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A Maximum Entropy Method Based on Orthogonal Polynomials for Frobenius-Perron Operators

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Abstract. Let S: $[0,1]\to[0,1]$ be a chaotic map and let f^* be a stationary density of the Frobenius-Perron operator P_S: $L^1\to L^1$ associated with S. We develop a numerical algorithm for approximating f^*, using the maximum entropy approach to an under-determined moment problem and the Chebyshev polynomials for the stability consideration. Numerical experiments show considerable improvements to both the original maximum entropy method and the discrete maximum entropy method.

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1 Introduction

In the past fifty years, since the publication of the pioneering work of Jayne (see [8]), the idea of maximum entropy method has been widely applied to solving density function recovering problems in mathematical physics and stochastic analysis. This idea was first adopted in [4] to numerically compute a stationary density of a chaotic map S from the interval $[0,1]$ to itself, based on the classic Hausdorff moment problems.

The maximum entropy method developed in [4] has been applied to the computation of Lyapunov exponents of chaotic maps in [5], which is closely related to the computation of the stationary density f^* since the Lyapunov exponent can be calculated by

$$\lambda = \int_0^1 f^*(x)\ln|S'(x)|dx,$$

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that quantitatively describes the sensitivity of the orbits on the initial conditions for the chaotic dynamics. The numerical experiments in [4,5] suggest that for relatively small number of moments, the algorithm can produce better approximations of the stationary density and exact Lyapunov exponent than the famous Ulam's method (see [7,11]). But due to the ill-conditioning resulting from employing the standard monomial basis of $\{1, x, x^2, \cdots, x^n\}$ (the condition number may reach the order of 10^{17} for n=12), round-off errors dominate the computation of the algorithm even if a high precision Gauss quadrature is used in numerical integration.

Recently, the authors of [2] proposed a discrete version of a maximum entropy method for computing stationary densities and Lyapunov exponents. Basically they first approximate the Boltzmann entropy functional, which is the objective function of the maximum entropy optimization problem, by a high precision Gauss quadrature, and do the same thing for the moment constraints. The resulting optimization problem is still finite dimensional, but integration is avoided, which is natural since the Gauss quadrature numerical integration had been done before solving the discretized optimization problem. In their implementation of the algorithm, the monomial basis of polynomials is replaced with the Chebeshev polynomial basis. The computationally needed moments of the unknown stationary density with respect to the Chebeshev polynomials are estimated by the average values of the polynomials along the orbit of an initial point under the repeated iteration of the map S. This is justified in theory by the classic Birkhoff individual ergodic theorem, which says that the time average equals the space average for ergodic maps. As many as 150 moments can be used in [2] for the implementation of the algorithm. However, there is an approximation accuracy issue here, that is, some additional errors occur from approximating the Boltzmann entropy functional and the constraint equations. Such errors explain why a relatively large number of moments are needed for the numerical recovery of the stationary density to a prescribed precision.

In this paper, we intend to overcome the two main drawbacks of the original maximum entropy method for solving the stationary density problem of Frobenius-Perron operators. The first drawback is the ill-conditioning of the monomials, so we employ orthogonal polynomials in our numerical computation. The second drawback is related to the "homogeneous moment problem" proposed in [4] since the maximum entropy solution involves the underlying map which is only piecewise continuous in general. Thus, a good accuracy of the computed stationary density may not be guaranteed. To solve this problem, as is done in the paper [2], we use the same idea of Birkhoff's individual ergodic theorem, and consequently we solve a "nonhomogeneous moment problem" whose solution is a smooth function. Thus we propose a new practical algorithm for solving the stationary density problem of Frobenius-Perron operators, which combines the original idea of the maximum entropy method [4] and the idea of solving a nonhomogeneous moment problem [2], using the good stability property of the orthogonal polynomials. From the reported numerical experiment results one can see that the present algorithm can not only use as many moments as needed, but also give a faster convergence. For some maps our algorithm uses much fewer moments to achieve the same accuracy as the one from [2].

After giving some preliminaries in the next section, we present a general maximum entropy algorithm in Section 3. A special algorithm using orthogonal polynomials such as Chebyshev or Legendre ones will be presented in Section 4. Numerical experiments using Chebyshev polynomials will be presented in Section 5 and we conclude in Section 6.

2 Preliminaries

Let a measurable transformation S: $[0,1]\to[0,1]$ be nonsingular, that is, $m(A)=0$ implies

$$m\big(S^{-1}(A)\big) = 0,$$

for any Lebesgue measurable subset A of $[0,1]$, where m denotes the Lebesgue measure. The linear operator P_S: $L^1(0,1)\to L^1(0,1)$ defined by

$$\int_A P_S f(x)dx = \int_{S^{-1}(A)} f(x)dx, \tag{2.1}$$

for every measurable $A\subset[0,1]$ is called the *Frobenius-Perron operator* associated with S.

