-P_h\phi)=&(P_ha_0y_h-a_0y(u_h),\phi-P_h\phi)\\=&(P_h(a_0y(u_h))-a_0y(u_h),\phi-P_h\phi)\\\leq&C\|P_h(a_0y(u_h))-a_0y(u_h)\|\cdot\|\phi-P_h\phi\|\\\leq&Ch^2\|a_0\|_{1,\infty}\|y(u_h)\|_1\|\phi\|_1.\end{aligned} \tag{3.12}$$

For the third and the fourth terms on the right side of (3.8), using (2.12b), we get

$$(a_0\tau,\phi-P_h\phi)\leq Ch\|\tau\|\cdot\|\phi\|_1, \tag{3.13a}$$
$$(a_0(y(u_h)-P_h(y(u_h))),\phi-P_h\phi)\leq Ch^2\|y(u_h)\|_1\|\phi\|_1. \tag{3.13b}$$

Moreover, by (2.12b), we find that

$$\begin{aligned}(a_0(y(u_h)-P_h(y(u_h))),\phi)=&(y(u_h)-P_h(y(u_h)),a_0\phi)\\\leq&C\|y(u_h)-P_h(y(u_h))\|_{-1}\|a_0\phi\|_1\\\leq&Ch^2\|a_0\|_{1,\infty}\|y(u_h)\|_1\|\phi\|_1.\end{aligned} \tag{3.14}$$

For sufficiently small h, by (3.5), (3.8)-(3.9) and (3.12)-(3.14), we derive

$$\|P_hy(u_h)-y_h\|\leq Ch\|\boldsymbol{p}(u_h)-\boldsymbol{p}_h\|+Ch^2. \tag{3.15}$$

Choosing $\boldsymbol{v}_h=\Pi_h\boldsymbol{p}(u_h)-\boldsymbol{p}_h$ in (3.3a) and $w_h=P_hy(u_h)-y_h$ in (3.3b), respectively. Then adding the two equations to get

$$\begin{aligned}&(A^{-1}(\Pi_h\boldsymbol{p}(u_h)-\boldsymbol{p}_h),\Pi_h\boldsymbol{p}(u_h)-\boldsymbol{p}_h)+(a_0(P_hy(u_h)-y_h),P_hy(u_h)-y_h)\\=&-(A^{-1}(\boldsymbol{p}(u_h)-\Pi_h\boldsymbol{p}(u_h)),\Pi_h\boldsymbol{p}(u_h)-\boldsymbol{p}_h)\\&-(a_0(y(u_h)-P_hy(u_h)),P_hy(u_h)-y_h).\end{aligned} \tag{3.16}$$

Using (3.16), (2.12b), (2.13b) and the assumptions on A and a_0, we find that

$$\|\Pi_h\boldsymbol{p}(u_h)-\boldsymbol{p}_h\|\leq Ch+\|P_hy(u_h)-y_h\|. \tag{3.17}$$

Substituting (3.17) into (3.15), using (2.13b), for sufficiently small h, we have

$$\|P_h y(u_h) - y_h\| \leq Ch^2, \tag{3.18}$$

which yields (3.1a).

Part II. Since

$$\|e\| = \sup_{\psi \in L^2(\Omega), \psi \neq 0} \frac{(e,\psi)}{\|\psi\|}, \tag{3.19}$$

we then need to bound (e,ψ) for $\psi \in L^2(\Omega)$. From (2.13a) and (3.3c), we can see that

$$\begin{aligned}(e,\psi) =&(e, -\mathrm{div}(A\nabla\phi)) + (e, a_0\phi)\\ =&-(e, \mathrm{div}(\Pi_h(A\nabla\phi))) + (e, a_0\phi)\\ =&-(A^{-1}(\boldsymbol{q}(u_h) - \boldsymbol{q}_h), \Pi_h(A\nabla\phi)) + (e, a_0\phi)\\ &-(\boldsymbol{p}(u_h) - \boldsymbol{p}_h, \Pi_h(A\nabla\phi)).\end{aligned} \tag{3.20}$$

Note that

$$(\mathrm{div}(\boldsymbol{q}(u_h) - \boldsymbol{q}_h), \phi) + (A^{-1}(\boldsymbol{q}(u_h) - \boldsymbol{q}_h), A\nabla\phi) = 0. \tag{3.21}$$

