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SEYMOUR'S SECOND NEIGHBORHOOD IN 3-FREE DIGRAPHS *

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Abstract

In this paper, we consider Seymour's Second Neighborhood Conjecture in 3-free digraphs, and prove that for any 3-free digraph D, there exists a vertex say v, such that $d^{++}(v) \geq \lfloor \lambda d^+(v) \rfloor$, $\lambda = 0.6958\cdots$. This slightly improves the known results in 3-free digraphs with large minimum out-degree.

Keywords Seymour's second neighborhood conjecture; 3-free digraph
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1 Introduction

All digraphs considered in this paper are finite, simple and digonless. Let $D = (V, A)$ be a digraph with vertex set $V(D)$ and arc set $A(D)$. For any vertex $v \in V$, the out-neighbourhood of v is the set $N^+(v)=\{u \in V(D):(v,u) \in A(D)\}$, and the out-degree of v is $d^+(v)= |N^+(v)|$. The in-neighbourhood of v is the set $N^-(v)=\{u \in V(D):(u,v) \in A(D)\}$, and the in-degree of v is $d^-(v)= |N^-(v)|$. The set $N(v) = N^+(v) \cup N^-(v)$ is called the neighbourhood of v. We call the vertices in $N^+(v), N^-(v)$ and $N(v)$ the out-neighbours, in-neighbours and neighbours of v respectively. The minimum out-degree (minimum in-degree) of D is $\delta^+(D) = \min\{d^+(v) : v \in V(D)\}$ ($\delta^-(D) = \min\{d^-(v) : v \in V(D)\}$). For a set $S \subseteq V$, we let $N^+(S) = \bigcup\limits_{v\in S} N^+(v) - S$, $N^-(S) = \bigcup\limits_{v\in S} N^-(v) - S$. For any vertex v, let $N^{++}(v) = N^+(N^+(v))$ and $d^{++}(v) = |N^{++}(v)|$. Similarly, one can define the maximum out-degree of D, $\Delta^+(D)$, and the maximum in-degree, $\Delta^-(D)$, $N^{--}(v)$, $d^{--}(v)$.

For the purpose of this paper, all cycles considered here are direct cycles. The girth $g(D)$ of D is the minimum length of the cycles of D. A digraph D is k-free means that $g(D) \geq k + 1$ for $k \geq 2$, that is, there is no cycle whose length is less than k in D.

In 1990, Seymour [1] put forward the following conjecture:

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Conjecture 1.1(Seymour's Second Neighborhood Conjecture) *For any digraph D, there exists a vertex v such that $d^{++}(v) \geq d^{+}(v)$.*

We call the vertex v in Conjecture 1.1 a Seymour vertex. In 1996, Fisher [3] proved that Conjecture 1.1 is true if D is a tournament. In 2007, Fidler and Yuster [2] showed that any tournament minus a star or a sub-tournament, and any digraph D with minimum degree $|V(D)|-2$ has a Seymour vertex. In 2016, Cohn et al [8] proved that almost surely there are a large number of Seymour vertices in random tournaments and even more in general random digraphs. However, Conjecture 1.1 is still an open problem for general digraphs.

Another approach to Conjecture 1.1 is to determine the maximum value of λ such that there is a vertex v in D satisfying $d^{++}(v) \geq \lambda d^{+}(v)$ for any digraph D. Chen, Shen and Yuster [4] proved that $d^{++}(v) \geq \lambda d^{+}(v)$, $\lambda = 0.6572\cdots$ is the unique real root of the equation $2x^3+x^2-1=0$. In 2010, Zhang and Zhou [7] proved that for any 3-free digraph D, there exists a vertex v in D such that $d^{++}(v) \geq \lambda d^{+}(v)$, where $\lambda = 0.6751\cdots$ is the only real root in the interval $(0,1)$ of the polynomial $x^3+3x^2-x-1=0$. Liang and Xu [6] considered k-free digraphs, $k \geq 3$, and proved that $d^{++}(v) \geq \lambda_k d^{+}(v)$, where λ_k is the only real root in the interval (0,1) of the polynomial

$$g_k(x) = 2x^3-(k-3)x^2+(2k-4)x-(k-1).$$

Furthermore, λ_k is increasing with k, and $\lambda_k \to 1$ while $k \to \infty$. When k=3, $\lambda_3 = 0.6823\cdots$ is the only real root in the interval $(0,1)$ of the polynomial $x^3+x-1=0$.

In this paper, we consider Seymour's second neighborhood conjecture in 3-free digraphs, and our result slightly improves the known results in 3-free digraphs with large minimum out-degree.