From the definition (2.1), we see that

$$P_{S^k} = (P_S)^k,$$

for any k. It is well-known that Frobenius-Perron operators are Markov operators, that is

$$f \geq 0 \ \Rightarrow \ P_S f \geq 0 \quad \text{and} \quad \|P_S f\|_1 = \|f\|_1,$$

where

$$\|f\|_1 = \int_0^1 |f(x)|dx.$$

If

$$f^* \geq 0 \quad \text{and} \quad \|f^*\|_1 = 1,$$

then the absolutely continuous probability measure μ_{f^*} defined by

$$\mu_{f^*}(A) = \int_A f^*(x)dx, \quad \forall \text{ measurable sets } A \subset [0,1],$$

satisfies the invariance property

$$\mu_{f^*}\big(S^{-1}(A)\big) = \mu_{f^*}(A), \quad \forall \text{ measurable sets } A \subset [0,1],$$

if and only if f^* is a fixed point of P_S. This fixed point density f^* is called a *stationary density* of P_S, which gives the asymptotic statistical properties of chaotic orbits of S. See [1,6] for details related to Frobenus-Perron operators in ergodic theory.

The dual of P_S is the operator U_S: $L^\infty(0,1)\to L^\infty(0,1)$ defined by

$$U_S g(x) = g\big(S(x)\big), \tag{2.2}$$

which is called the *Koopman operator* with respect to S. Thus, we have

$$\int_0^1 g(x) P_S f(x) dx = \int_0^1 U_S g(x) f(x) dx, \tag{2.3}$$

for all $f\in L^1(0,1)$ and $g\in L^\infty(0,1)$, which will be used in the next section.

The number

$$H(f) = -\int_0^1 f(x) \ln f(x) dx, \tag{2.4}$$

is called the *Boltzmann entropy*, this number is either finite or $-\infty$, and the functional

$$H : \big\{ f \geq 0 : f \in L^1(0,1) \big\} \to [-\infty, \infty),$$

defined via (2.4) is a proper, upper semi-continuous concave function, strictly concave on its domain that consists of all functions $f\geq 0$, with $H(f)>-\infty$. Moreover, for any $\alpha>-\infty$, the upper level set

$$\big\{ f \geq 0 : H(f) \geq \alpha \big\},$$

is weakly compact in $L^1(0,1)$ (see [3]).

Let

$$D \equiv D(0,1) = \big\{ f \geq 0 : f \in L^1(0,1),\ \ \|f\|_1 = 1 \big\},$$

be the set of all densities. The theoretical background of the maximum entropy method is the following maximization problem

$$\max \Big\{ H(f) : f \in D,\ \int_0^1 f(x) g_k(x) dx = \mu_k,\ \ k = 1, \cdots, n \Big\}, \tag{2.5}$$

where

$$\big\{ g_1, g_2, \cdots, g_n \big\} \subset L^\infty(0,1),$$

and $\mu_1, \cdots, \mu_n$ are given constants. The maximum entropy method is justified by the next result [6].

Theorem 2.1. *Suppose the numbers $\lambda_1, \cdots, \lambda_n$ are chosen, such that the density function*

$$f_n(x) = \frac{e^{\sum_{k=1}^n \lambda_k g_k(x)}}{\int_0^1 e^{\sum_{k=1}^n \lambda_k g_k(x)} dx},$$

satisfies the equalities

$$\int_0^1 g_j(x) f_n(x) dx = \mu_j, \quad j = 1, \cdots, n,$$

then f_n is the solution to the maximum entropy problem (2.5).

Proof. (see [6]) The following Gibbs inequality

$$u - u\ln u \leq v - u\ln v,$$

for $u, v \geq 0$, which can be proved by elementary calculus, implies the inequality

$$\int_0^1 f(x)\ln f(x)dx \geq \int_0^1 f(x)\ln g(x)dx, \quad \forall f, g \in D. \tag{2.6}$$

Denote

$$Z = \int_0^1 e^{\sum_{k=1}^n \lambda_k g_k(x)} dx,$$

then

$$H(f_n) = \ln Z - \sum_{k=1}^n \lambda_k \mu_k.$$

Given $f \in \mathcal{D}$ that satisfies the constraints of (2.5), then from (2.6), we have

$$\begin{aligned} H(f) &\leq -\int_0^1 f(x)\ln f_n(x)dx \\ &= -\int_0^1 f(x)\Big(-\ln Z + \sum_{k=1}^n \lambda_k g_k(x)\Big)dx \\ &= \ln Z - \sum_{k=1}^n \lambda_k \int_0^1 f(x) g_k(x)dx \\ &= \ln Z - \sum_{k=1}^n \lambda_k \mu_k = H(f_n). \end{aligned}$$

Since the Boltzmann entropy is strictly concave, f_n is the unique maximum entropy solution. □

3 A general maximum entropy method

Now we develop a general maximum entropy method for solving the Frobenius-Perron operator equation. First, we need a lemma.