Thus, from (3.3d), (3.20) and (3.21), we derive

$$\begin{aligned}(e,\psi) =&(A^{-1}(\boldsymbol{q}(u_h) - \boldsymbol{q}_h), A\nabla\phi - \Pi_h(A\nabla\phi))\\ &+ (\mathrm{div}(\boldsymbol{q}(u_h) - \boldsymbol{q}_h), \phi - P_h\phi) + (a_0 e, \phi - P_h\phi)\\ &+ (a_0(z(u_h) - P_h(z(u_h))), \phi - P_h\phi) - (a_0(z(u_h) - P_h(z(u_h))), \phi)\\ &- (\tau, P_h\phi) - (\boldsymbol{p}(u_h) - \boldsymbol{p}_h, \Pi_h(A\nabla\phi))\\ =:& \sum_{i=1}^{7} I_i.\end{aligned} \tag{3.22}$$

Let $\tilde{u} = u_h$ and $w = \mathrm{div}\boldsymbol{q}(u_h) + a_0 z(u_h) - y(u_h) + y_d$ in (2.18d), we can find that

$$\mathrm{div}\boldsymbol{q}(u_h) + a_0 z(u_h) = y(u_h) - y_d. \tag{3.23}$$

Similarly, by (2.12a) and (2.16d), it is easy to see that

$$\mathrm{div}\boldsymbol{q}_h = y_h - P_h y_d - P_h a_0 z_h. \tag{3.24}$$

By (2.12a)-(2.12b) and (3.23)-(3.24), we have

$$\begin{aligned}I_2 =&(P_h a_0 z_h - a_0 z(u_h), \phi - P_h\phi) + (P_h y_d - y_d, \phi - P_h\phi)\\ &+ (y(u_h) - P_h y(u_h), \phi - P_h\phi) + (P_h y(u_h) - y_h, \phi - P_h\phi)\\ =&(P_h(a_0 z(u_h)) - a_0 z(u_h), \phi - P_h\phi) + (P_h y_d - y_d, \phi - P_h\phi)\\ &+ (y(u_h) - P_h y(u_h), \phi - P_h\phi)\\ \leq& Ch^2(\|a_0\|_{1,\infty}\|z(u_h)\|_1 + \|y_d\|_1 + \|y(u_h)\|_1)\|\phi\|_1.\end{aligned} \tag{3.25}$$

Similar to the estimates (3.9) and (3.13a)-(3.14), we estimate I_1, I_3, I_4 and I_5 as follows

$$I_1 \le Ch\|\boldsymbol{q}(u_h)-\boldsymbol{q}_h\|\cdot\|\phi\|_2, \qquad I_3 \le Ch\|e\|\cdot\|\phi\|_1, \tag{3.26a}$$

$$I_4 \le Ch^2\|z(u_h)\|_1\|\phi\|_1, \qquad I_5 \le Ch^2\|a_0\|_{1,\infty}\|z(u_h)\|_1\|\phi\|_1. \tag{3.26b}$$

For I_6, by use of (3.1a), we get

$$I_6 \le C\|\tau\|\cdot\|\phi\| \le Ch^2\|\phi\|. \tag{3.27}$$

Finally, for I_7, from (2.13b), (2.13c), (3.1a), (3.3a) and (3.17), we have

$$\begin{aligned} I_7 =&(\boldsymbol{p}(u_h)-\boldsymbol{p}_h, A\nabla\phi-\Pi_h(A\nabla\phi))-(A^{-1}(\boldsymbol{p}(u_h)-\boldsymbol{p}_h), A^2\nabla\phi)\\ =&(\boldsymbol{p}(u_h)-\boldsymbol{p}_h, A\nabla\phi-\Pi_h(A\nabla\phi))-(A^{-1}(\boldsymbol{p}(u_h)-\boldsymbol{p}_h), A^2\nabla\phi-\Pi_h(A^2\nabla\phi))\\ &-(P_hy(u_h)-y_h, \mathrm{div}(\Pi_h(A^2\nabla\phi)))\\ \le& Ch^2\|\phi\|_2. \end{aligned} \tag{3.28}$$