Theorem 1.1 *Let D be an n order 3-free digraph, then there exists a vertex $v \in V(D)$ such that $d^{++}(v) \geq \lfloor \lambda d^{+}(v) \rfloor$, where $\lambda = 0.6958\cdots$ is the only real root in the interval $(0,1)$ of the polynomial $x^3+\frac{1}{2}x^2-\frac{(1-x)^2}{1.17}-\frac{1}{2}=0$.*

This paper is organized as follows. In Section 2, we first introduce some definitions and notations used in the paper, and give some lemmas in order to prove Theorem 1.1. In Section 3, we will prove Theorem 1.1.

2 Preparation

A digraph G is a subdigraph of a digraph D if $V(G) \subseteq V(D)$, $A(G) \subseteq A(D)$. For any subdigraph G of D, let $N_G^+(v) = N_D^+(v) \cap V(G)$ and $d_G^+(v) = |N_G^+(v)|$. For a set $W \subseteq V$, we let $D[W]$ denote the subgraph induced by W and $N_W^+(v) = N_{D[W]}^+(v)$, $d_W^+(v) = d_{D[W]}^+(v)$. Similarly, we can define $N_G^-(v)$, $d_G^-(v)$, $N_W^-(v)$, $d_W^-(v)$. For any two vertex disjoint vertex sets X and Y, denote $A(X,Y)$ as the arc set between X and Y, every arc $(x,y) \in A(X,Y)$ with $x \in X$ and $y \in Y$.

We need the following lemmas to prove Theorem 1.1.

Lemma 2.1 *The polynomial* $f(x) = x^3 + \frac{1}{2}x^2 - \frac{(1-x)^2}{1.17} - \frac{1}{2} = 0$ *is strictly increasing and has a unique real root in the interval* $(0,1)$.

Proof Since $f(x) = x^3 + \frac{1}{2}x^2 - \frac{(1-x)^2}{1.17} - \frac{1}{2} = 0$, we have

$$f'(x) = 3x^2 + x + 2\frac{(1-x)^2}{1.17}.$$

$f'(x) > 0$ when $x \in (0,1)$, which implies $f(x)$ is strictly increasing in $(0,1)$. Since $f(0) < 0$ and $f(1) > 0$, $f(x)$ has a unique real root in the interval $(0,1)$. The proof is completed.

Lemma 2.2 *If* $x < \lceil y \rceil$, *then* $x < y$, *where* x *is an integer and* y *is a real number.*

Proof (i) If y is an integer, then $x < \lceil y \rceil = y$.

(ii) If y is not an integer, denote $y = y' + c$, here y' is an integer and $0 < c < 1$. Since $x < \lceil y \rceil = \lceil y' + c \rceil = y' + 1$, so we have $x \le y' < y' + c = y$. The proof is completed.

Lemma 2.3 *If* $\lceil x \rceil > \lceil y \rceil$, *then* $x > y$, *where both* x *and* y *are real numbers.*

Proof If not, we have $x \le y$. It is easy to see that $\lceil x \rceil \le \lceil y \rceil$, a contradiction. The proof is completed.

Lemma 2.4 $x - \lceil y \rceil = \lfloor x - y \rfloor$, *where* x *is an integer and* y *is a real number.*

Proof (i) If y is an integer, then $x - \lceil y \rceil = x - y = \lfloor x - y \rfloor$.

(ii) If y is not an integer, denote $y = y' + c$, here y' is an integer and $0 < c < 1$, then $x - \lceil y \rceil = x - \lceil y' + c \rceil = x - y' - 1 = \lfloor x - y \rfloor$. The proof is completed.

Lemma 2.5 $\lfloor x \rfloor - \lfloor y \rfloor \le \lceil x - y \rceil$, *where both* x *and* y *are real numbers.*

Proof (i) If both x and y are integers, then $\lfloor x \rfloor - \lfloor y \rfloor = x - y = \lceil x - y \rceil$.

(ii) If x is an integer and y is not an integer, denote $y = y' + c$, here y' is an integer and $0 < c < 1$, then $\lfloor x \rfloor - \lfloor y \rfloor = x - y' = \lceil x - y \rceil$.

(iii) If x is not an integer and y is an integer, denote $x = x' + c$, here x' is an integer and $0 < c < 1$, then $\lfloor x \rfloor - \lfloor y \rfloor = x' - y < x' - y + 1 = \lceil x - y \rceil$.

(iv) If both x and y are not integers, denote $x = x' + c_1$, $y = y' + c_2$, here both x' and y' are integers and $0 < c_i < 1$, where $i = 1, 2$, then $\lfloor x \rfloor - \lfloor y \rfloor = x' - y' \le \lceil x - y \rceil$.