Lemma 3.1. *Let $\{q_n\}_{n=0}^\infty$ be a sequence of differentiable functions defined on $[0,1]$, such that $q_0(x) \equiv 1$ and its derivative function sequence $\{q_n'\}_{n=1}^\infty$ is dense in $L^2(0,1)$. If a function $f \in L^1(0,1)$ satisfies the equalities*

$$\int_0^1 q_n(x) f(x)dx = 0, \quad n = 0, 1, \cdots,$$

then

$$f = 0.$$

Proof. Let

$$\hat{f}(x) = \int_0^x f(t)dt,$$

for $0 \le x \le 1$, then $\hat{f}$ is a continuous function and

$$\hat{f}(0) = \hat{f}(1) = 0.$$

Integration by parts gives

$$\begin{aligned} 0 &= \int_0^1 q_n(x)f(x)dx = \Big[q_n(x)\hat{f}(x)\Big]\Big|_0^1 - \int_0^1 q_n'(x)\hat{f}(x)dx \\ &= -\int_0^1 q_n'(x)\hat{f}(x)dx, \quad \forall n = 1,2,\cdots. \end{aligned}$$

Since $\hat{f} \in L^2(0,1)$ and the set $\{q_n'\}_{n=1}^{\infty}$ is dense in $L^2(0,1)$, we have $\hat{f}(x) \equiv 0$, which implies that $f=0$. □

Remark 3.1. The special case that $q_n(x)=x^n$ was proved in [9].

Let S: $[0,1] \to [0,1]$ be a nonsingular transformation, such that the fixed point equation $P_S f = f$ of the corresponding Frobenius-Perron operator P_S: $L^1(0,1) \to L^1(0,1)$ has a density solution f^*.

Choose a sequence of linearly independent functions $\{q_n\}$ that satisfies the condition of the above lemma. For any positive integer N, we have

$$\frac{1}{N}\sum_{i=1}^{N} P_S^i f^* = f^*,$$

then

$$\int_0^1 q_n(x)\frac{1}{N}\sum_{i=1}^{N} P_S^i f^*(x)dx = \int_0^1 q_n(x)f^*(x)dx,$$

for $n=1,2,\cdots$, by the dual relation (2.3) and the definition (2.2) of the Koopman operator

$$\int_0^1 q_n(x)\frac{1}{N}\sum_{i=1}^{N} P_S^i f^*(x)dx = \int_0^1 \frac{1}{N}\sum_{i=1}^{N} q_n\big(S^i(x)\big)f^*(x)dx,$$

so f^* solves the following system

$$\int_0^1 \Big[q_n(x) - \frac{1}{N}\sum_{i=1}^{N} q_n\big(S^i(x)\big)\Big]f(x)dx = 0, \quad n = 1,2,\cdots, \tag{3.1}$$

of infinitely many equations. Conversely, if a density function f^* solves the above system, then, by the duality relation (2.3) of the Frobenius-Perron operator P_S and the Koopman operator U_S defined by (2.2)

$$\int_0^1 q_n(x)\Big[\frac{1}{N}\sum_{i=1}^{N} P_S^i f^*(x) - f^*(x)\Big]dx = 0, \quad n = 1,2,\cdots.$$

Thus

$$\frac{1}{N}\sum_{i=1}^{N} P_S^i f^* = f^*,$$

by Lemma 3.1. In many cases, the above equality implies

$$P_S f^* = f^*,$$

hence, finding a stationary density of the Frobenius-Perron operator is the same as finding a density solution of the *homogeneous* moment problem (3.1).

The above equivalent condition for a stationary density naturally motivates a maximum entropy method for solving (3.1) numerically as follows: choose N and n and solve the maximum entropy problem

$$\max\Big\{H(f) : f \in D,\ \int_0^1 \Big[q_k(x) - \frac{1}{N}\sum_{i=1}^{N} q_k\big(S^i(x)\big)\Big] f(x)dx = 0,\ k = 1,\cdots,n\Big\}, \tag{3.2}$$

to get an approximate stationary density f_n to the exact stationary density f^* of the Frobenius-Perron operator P_S.