Substituting the estimates I_1-I_7 in (3.22), for sufficiently small h, by (3.19), we derive

$$\|P_hz(u_h)-z_h\| \le Ch\|\boldsymbol{q}(u_h)-\boldsymbol{q}_h\|+Ch^2. \tag{3.29}$$

Next, using (2.13a), we rewrite (3.3c)-(3.3d) as

$$\begin{aligned} &(A^{-1}(\Pi_h\boldsymbol{q}(u_h)-\boldsymbol{q}_h),\boldsymbol{v}_h)-(P_hz(u_h)-z_h,\mathrm{div}\boldsymbol{v}_h)\\ =&(A^{-1}(\boldsymbol{q}(u_h)-\Pi_h\boldsymbol{q}(u_h)),\boldsymbol{v}_h)-(\boldsymbol{p}(u_h)-\Pi_h\boldsymbol{p}(u_h),\boldsymbol{v}_h)\\ &-(\Pi_h\boldsymbol{p}(u_h)-\boldsymbol{p}_h,\boldsymbol{v}_h), \qquad \forall\boldsymbol{v}_h\in\boldsymbol{V}_h, \end{aligned} \tag{3.30a}$$

$$\begin{aligned} &(\mathrm{div}(\Pi_h\boldsymbol{q}(u_h)-\boldsymbol{q}_h),w_h)+(a_0(P_hz(u_h)-z_h),w_h)\\ =&-(a_0(z(u_h)-P_hz(u_h)),w_h)+(P_hy(u_h)-y_h,w_h), \qquad \forall\, w_h\in W_h. \end{aligned} \tag{3.30b}$$

Similar to (3.17), we can get

$$\|\Pi_h\boldsymbol{q}(u_h)-\boldsymbol{q}_h\| \le Ch+\|P_hz(u_h)-z_h\|. \tag{3.31}$$

Substituting (3.31) into (3.29), using (2.13b), for sufficiently small h, we have

$$\|P_hz(u_h)-z_h\| \le Ch^2. \tag{3.32}$$

Thus, we complete the proof. $\square$

In order to derive the main results, we need the following error estimates.

Lemma 3.2. *Let* $(\boldsymbol{p}(P_hu),y(P_hu),\boldsymbol{q}(P_hu),z(P_hu))$ *and* $(\boldsymbol{p}(u),y(u),\boldsymbol{q}(u),z(u))$ *be the solutions of* (2.18a)-(2.18d) *with* $\tilde{u}=P_hu$ *and* $\tilde{u}=u$*, respectively. Assume that* $u\in H^1(\Omega)$*. Then we have*

$$\|y(u)-y(P_hu)\|+\|\boldsymbol{p}(u)-\boldsymbol{p}(P_hu)\| \le Ch^2, \tag{3.33a}$$

$$\|z(u)-z(P_hu)\|+\|\boldsymbol{q}(u)-\boldsymbol{q}(P_hu)\| \le Ch^2. \tag{3.33b}$$

Proof. First, we choose $\tilde{u} = P_h u$ and $\tilde{u} = u$ in (2.18a)-(2.18d) respectively, then we obtain the following error equations

$$(A^{-1}(\boldsymbol{p}(P_h u) - \boldsymbol{p}(u)), \boldsymbol{v}) - (y(P_h u) - y(u), \mathrm{div}\boldsymbol{v}) = 0, \tag{3.34a}$$
$$(\mathrm{div}(\boldsymbol{p}(P_h u) - \boldsymbol{p}(u)), w) + (a_0(y(P_h u) - y(u)), w) = (P_h u - u, w), \tag{3.34b}$$
$$(A^{-1}(\boldsymbol{q}(P_h u) - \boldsymbol{q}(u)), \boldsymbol{v}) - (z(P_h u) - z(u), \mathrm{div}\boldsymbol{v}) = -(\boldsymbol{p}(P_h u) - \boldsymbol{p}(u), \boldsymbol{v}), \tag{3.34c}$$
$$(\mathrm{div}(\boldsymbol{q}(P_h u) - \boldsymbol{q}(u)), w) + (a_0(z(P_h u) - z(u)), w) = (y(P_h u) - y(u), w), \tag{3.34d}$$

for any $\boldsymbol{v} \in \boldsymbol{V}$ and $w \in W$.