The proof is completed.

Lemma 2.6[5] *Let* D *be an* n *order digraph with* $\delta^+(D) \ge \lceil 0.3465n \rceil$, *then* D *contains a directed triangle.*

Lemma 2.7 *Let* D *be an* n *order* 3*-free digraph with* $|A(D)| = \binom{n}{2} - k$ *and* $\delta^+(D) = \lceil \alpha n \rceil$, $0.3465/2 < \alpha < 0.3465$, *then*

$$\delta^+(D) < \frac{3.6535\alpha - 0.693}{2(2\alpha - 0.3465)} + \sqrt{\frac{(0.3465\alpha)^2}{4(2\alpha - 0.3465)^2} + \frac{3.307\alpha^2 k}{2\alpha - 0.3465}}.$$

Proof For any vertex $v \in V(D)$, denote $X = N^+(v)$, $Y = N^-(v)$. So $|X| = d^+(v)$ and $|Y| = d^-(v)$. There exists a vertex say $x \in X$, such that $d_X^+(x) < \lceil 0.3465|X|\rceil$ since D is 3-free by Lemma 2.6. And we have $d_X^+(x) < 0.3465|X|$ by Lemma 2.2. Denote S as $V - X - Y - v$. Since D is 3-free and $\delta^+(D) = \lceil \alpha n\rceil$, we have

$$|S| \geq d_S^+(x) = d^+(x) - d_X^+(x) > \lceil \alpha n\rceil - 0.3465|X| = \lceil \alpha n\rceil - 0.3465 d^+(v).$$

Since X, Y and S are pairwise-disjoint sets since D is 3-free, we can acquire that

$$\begin{aligned} n &\geq |X| + |Y| + |S| + 1 \geq d^+(v) + d^-(v) + \lceil \alpha n\rceil - 0.3465 d^+(v) + 1 \\ &= 0.6535 d^+(v) + d^-(v) + \lceil \alpha n\rceil + 1. \end{aligned}$$

We sum this inequality over all vertex $v \in V(D)$, then

$$\begin{aligned} \sum_{v\in V} n &> \sum_{v\in V}(0.6535 d^+(v) + d^-(v) + \lceil \alpha n\rceil + 1) \\ &= \sum_{v\in V} 0.6535 d^+(v) + \sum_{v\in V} d^-(v) + \sum_{v\in V}\lceil \alpha n\rceil + n \\ &= 1.6535|A(D)| + n\lceil \alpha n\rceil + n, \end{aligned}$$

because $\sum\limits_{v\in V} d^+(v) = \sum\limits_{v\in V} d^-(v) = |A(D)|$.

Since $|A(D)| = \binom{n}{2} - k$ and $\alpha n \leq \lceil \alpha n\rceil < \alpha n + 1$, the following inequality holds,

$$\begin{aligned} k &> \frac{1.6535\binom{n}{2} + n\lceil \alpha n\rceil - n^2 + n}{1.6535} = \frac{-0.3465(n^2 - n) + 2n\lceil \alpha n\rceil}{3.307} \\ &\geq \frac{(2\alpha - 0.3465)n^2 + 0.3465 n}{3.307}. \end{aligned}$$

So we obtain that

$$\frac{\delta^+(D) - 1}{\alpha} < n < \frac{-0.3465}{2(2\alpha - 0.3465)} + \frac{\sqrt{0.3465^2 + 13.228(2\alpha - 0.3465)k}}{2(2\alpha - 0.3465)}.$$

Since $0.3465/2 < \alpha < 0.3465$ and $\delta^+(D) = \lceil \alpha n\rceil < \alpha n + 1$, rearranging the inequality, we obtain

$$\delta^+(D) < \frac{3.6535\alpha - 0.693}{2(2\alpha - 0.3465)} + \sqrt{\frac{(0.3465\alpha)^2}{4(2\alpha - 0.3465)^2} + \frac{3.307\alpha^2 k}{2\alpha - 0.3465}}.$$

The proof is completed.

3 Proof of Theorem 1.1

Proof of Theorem 1.1 We prove Theorem 1.1 by induction on the number of vertices. It is trivial for $n = 1, 2, 3$. Assume that Theorem 1.1 holds for all 3-free digraphs with less than n vertices. When $|V(D)| = n$, assume to the contrary that for any vertex $v \in V(D)$, $d^{++}(v) < \lfloor \lambda d^+(v) \rfloor$. We will show that the assumption leads to a contradiction.