Remark 3.2. The original maximum entropy method of [4] just chose N=1 in the above approach.

If the following algebraic system

$$\int_0^1 \Big[q_j(x) - \frac{1}{N}\sum_{i=1}^{N} q_j\big(S^i(x)\big)\Big] \exp\Big\{\sum_{k=1}^{n}\lambda_k\Big[q_k(x) - \frac{1}{N}\sum_{i=1}^{N} q_k\big(S^i(x)\big)\Big]\Big\}dx = 0,$$

with j=1, $\cdots$, n, of equations has a solution $(\lambda_1,\cdots,\lambda_n)$, then, by Theorem 2.1, the corresponding density

$$f_n(x) = \frac{e^{\sum_{k=1}^{n}\lambda_k\left[q_k(x)-\frac{1}{N}\sum_{i=1}^{N} q_k(S^i(x))\right]}}{\int_0^1 e^{\sum_{k=1}^{n}\lambda_k\left[q_k(x)-\frac{1}{N}\sum_{i=1}^{N} q_k(S^i(x))\right]}dx}, \tag{3.3}$$

is a solution to (3.2). The above method, however, has a serious smoothness issue. That is, the maximum entropy solution (3.3) of the homogeneous moment problem contains the map S and its iterates which may only be piecewise smooth, so it may not provide a good approximation of the stationary density f^* of P_S. In order to address this issue, we need Birkhoff's individual ergodic theorem. This classic result asserts that if S is ergodic with respect to the invariant measure μ_{f^*}, then

$$\lim_{N\to\infty}\frac{1}{N}\sum_{i=1}^{N} q_k\big(S^i(x)\big) = \int_0^1 q_k(x)f^*(x)dx, \quad \forall x\in[0,1], \text{ a.e. } \mu_{f^*},$$

which is just the kth moment of f^* with respect to q_k. Therefore, we can pick a large integer N, so that

$$\frac{1}{N}\sum_{i=1}^{N} q_k\big(S^i(x)\big) \approx \int_0^1 q_k(x)f^*(x)dx.$$

This leads to the following *nonhomogeneous* maximum entropy algorithm:

Algorithm 3.1. Approximate the moments

$$\mu_k = \lim_{N\to\infty} \frac{1}{N} \sum_{i=1}^{N} q_k\big(S^i(x)\big), \tag{3.4}$$

with a large integer N for $k{=}1,\cdots,n$, and solve the following maximum entropy problem with nonhomogeneous moments

$$\max\Big\{H(f) : f \in D,\ \int_0^1 q_k(x)f(x)dx = \mu_k,\ \ k = 1,\cdots,n\Big\}. \tag{3.5}$$

The solution of the above nonhomogeneous moment problem is

$$f_n(x) = \frac{e^{\sum_{k=1}^n \lambda_k q_k(x)}}{\int_0^1 e^{\sum_{k=1}^n \lambda_k q_k(x)}dx},$$

where the numbers λ_k's satisfy

$$\frac{\int_0^1 q_j(x)e^{\sum_{k=1}^n \lambda_k q_k(x)}dx}{\int_0^1 e^{\sum_{k=1}^n \lambda_k q_k(x)}dx} = \mu_j, \quad j = 1,\cdots,n. \tag{3.6}$$

From now on, we assume that system (3.6) is consistent for all n for the given map S, which guarantees that the maximum entropy method is well-posed. A sufficient condition that guarantees the consistency of the above system is that f^* is positive on $[0,1]$ almost everywhere, as proved in [4].

Note that the limiting case of the homogeneous moment problem (3.2) is equivalent to the nonhomogeneous moment problem (3.5) because due to (3.4) the equations

$$\int_0^1 \Big[q_k(x) - \lim_{N\to\infty}\frac{1}{N}\sum_{i=1}^{N} q_k\big(S^i(x)\big)\Big] f(x)dx = 0,$$

for $k{=}1,\cdots,n$, are equvalent to

$$\int_0^1 \big(q_k(x) - \mu_k\big) f(x)dx = 0,$$

for $k{=}1,\cdots,n$, which are

$$\int_0^1 q_k(x)f(x)dx = \mu_k,$$

for $k{=}1,\cdots,n$, because $f{\in}D$.

Using the abstract convergence theory for general moment problems from [3] as was done in the paper [4], we have the following convergence result for our Algorithm. Its proof is basically the same as that given in [4], so it is omitted here.