Setting $\boldsymbol{v} = \boldsymbol{p}(P_h u) - \boldsymbol{p}(u)$ and $w = y(P_h u) - y(u)$ in (3.34a) and (3.34b) respectively and adding the two equations to get

$$\begin{aligned}&(A^{-1}(\boldsymbol{p}(P_h u) - \boldsymbol{p}(u)), \boldsymbol{p}(P_h u) - \boldsymbol{p}(u)) + (a_0(y(P_h u) - y(u)), y(P_h u) - y(u))\\ =&(P_h u - u, y(P_h u) - y(u)).\end{aligned} \tag{3.35}$$

Then, we estimate the right side of (3.35). Note that $\boldsymbol{p}(P_h u) - \boldsymbol{p}(u) = -\nabla(y(P_h u) - y(u))$, by (2.12b) and Poincare' inequality, we have

$$\begin{aligned}(P_h u - u, y(P_h u) - y(u)) \leq& C\|P_h u - u\|_{-1}\|y(P_h u) - y(u))\|_1\\ \leq& Ch^2\|u\|_1\|\boldsymbol{p}(P_h u) - \boldsymbol{p}(u)\|.\end{aligned} \tag{3.36}$$

It follows from the assumptions on A and a_0, (3.35) and (3.36) that

$$\|\boldsymbol{p}(P_h u) - \boldsymbol{p}(u)\| \leq Ch^2. \tag{3.37}$$

By the Poincare's inequality again, we have

$$\|y(P_h u) - y(u)\| \leq C\|\boldsymbol{p}(P_h u) - \boldsymbol{p}(u)\| \leq Ch^2. \tag{3.38}$$

Similarly, selecting $\boldsymbol{v} = \boldsymbol{q}(P_h u) - \boldsymbol{q}(u)$ and $w = z(P_h u) - z(u)$ in (3.34c) and (3.34d), respectively. It follows from the standard stability argument that

$$\|z(P_h u) - z(u)\| + \|\boldsymbol{q}(P_h u) - \boldsymbol{q}(u)\| \leq C(\|y(P_h u) - y(u)\| + \|\boldsymbol{p}(P_h u) - \boldsymbol{p}(u)\|). \tag{3.39}$$

Therefore Lemma 3.2 is proved from (3.37)-(3.39). $\square$

Now, we will discuss the superconvergence for the control variable. Let

$$\begin{aligned}&\Omega^+ = \Big\{\bigcup T \,:\, T \subset \Omega, u(x)|_T > 0\Big\},\\ &\Omega^0 = \Big\{\bigcup T \,:\, T \subset \Omega, u(x)|_T \equiv 0\Big\},\\ &\Omega^- = \Omega \setminus (\Omega^+ \cup \Omega^0).\end{aligned}$$

It is easy to check that the three parts do not intersect on each other, and $\Omega = \Omega^+ \cup \Omega^0 \cup \Omega^-$. In this paper we assume that u and $\mathcal{T}_h$ are regular such that $\mathrm{meas}(\Omega^-) \leq Ch$ (see [21]).

Theorem 3.1. *Let u be the solution of* (2.8a)-(2.8e) *and u_h be the solution of* (2.16a)-(2.16e)*, respectively. Assume that all the assumptions in Lemma 3.1 are valid and $u, z \in W^{1,\infty}(\Omega)$. Then, we have*

$$\|P_h u - u_h\| \le Ch^{\frac{3}{2}}. \tag{3.40}$$

Proof. We choose $\tilde{u} = u_h$ in (2.8e) and $\tilde{u}_h = P_h u$ in (2.16e) to get the following two inequalities:

$$(\nu u + z, u_h - u) \ge 0 \tag{3.41}$$

and

$$(\nu u_h + z_h, P_h u - u_h) \ge 0. \tag{3.42}$$

Note that $u_h - u = u_h - P_h u + P_h u - u$. Adding the two inequalities (3.41) and (3.42), we have

$$(\nu u_h + z_h - \nu u - z, P_h u - u_h) + (\nu u + z, P_h u - u) \ge 0. \tag{3.43}$$