Let $u \in V(D)$ be a vertex with minimum out-degree, that is $d^+(u) = \delta^+(D)$. Let $R = N^+(u)$, $S = N^{++}(u)$, $r = d^+(u)$ and $s = d^{++}(u)$. It is easy to check that Theorem 1.1 holds for all 3-free digraphs with minimum out-degree less than 7, and every vertex with minimum out-degree satisfies $d^{++}(v) \geq \lfloor \lambda d^+(v) \rfloor$, where $\lambda = 0.6958\cdots$ is the only real root in the interval (0, 1) of the polynomial $x^3 + \frac{1}{2}x^2 - \frac{(1-x)^2}{1.17} - \frac{1}{2} = 0$. So we can assume that $r \geq 7$. By our assumption, we have the following inequality:

$$s = d^{++}(u) < \lfloor \lambda d^+(u) \rfloor = \lfloor \lambda r \rfloor \leq \lambda r. \tag{1}$$

Since D is 3-free, $D(R)$ is 3-free. There exists at least one vertex say $x \in R$, such that $d_R^+(x) < \lceil 0.3465|R| \rceil$ by Lemma 2.6. We have that $d_R^+(x) < 0.3465|R| = 0.3465r$ by Lemma 2.2. So we obtain that

$$d_S^+(x) = d^+(x) - d_R^+(x) > r - 0.3465r = 0.6535r.$$

Thus

$$\lambda r \geq \lfloor \lambda r \rfloor > s = |S| \geq d_S^+(x) > 0.6535r,$$

which implies that

$$\lambda > 0.6535. \tag{2}$$

Suppose that $|A(D(R))| = \binom{r}{2} - m$ and let $\sigma = \frac{m}{r^2}$, then

$$|A(R,S)| = \sum_{v \in R} d^+(v) - |A(D(R))| \geq r^2 - \binom{r}{2} + m = \Big(\frac{1}{2} + \sigma\Big) r^2 + \frac{r}{2}.$$

Assume that there exists a vertex $t \in R$ such that $d_R^+(t) \leq \lceil (1-\lambda)r \rceil$, then $d_S^+(t) = d^+(t) - d_R^+(t) \geq r - \lceil (1-\lambda)r \rceil = \lfloor \lambda r \rfloor$ by Lemma 2.6, which implies that $s \geq d_S^+(y) \geq \lfloor \lambda r \rfloor$, a contradiction.

Now we suppose that for every vertex $t \in R$, $d_R^+(t) > \lceil (1-\lambda)r \rceil$. Assume that $\delta^+(D(R)) = \lceil \alpha r \rceil$, where $\alpha < 0.3465$ by Lemma 2.6, so $\delta^+(D(R)) = \lceil \alpha r \rceil > \lceil (1-\lambda)r \rceil$. By Lemma 2.3, we have $\alpha r > (1-\lambda)r$. On the other hand, $\alpha > 0.304$, otherwise $\lambda \geq 0.696$, a contradiction.

Since $D(R)$ is 3-free, $|A(D(R))| = \binom{r}{2} - m$ and $\delta^+(D(R)) = \lceil \alpha r \rceil$, where $0.304 < \alpha < 0.3465$. By Lemma 2.7, the following inequality holds,

$$\delta^+(D(R)) < \frac{3.6535\alpha - 0.693}{2(2\alpha - 0.3465)} + \sqrt{\frac{(0.3465\alpha)^2}{4(2\alpha - 0.3465)^2} + \frac{3.307\alpha^2 m}{2\alpha - 0.3465}} < 0.8 + \sqrt{0.05 + 1.17m}.$$

Thus

$$(1-\lambda)r \leq \lceil (1-\lambda)r \rceil < \delta^+(D(R)) < 0.8 + \sqrt{0.05 + 1.17m}.$$

We have that

$$\sigma > \frac{(1-\lambda)^2}{1.17} - \frac{1.6(1-\lambda)}{1.17r},$$

since $(1-\lambda)r - 0.8 > 0$.

Thus, the following inequality holds

$$|A(R,S)| > \Big(\frac{1}{2} + \frac{(1-\lambda)^2}{1.17} - \frac{1.6(1-\lambda)}{1.17r}\Big)r^2 + \frac{r}{2}. \tag{3}$$

By induction hypothesis, there exists at least one vertex say $w \in R$ such that $d_R^{++}(w) \geq \lfloor \lambda d_R^+(w) \rfloor$. Denote $X = N_R^+(w)$, $Y = N^+(w) - R = N^+(w) \cap S$ and $d = |Y|$. Since $|R - X| \geq d_R^{++}(w) \geq \lfloor \lambda d_R^+(w) \rfloor = \lfloor \lambda |X| \rfloor$, we have $|X| \leq \frac{r+1}{1+\lambda}$. Thus,

$$d = |Y| = d^+(w) - |X| \geq r - \frac{r+1}{1+\lambda} = \frac{\lambda r - 1}{1+\lambda} > 0.3r,$$

since $r \geq 7$ and $\lambda > 0.6535$.