Theorem 3.1. *Suppose f^* is a unique stationary density of the Frobenius-Perron operator P_S, such that $H(f^*)$ is finite. Let $\{f_n\}$ be the sequence of the maximum entropy solutions from the Algorithm. Then*

$$\lim_{n\to\infty} f_n = f^*,$$

under the L^1-norm. Furthermore, we have an L^1-norm error upper bound

$$\|f_n - f^*\|_1 \equiv \int_0^1 |f_n(x) - f^*(x)|dx \le b_n e^{\frac{b_n}{2}},$$

where b_n is the L^∞-norm distance of the function $1 + \ln f^*$ to the subspace of $L^\infty(0,1)$ that is spanned by $q_1, \cdots, q_n$.

The original maximum entropy method for computing a stationary density of the Frobenius-Perron operator as developed in [4] chose $q_k(x)=x^k$ for all k. Although this choice of the "monomial moment functions" is natural and simple, the main problem of instability for relatively large n limited its use for $n>10$, due to the huge condition number and resulting round-off error dominance. In the next section, we use a sequence of orthogonal polynomials as q_k functions. In particular, we use re-scaled Chebyshev polynomials or Legendre polynomials.

4 Practical maximum entropy algorithms using orthogonal polynomials

Let w be a nonnegative function defined on an open interval (a,b). A sequence of polynomials p_n is said to be orthogonal with respect to the weight function w over $[a,b]$ if

$$\int_a^b p_n(x)p_k(x)w(x)dx = 0,$$

for all $n \neq k$. For example, the sequence of Chebyshev polynomials

$$T_n(x) = \cos(n \arccos x),$$

is orthogonal with respect to the weight function

$$w(x) = \frac{1}{\sqrt{1-x^2}},$$

over $[-1,1]$, and the Legendre polynomials

$$L_n(x) = \frac{d^n}{dx^n}(x^2-1)^n,$$

are orthogonal with respect to the weight function $w(x) \equiv 1$ over $[-1,1]$.

The first several Chebyshev polynomials are

$$T_0(x)=1, \qquad T_1(x)=x, \qquad T_2(x)=2x^2-1, \qquad T_3(x)=4x^3-3x,$$

the first Legendre polynomials are

$$L_0(x)=1, \qquad L_1(x)=2x, \qquad L_2(x)=12x^2-4, \qquad L_3(x)=120x^3-72x.$$

In general, T_n and L_n are even functions, if n is even and odd functions, if n is odd.

Suppose $\{p_n\}$ is a sequence of orthogonal polynomials with respect to the weight function w over $[a,b]$. Let

$$q_n(x)=p_n\big((b-a)x+a\big),$$

then $\{q_n\}$ is an orthogonal polynomial sequence with respect to the weight function $w((b-a)x+a)$ over $[0,1]$. Since the set $\{q_0,q_1,\cdots,q_n\}$ forms a basis for the space of all polynomials of degree less than or equal to n for all n, it is easy to see that the sequence $\{q_n\}$ of the re-scaled and shifted orthogonal polynomials satisfies the condition of Lemma 3.1.

With the choice of $\{q_n\}$ in the general algorithm of the previous section, the moment constraint equations (3.6) become

$$\frac{\int_0^1 p_j\big((b-a)x+a\big)e^{\sum_{k=1}^n \lambda_k p_k\big((b-a)x+a\big)}dx}{\int_0^1 e^{\sum_{k=1}^n \lambda_k p_k\big((b-a)x+a\big)}dx}=\mu_j, \quad j=1,\cdots,n. \tag{4.1}$$

As an example, if we choose the Chebyshev polynomials, then the above nonlinear equations are

$$\frac{\int_0^1 T_j(2x-1)e^{\sum_{k=1}^n \lambda_k T_k(2x-1)}dx}{\int_0^1 e^{\sum_{k=1}^n \lambda_k T_k(2x-1)}dx}=\mu_j, \quad j=1,\cdots,n. \tag{4.2}$$

The main numerical work of the orthogonal polynomial maximum entropy method here is solving the system (4.1) of nonlinear equations for $\lambda_1,\cdots,\lambda_n$. There are many numerical methods to solve the above nonlinear system, such as Newton's method or the quasi-Newton method [10]. In applying any nonlinear system solver, we can implement a high precision numerical integration scheme, such as a composite Gaussian quadrature, to approximate the involved integrals with high accuracy.

After a numerical solution $(\lambda_1,\cdots,\lambda_n)$ is obtained, the density function

$$f_n(x)=\frac{e^{\sum_{k=1}^n \lambda_k p_k\big((b-a)x+a\big)}}{\int_0^1 e^{\sum_{k=1}^n \lambda_k p_k\big((b-a)x+a\big)}dx},$$

is a maximum entropy approximation to the stationary density f^* of the Frobenius-Perron operator P_S associated with S: $[0,1]\to[0,1]$.