Thus, by (3.43) and (2.12a), we find that

$$\begin{aligned}
\nu\|P_h u - u_h\|^2 =& \nu(P_h u - u_h, P_h u - u_h) \\
=& \nu(P_h u - u, P_h u - u_h) + \nu(u - u_h, P_h u - u_h) \\
\le& (z_h - z, P_h u - u_h) + (\nu u + z, P_h u - u) \\
=& (z_h - P_h z(u_h), P_h u - u_h) + (\nu u + z, P_h u - u) \\
& + (z(P_h u) - z(u), P_h u - u_h) + (z(u_h) - z(P_h u), P_h u - u_h).
\end{aligned} \tag{3.44}$$

By Lemma 3.1 and Lemma 3.2, we find that

$$(z_h - P_h z(u_h), P_h u - u_h) \le Ch^4 + \frac{\nu}{4}\|P_h u - u_h\|^2 \tag{3.45}$$

and

$$(z(P_h u) - z(u), P_h u - u_h) \le Ch^4 + \frac{\nu}{4}\|P_h u - u_h\|^2. \tag{3.46}$$

For the second term at the right side of (3.44), by Theorem 5.1 in [8], we have

$$(\nu u + z, P_h u - u) \le Ch^3(\|u\|^2_{1,\infty} + \|z\|^2_{1,\infty}). \tag{3.47}$$

For the last term at the right side of (3.44), it is easy to see that

$$(z(u_h) - z(P_h u), P_h u - u_h) = -\|y(u_h) - y(P_h u)\|^2 - \|\boldsymbol{p}(u_h) - \boldsymbol{p}(P_h u)\|^2 \le 0. \tag{3.48}$$

Combining (3.44)-(3.48), we derive (3.40). □

Now, we can derive the L^2 and L^∞-error estimates for the control variable.

Theorem 3.2. *Let* (y,z,u) *and* (y_h,z_h,u_h) *be the solutions of* (2.8a)-(2.8e) *and* (2.16a)-(2.16e) *respectively. Assume that all the assumptions in Theorem 3.1 are valid. Then we have*

$$\|u-u_h\| \leq Ch, \tag{3.49a}$$

$$\|u-u_h\|_{0,\infty} \leq Ch. \tag{3.49b}$$

Proof. Using (2.12b) and Theorem 3.1, it is easy to see that

$$\begin{aligned}\|u-u_h\| \leq&\|u-P_hu\|+\|P_hu-u_h\|\\ \leq&Ch\|u\|_1+\|P_hu-u_h\|\\ \leq&Ch.\end{aligned} \tag{3.50}$$

From (1.2), we have the following error equation

$$-\mathrm{div}(A\nabla(y-y(u_h)))+a_0(y-y(u_h))=u-u_h. \tag{3.51}$$

Using Lemma 2.1, (3.50) and the classical imbedding theorem, we can see that

$$\|y-y(u_h)\|_{0,\infty} \leq C\|y-y(u_h)\|_2 \leq C\|u-u_h\| \leq Ch. \tag{3.52}$$

Thus, by use of Lemma 3.1, (2.12b), (3.52) and the inverse estimate, we find that

$$\begin{aligned}\|y-y_h\|_{0,\infty} \leq&\|y-y(u_h)\|_{0,\infty}+\|y(u_h)-P_hy(u_h)\|_{0,\infty}+\|P_hy(u_h)-y_h\|_{0,\infty}\\ \leq&Ch\|y(u_h)\|_{1,\infty}+Ch^{-1}\|P_hy(u_h)-y_h\|\\ \leq&Ch.\end{aligned} \tag{3.53}$$

Similarly, from (2.1), we have the following error equation

$$-\mathrm{div}(A\nabla(z-z(u_h)))+a_0(z-z(u_h))=-\mathrm{div}(A^2\nabla(y-y(u_h)))+y-y(u_h). \tag{3.54}$$

Using Lemma 2.1 and the classical imbedding theorem, we can see that

$$\begin{aligned}\|z-z(u_h)\|_{0,\infty} \leq&C\|z-z(u_h)\|_2\\ \leq&C\|\mathrm{div}(A^2\nabla(y-y(u_h)))-y+y(u_h)\|\\ \leq&C\|\mathrm{div}(A^2\nabla(y-y(u_h)))\|+C\|y-y(u_h)\|\\ \leq&C\|A^2\nabla(y-y(u_h))\|_1+C\|y-y(u_h)\|\\ \leq&C\|A\|^2_{1,\infty}\|y-y(u_h)\|_2+C\|y-y(u_h)\|\\ \leq&C\|y-y(u_h)\|_2.\end{aligned} \tag{3.55}$$