Since $d = |Y| < |S| = s < \lfloor \lambda r \rfloor < \lambda r$, we conclude that

$$0.3r < d < \lambda r. \tag{4}$$

For every vertex $y \in Y$, use $d_{V-R-Y}^+(y)$ to denote the number of out-neighbors of y in D not in $R \cup Y$. Since $d^{++}(w) < \lfloor \lambda d^+(w) \rfloor$ and $d_R^{++}(w) \geq \lfloor \lambda d_R^+(w) \rfloor$, by Lemma 2.5, we have that

$$d_{V-R-Y}^+(y) \leq d^{++}(w) - d_R^{++}(w) < \lfloor \lambda d^+(w) \rfloor - \lfloor \lambda d_R^+(w) \rfloor \leq \lceil \lambda d^+(w) - \lambda d_R^+(w) \rceil = \lceil \lambda d \rceil.$$

By Lemma 2.2, we have $d_{V-R-Y}^+(y) < \lambda d$.

Since $d^+(y) \geq d^+(u) = r$ and $\sum\limits_{y \in Y} d_Y^+(y) \leq \binom{d}{2}$, we obtain that

$$\begin{aligned}|A(Y,R)| &= \sum_{y \in Y} d_R^+(y) \geq \sum (r - d_{V-R-Y}^+(y) - d_Y^+(y)) \\ &> (r - \lambda d)d - \frac{d(d-1)}{2} > \Big(r - \lambda d - \frac{d}{2}\Big)d.\end{aligned}$$

That is,

$$|A(Y,R)| > \Big(r - \lambda d - \frac{d}{2}\Big)d. \tag{5}$$

Combining (1), (3) and (5), $\frac{1.6(1-\lambda)}{1.17}r < \frac{r}{2}$ since $\lambda > 0.6535$, therefore,

$$\begin{aligned}\lambda r^2 &\geq \lfloor \lambda r \rfloor r > rs \geq |A(R,S)| + |A(S,R)| \geq |A(R,S)| + |A(Y,R)| \\ &> \Big(\frac{1}{2} + \frac{(1-\lambda)^2}{1.17} - \frac{1.6(1-\lambda)}{1.17r}\Big)r^2 + \frac{r}{2} + \Big(r - \lambda d - \frac{d}{2}\Big)d \\ &> -\Big(\lambda + \frac{1}{2}\Big)d^2 + rd + \Big(\frac{1}{2} + \frac{(1-\lambda)^2}{1.17}\Big)r^2.\end{aligned}$$

That is,

$$\lambda r^2 > -\Big(\lambda + \frac{1}{2}\Big)d^2 + rd + \Big(\frac{1}{2} + \frac{(1-\lambda)^2}{1.17}\Big)r^2, \tag{6}$$

where $0.3r < d < \lambda r$.

Let

$$f(z) = -\Big(\lambda + \frac{1}{2}\Big)z^2 + rz + \Big(\frac{1}{2} + \frac{(1-\lambda)^2}{1.17}\Big)r^2,$$

where $0.3r < z < \lambda r$. Since $f(z)$ is a quadratic function with a negative leading coefficient, the following inequality holds,

$$f(z) \geq \min\{f(0.3r), f(\lambda r)\}, \tag{7}$$

for all $z \in (0.3r, \lambda r)$.

Combining (6) with (7), we have

$$\lambda r^2 > f(d) \geq \min\{f(0.3r), f(\lambda r)\}. \tag{8}$$

A simple calculation shows that if $\lambda r^2 > f(0.3r)$, then $\lambda > 0.744$, a contradiction. Similarly, if $\lambda r^2 > f(\lambda r)$, by simplifying there is

$$\lambda r^2 > -\Big(\lambda + \frac{1}{2}\Big)\lambda^2 r^2 + \lambda r^2 + \Big(\frac{1}{2} + \frac{(1-\lambda)^2}{1.17}\Big)r^2.$$

We obtain that $\lambda^3 + \frac{1}{2}\lambda^2 - \frac{(1-\lambda)^2}{1.17} - \frac{1}{2} > 0$, which contradicts that λ is the unique real root of the equation $x^3 + \frac{1}{2}x^2 - \frac{(1-x)^2}{1.17} - \frac{1}{2} > 0$. The proof is completed.

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