Even if the map S: $[0,1]\to[0,1]$ is only piecewise continuous, the above computed density function f_n is still smooth that can approximate the exact stationary density well. Numerical experiments in the next section show that the proposed approach works nicely for non-smooth S.

5 Numerical results

In this section, we present some numerical results. And we also compare them with those obtained in earlier papers. The test maps are

$$S_1(x) = \begin{cases} \dfrac{2x}{1-x^2}, & 0 \le x \le \sqrt{2}-1, \\ \dfrac{1-x^2}{2x}, & \sqrt{2}-1 \le x \le 1, \end{cases} \qquad S_2(x) = \begin{cases} \dfrac{2x}{1-x}, & 0 \le x \le \dfrac{1}{3}, \\ \dfrac{1-x}{2x}, & \dfrac{1}{3} \le x \le 1, \end{cases}$$
$$S_3(x) = 4x(1-x).$$

The unique stationary densities of S_i are given by

$$f_1^*(x) = \frac{4}{\pi(1+x^2)}, \qquad f_2^*(x) = \frac{2}{(1+x)^2}, \qquad f_3^*(x) = \frac{1}{\pi\sqrt{x(1-x)}},$$

here, we use the maximum entropy method with various n to approximate the stationary density. We used the Chebyshev polynomials, and we used Newton's method to solve the system (4.2) of nonlinear equations for $\lambda_1, \lambda_2, \cdots, \lambda_n$. For the evaluation of each equation we used the composite 3-node Gaussian quadrature. In Tables 1-9, Algorithm I stands for our algorithm that computes the exact moments using $f_i^*(x)$ (i=1, 2, 3) with absolute errors not more than 10^{-8}. Algorithm II and Algorithm III stand for our algorithm that approximates the exact moments using (3.4) with N=10^6 and N=10^7, respectively.

In Tables 1-3, we compared the quantities

$$\alpha_n = \int_0^1 f_n(x)\frac{\sqrt{x}}{2}dx,$$

with the exact average

$$\alpha^* = \int_0^1 f^*(x)\frac{\sqrt{x}}{2}dx,$$

Table 1: Relative errors in α_n for S_1.

n	Algorithm in [5]	Algorithm I	Algorithm II	Algorithm III
1		2.9×10^{-3}	3.1×10^{-3}	3.1×10^{-3}
2	1.5×10^{-2}	2.3×10^{-4}	4.4×10^{-4}	3.7×10^{-4}
3	1.4×10^{-2}	4.6×10^{-7}	2.5×10^{-4}	1.4×10^{-4}
4	1.2×10^{-2}	2.0×10^{-6}	2.7×10^{-4}	1.4×10^{-4}
5	1.2×10^{-2}	2.0×10^{-7}	2.7×10^{-4}	1.4×10^{-4}
6	1.2×10^{-2}	1.0×10^{-8}	2.9×10^{-4}	1.4×10^{-4}
7	1.2×10^{-2}	4.8×10^{-9}	2.8×10^{-4}	1.4×10^{-4}
8	1.2×10^{-2}	3.4×10^{-10}	2.9×10^{-4}	1.4×10^{-4}
9	1.1×10^{-2}	5.3×10^{-11}	2.9×10^{-4}	1.4×10^{-4}
10	5.2×10^{-3}	4.1×10^{-11}	2.8×10^{-4}	1.4×10^{-4}

Table 2: Relative errors in α_n for S_2.

n	Algorithm in [5]	Algorithm I	Algorithm II	Algorithm III
1		3.1×10^{-3}	2.7×10^{-3}	3.1×10^{-3}
2	4.5×10^{-2}	1.5×10^{-4}	4.0×10^{-4}	1.6×10^{-4}
3	4.0×10^{-2}	1.1×10^{-5}	5.5×10^{-4}	1.5×10^{-5}
4	3.5×10^{-2}	9.1×10^{-7}	5.8×10^{-4}	1.0×10^{-5}
5	3.3×10^{-2}	8.7×10^{-8}	5.7×10^{-4}	8.5×10^{-6}
6	3.0×10^{-2}	9.2×10^{-9}	5.9×10^{-4}	2.8×10^{-6}
7	3.0×10^{-2}	1.1×10^{-9}	5.9×10^{-4}	3.6×10^{-6}
8	3.0×10^{-2}	2.5×10^{-10}	5.8×10^{-4}	2.9×10^{-6}
9		1.5×10^{-10}	5.8×10^{-4}	2.4×10^{-6}
10		1.4×10^{-10}	5.8×10^{-4}	1.8×10^{-6}