Thus, by use of (2.12b), (3.52), (3.55), Lemma 3.1 and the inverse estimate, we find that

$$\begin{aligned}\|z-z_h\|_{0,\infty} \leq&\|z-z(u_h)\|_{0,\infty}+\|z(u_h)-P_hz(u_h)\|_{0,\infty}+\|P_hz(u_h)-z_h\|_{0,\infty}\\ \leq&Ch.\end{aligned} \tag{3.56}$$

Finally, from (2.9), (2.17) and (3.56), we get

$$\|u-u_h\|_{0,\infty} \le C\|z-z_h\|_{0,\infty} \le Ch. \tag{3.57}$$

We complete the proof. □

Now, let us construct the recovery operator G_h. Let G_hv be a continuous piecewise linear function (without zero boundary constraint). The value of G_hv on the nodes are defined by least-squares argument on an element patches surrounding the nodes, the details can be refer to the definition of R_h in [18].

Theorem 3.3. *Let u and u_h be the solutions of* (2.8a)-(2.8e) *and* (2.16a)-(2.16e)*, respectively. Assume that all the conditions in Theorem 3.1 are valid and $u \in W^{1,\infty}(\Omega)$. Then we have*

$$\|u-G_hu_h\| \le Ch^{\frac{3}{2}}. \tag{3.58}$$

Proof. Let P_hu be defined in (2.12a). Then

$$\|u-G_hu_h\| \le \|u-G_hu\| + \|G_hu-G_hP_hu\| + \|G_hP_hu-G_hu_h\|. \tag{3.59}$$

According to Lemma 4.2 in [18], we have

$$\|u-G_hu\| \le Ch^{\frac{3}{2}}. \tag{3.60}$$

Using the definition of G_h, we find that

$$G_hu = G_hP_hu \tag{3.61}$$

and

$$\|G_hP_hu-G_hu_h\| \le C\|P_hu-u_h\|. \tag{3.62}$$

Combining (3.59)-(3.62) with Theorem 3.1, we complete the proof. □

Moreover, as in [21] we construct a postprocessing projection operator of the discrete co-state to the admissible set

$$\hat{u} = \max\{0, -G_hz_h\}/\nu. \tag{3.63}$$

Now, we can prove the second global superconvergence result for the control variable.

Theorem 3.4. *Assume that all the conditions in Theorem 3.1 and Lemma 3.1 are valid. Moreover, we assume that $\boldsymbol{p},\boldsymbol{q} \in H^1(\Omega)$ and $y,z \in H^2(\Omega)\cap W^{1,\infty}(\Omega)$. Let u be the solution of* (2.8a)-(2.8e) *and $\hat{u}$ be the function constructed in* (3.63)*. Then we have*

$$\|u-\hat{u}\| \le Ch^{\frac{3}{2}}. \tag{3.64}$$

Proof. Similar to Lemma 3.1, we can derive

$$\|P_h z - z_h\| \leq Ch^{\frac{3}{2}}. \tag{3.65}$$

Using the same method as in Theorem 3.3, we find that

$$\|z - G_h z_h\| \leq Ch^{\frac{3}{2}}. \tag{3.66}$$

Then, by use of (2.9) and (3.63), we arrive at

$$|u - \hat{u}| \leq C|z - G_h z_h|. \tag{3.67}$$

Thus, (3.64) can be proved by (3.66) and (3.67). □

4 Numerical experiments

In this section, we present below an example to illustrate the theoretical results. The optimization problems were solved numerically by projected gradient methods, with codes developed based on AFEPack [16]. The discretization was already described in previous sections: the control function u was discretized by piecewise constant functions, whereas the state $(y, \boldsymbol{p})$ and the co-state $(z, \boldsymbol{q})$ were approximated by the lowest order Raviart-Thomas mixed finite element functions. In our examples, we choose the domain $\Omega = [0,1] \times [0,1]$, $\nu = 1$, $a_0 = 0$ and $A = I$.