Table 3: Relative errors in α_n for S_3.

n	Algorithm in [5]	Algorithm I	Algorithm II	Algorithm III
1		4.7×10^{-2}	4.7×10^{-2}	4.7×10^{-2}
2	1.2×10^{-1}	8.0×10^{-3}	7.7×10^{-3}	8.1×10^{-3}
3	1.2×10^{-1}	8.0×10^{-3}	7.7×10^{-3}	8.0×10^{-3}
4	1.1×10^{-1}	3.1×10^{-3}	2.7×10^{-3}	3.1×10^{-3}
5	1.1×10^{-1}	3.1×10^{-3}	2.7×10^{-3}	3.1×10^{-3}
6	1.1×10^{-1}	1.6×10^{-3}	1.3×10^{-3}	1.6×10^{-3}
7	1.1×10^{-1}	1.6×10^{-3}	1.3×10^{-3}	1.6×10^{-3}
8	9.6×10^{-2}	9.7×10^{-4}	6.3×10^{-4}	1.0×10^{-3}
9	9.6×10^{-2}	9.7×10^{-4}	6.4×10^{-4}	1.0×10^{-3}
10	9.6×10^{-2}	6.5×10^{-4}	3.2×10^{-4}	6.8×10^{-4}

to estimate the errors in the sense of weak convergence, as in the paper [5]. For S_3, since we are using re-scaled and shifted Chebyshev polynomials, all the moments are zero. Hence the exact zero moments are used in Algorithm I. Note that for all S_i Algorithms I, II and III with n=1 give smaller relative errors than those of the algorithm in [5] with n=10 for S_1 and S_3, and with n=8 for S_2,

In Tables 4-6, we compared the approximated Lyapunov exponents

$$\lambda_n = \int_0^1 f_n(x) \ln |S'(x)| dx,$$

with the exact Lyapunov exponent

$$\lambda^* = \int_0^1 f^*(x) \ln |S'(x)| dx.$$

It is striking to note that according to Table 4 Algorithms I, II and III with n=4 gave the comparable results with the Algorithm in [2], when n=100.

In Tables 7-9, we compiled L^1-norm errors

$$\int_0^1 |f^*(x) - f_n(x)| dx,$$

Table 4: Relative errors in λ_n for S_1.

n	Algorithm I	Algorithm II	Algorithm III
1	1.2×10^{-2}	1.2×10^{-2}	1.2×10^{-2}
2	6.7×10^{-4}	6.7×10^{-4}	7.3×10^{-4}
3	5.6×10^{-6}	9.9×10^{-5}	3.7×10^{-5}
4	1.2×10^{-5}	4.9×10^{-5}	3.3×10^{-5}
5	1.3×10^{-6}	7.4×10^{-5}	2.9×10^{-5}
6	8.1×10^{-8}	8.7×10^{-7}	1.8×10^{-5}
7	1.4×10^{-7}	1.6×10^{-5}	1.7×10^{-5}
8	1.3×10^{-7}	2.9×10^{-5}	1.8×10^{-5}
9	1.2×10^{-7}	1.3×10^{-5}	1.7×10^{-5}
10	1.2×10^{-7}	9.6×10^{-6}	1.7×10^{-5}

Table 5: Relative errors in λ_n for S_2.

n	Algorithm I	Algorithm II	Algorithm III
1	1.6×10^{-2}	1.7×10^{-2}	1.6×10^{-2}
2	9.5×10^{-4}	7.8×10^{-4}	9.6×10^{-4}
3	3.2×10^{-5}	2.0×10^{-4}	4.7×10^{-5}
4	8.7×10^{-6}	1.3×10^{-4}	3.6×10^{-5}
5	2.5×10^{-7}	1.5×10^{-4}	1.8×10^{-5}
6	8.1×10^{-8}	8.7×10^{-5}	6.9×10^{-6}
7	8.1×10^{-9}	5.5×10^{-5}	1.4×10^{-5}
8	1.8×10^{-9}	1.9×10^{-5}	6.2×10^{-6}
9	1.5×10^{-9}	1.6×10^{-5}	7.4×10^{-6}
10	1.4×10^{-9}	3.2×10^{-5}	1.5×10^{-5}

Table 6: Relative errors in λ_n for S_3.