Example 4.1. We consider the following two-dimensional elliptic optimal control problem

$$\min_{u \in U_{ad}} \left\{ \frac{1}{2}\|\boldsymbol{p} - \boldsymbol{p}_d\|^2 + \frac{1}{2}\|y - y_d\|^2 + \frac{1}{2}\|u - u_0\|^2 \right\} \tag{4.1}$$

subject to the state equation

$$\mathrm{div}\boldsymbol{p} = f + u, \quad \boldsymbol{p} = -\mathbf{grad} y, \tag{4.2}$$

where

$$y = \sin(\pi x_1)\sin(\pi x_2), \qquad z = \sin(\pi x_1)\sin(\pi x_2), \tag{4.3a}$$

$$u_0 = 1.0 - 0.8\sin\left(\frac{\pi x_1}{2}\right) - 0.8\sin(2\pi x_2), \qquad u = \max(u_0 - z, 0), \tag{4.3b}$$

$$f = 2\pi^2 y - u, \qquad y_d = y - 2\pi^2 y, \tag{4.3c}$$

$$\boldsymbol{p}_d = -\begin{pmatrix} \pi\cos(\pi x_1)\sin(\pi x_2) \\ \pi\sin(\pi x_1)\cos(\pi x_2) \end{pmatrix}. \tag{4.3d}$$

In Table 1, the errors $\|u - u_h\|$, $\|u - u_h\|_{0,\infty}$, $\|P_h u - u_h\|$, $\|u - G_h u_h\|$ and $\|u - \hat{u}\|$ obtained on a sequence of uniformly refined meshes are shown. Table 2 shows the convergence orders of these errors. In Fig. 1, the profile of the numerical solution of u on the 64×64 mesh grid is plotted. Theoretical results are clearly recognized from the data.

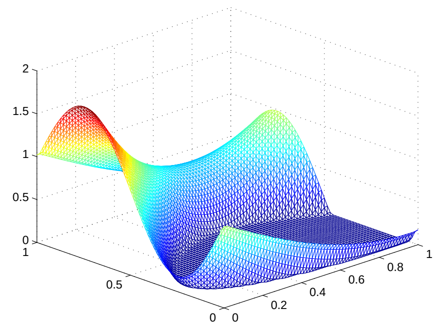

Figure 1: The profile of the numerical solution of u on 64×64 triangle mesh.

Table 1: The errors of $\|u-u_h\|$, $\|u-u_h\|_{0,\infty}$, $\|P_hu-u_h\|$, $\|u-G_hu_h\|$ and $\|u-\hat{u}\|$.

Resolution	$\|u-u_h\|$	$\|u-u_h\|_{0,\infty}$	$\|P_hu-u_h\|$	$\|u-G_hu_h\|$	$\|u-\hat{u}\|$
16×16	4.60625e-02	2.46735e-01	5.47148e-03	3.90463e-02	2.01596e-02
32×32	2.31954e-02	1.24657e-01	1.95519e-03	1.55743e-02	6.79340e-03
64×64	1.16692e-02	6.24724e-02	7.28564e-04	5.61971e-03	2.32197e-03
128×128	5.84874e-03	3.12553e-02	2.45686e-04	1.98680e-03	8.04781e-04

Table 2: Convergence orders of $\|u-u_h\|$, $\|u-u_h\|_{0,\infty}$, $\|P_hu-u_h\|$, $\|u-G_hu_h\|$ and $\|u-\hat{u}\|$.

h	$\|u-u_h\|$	$\|u-u_h\|_{0,\infty}$	$\|P_hu-u_h\|$	$\|u-G_hu_h\|$	$\|u-\hat{u}\|$
1/16	-	-	-	-	-
1/32	0.9898	0.9850	1.4846	1.3260	1.5693
1/64	0.9911	0.9967	1.4242	1.4706	1.5488
1/128	0.9965	0.9991	1.5682	1.5000	1.5287

5 Conclusions

In this paper, we discussed the lowest order Raviart-Thomas mixed finite element methods for an linear elliptic optimal control problem (1.1)-(1.4). Our superconvergence analysis and L^∞-error estimates for the linear elliptic optimal control problems by mixed finite element methods seems to be new, and these results can be extended to RT1 mixed finite element methods. In our future work, we will investigate the superconvergence of the lowest order mixed finite element methods for optimal control problems governed by nonlinear elliptic equations.

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