n	Algorithm in [2]	Algorithm I	Algorithm II	Algorithm III
20		2.5×10^{-3}	2.8×10^{-3}	2.5×10^{-3}
40	3.5×10^{-3}	6.7×10^{-4}	8.9×10^{-4}	7.0×10^{-4}
60	1.5×10^{-3}	3.0×10^{-4}	4.6×10^{-4}	3.1×10^{-4}
80	4.3×10^{-4}	1.7×10^{-4}	2.7×10^{-4}	1.7×10^{-4}
100	2.5×10^{-4}	1.1×10^{-4}	1.3×10^{-4}	1.1×10^{-4}

in f_n for S_1, S_2 and S_3. We note that the same maps were tested in the original paper [4] with the homogeneous moment approach and n=4, and the L^1-norm errors are

$$4.5\times10^{-2} \text{ for } S_1, \quad 1.3\times10^{-1} \text{ for } S_2 \quad \text{and} \quad 3.2\times10^{-1} \text{ for } S_3.$$

As a comparison, our Algorithm III errors from Tables 7-8 are 5.8×10^{-4} for S_1 and 2.6×10^{-4} for S_2. This shows a great improvement from the new approach. Also the L^1-norm error from our Algorithms I, II and III in Table 9 is 1.8×10^{-1} for S_3. This shows that even when S is smooth, nonhomegeneous moment approach has an advantage over homogeneous moment approach.

Table 7: L^1-norm errors in f_n for S_1.

n	Algorithm I	Algorithm II	Algorithm III
1	3.3×10^{-2}	3.3×10^{-2}	3.3×10^{-2}
2	6.7×10^{-3}	6.7×10^{-3}	6.7×10^{-3}
3	2.1×10^{-4}	1.1×10^{-3}	5.8×10^{-4}
4	1.8×10^{-4}	1.6×10^{-3}	5.8×10^{-4}
5	3.0×10^{-5}	1.5×10^{-3}	5.6×10^{-4}
6	1.6×10^{-6}	1.8×10^{-3}	6.1×10^{-4}
7	1.3×10^{-6}	1.8×10^{-3}	6.1×10^{-4}
8	1.6×10^{-7}	1.7×10^{-3}	6.1×10^{-4}
9	1.9×10^{-8}	1.7×10^{-3}	6.2×10^{-4}
10	8.3×10^{-9}	2.5×10^{-3}	6.2×10^{-4}

Table 8: L^1-norm errors in f_n for S_2.

n	Algorithm I	Algorithm II	Algorithm III
1	3.0×10^{-2}	3.0×10^{-2}	3.0×10^{-2}
2	3.5×10^{-3}	3.9×10^{-3}	3.6×10^{-3}
3	4.6×10^{-4}	1.9×10^{-3}	4.6×10^{-4}
4	6.4×10^{-5}	1.9×10^{-3}	2.6×10^{-4}
5	9.2×10^{-6}	1.9×10^{-3}	2.2×10^{-4}
6	1.4×10^{-6}	2.0×10^{-3}	7.3×10^{-4}
7	2.0×10^{-7}	2.0×10^{-3}	7.1×10^{-4}
8	2.7×10^{-8}	2.4×10^{-3}	7.2×10^{-4}
9	3.9×10^{-9}	2.4×10^{-3}	7.2×10^{-4}
10	6.4×10^{-10}	2.4×10^{-3}	7.4×10^{-4}

Table 9: L^1-norm errors in f_n for S_3.

n	Algorithm I	Algorithm II	Algorithm III
4	1.8×10^{-1}	1.8×10^{-1}	1.8×10^{-1}
20	6.5×10^{-2}	6.5×10^{-2}	6.5×10^{-2}
40	3.8×10^{-2}	3.8×10^{-2}	3.8×10^{-2}
60	2.7×10^{-2}	2.8×10^{-2}	2.8×10^{-2}
80	2.2×10^{-2}	2.3×10^{-2}	2.2×10^{-2}
100	1.8×10^{-2}	2.0×10^{-2}	1.8×10^{-2}

6 Conclusions

In this paper, we approximated the stationary densities of the Frobenius-Perron operators using the improved maximum entropy method. We modified the "homogeneous moment problem" proposed in [4] to become the "nonhomogeneous moment problem" using the Birkhoff individual ergodic theorem (see (3.4)). Then we combined it with the orthogonal polynomial technique and developed a stable maximum entropy method with much faster convergence. As the last section of the paper [4] mentioned, "The choice of the standard monomial sequence x^n in the paper is only for the sake of simplicity of presentation. Better conditioned ones, such as orthogonal polynomials or basic splines, can be used to improve the convergence and stability of the maximum entropy method." The numerical experiments presented in the current paper have shown the above prediction.

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