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Additive Maps Preserving the Star Partial Order on $\mathcal{B}(\mathcal{H})$

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Abstract: Let $\mathcal{B}(\mathcal{H})$ be the C^*-algebra of all bounded linear operators on a complex Hilbert space $\mathcal{H}$. It is proved that an additive surjective map φ on $\mathcal{B}(\mathcal{H})$ preserving the star partial order in both directions if and only if one of the following assertions holds. (1) There exist a nonzero complex number α and two unitary operators $\boldsymbol{U}$ and $\boldsymbol{V}$ on $\mathcal{H}$ such that $\varphi(\boldsymbol{X}) = \alpha\boldsymbol{UXV}$ or $\varphi(\boldsymbol{X}) = \alpha\boldsymbol{UX}^*\boldsymbol{V}$ for all $\boldsymbol{X} \in \mathcal{B}(\mathcal{H})$. (2) There exist a nonzero α and two anti-unitary operators $\boldsymbol{U}$ and $\boldsymbol{V}$ on $\mathcal{H}$ such that $\varphi(\boldsymbol{X}) = \alpha\boldsymbol{UXV}$ or $\varphi(\boldsymbol{X}) = \alpha\boldsymbol{UX}^*\boldsymbol{V}$ for all $\boldsymbol{X} \in \mathcal{B}(\mathcal{H})$.

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1 Introduction

In the last few decades, many researchers have studied properties of various partial orders on matrix algebras, or operator algebras acting on a complex infinite dimensional Hilbert space, such as minus partial order, star partial order, left and right star partial order and so on (see [1–6]). One of the orders on the algebra M_n of all $n \times n$ complex matrices is the star partial order "$\overset{*}{\leq}$" defined by Drazin in [5]. Let $\boldsymbol{A}, \boldsymbol{B} \in M_n$. Then we say that $\boldsymbol{A} \overset{*}{\leq} \boldsymbol{B}$ if $\boldsymbol{A}^*\boldsymbol{A} = \boldsymbol{A}^*\boldsymbol{B}$ and $\boldsymbol{A}\boldsymbol{A}^* = \boldsymbol{B}\boldsymbol{A}^*$. We note that this definition can be extended to a C^*-algebra by the same way. In particular, it can be extended to the C^*-algebra $\mathcal{B}(\mathcal{H})$ of all bounded linear operators on a complex Hilbert space $\mathcal{H}$. For example, motivated by Šemrl's approach presented in [7] for minus partial order, Dolinar and Marovt[4] gave an equivalent

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definition (see Definition 2 in [4]) of the star partial order and considered some properties of this partial order. We can refer [1, 4] to see more interesting properties.

On the other hand, as partially ordered algebraic structures on M_n and $\mathcal{B}(\mathcal{H})$, what are the automorphisms of M_n and $\mathcal{B}(\mathcal{H})$ with respect to those partial orders? These topics have been studied and some interesting results have been obtained. Šemrl[7] described the structure of corresponding automorphisms for the minus partial order. For the star partial order, Guterman[8] characterized linear bijective maps on M_n preserving the star partial order and Legiša[9] considered automorphisms of M_n with respect to the star partial order. Recently, several authors consider the automorphisms of certain subspaces of $\mathcal{B}(\mathcal{H})$ with respect to the star partial order when $\mathcal{H}$ is infinite dimensional. Dolinar and Guterman[10] studied the automorphisms of the algebra $\mathcal{K}(\mathcal{H})$ of compact operators on a separable complex Hilbert space $\mathcal{H}$ and they characterized the bijective, additive, continuous maps on $\mathcal{K}(\mathcal{H})$ which preserve the star partial order in both directions. On the other hand, characterizations of certain continuous bijections on the normal elements of a von Neumann algebra preserving the star partial order in both directions are obtained by Bohata and Hamhalter[11]. In this paper, we consider additive surjective maps preserving the star partial order in both directions on $\mathcal{B}(\mathcal{H})$ and characterizations of those maps are given. In particular, we improve the main result in [10].

Let $\mathcal{H}$ be a complex Hilbert space and denote by $\dim\mathcal{H}$ the dimension of $\mathcal{H}$. Let $\mathbb{C}$ and $\mathbb{Q}$ denote the complex field and the rational number field, respectively. Let $\mathcal{B}(\mathcal{H})$, $\mathcal{K}(\mathcal{H})$ and $\mathcal{F}(\mathcal{H})$ be the algebras of all bounded linear operators, the compact operators and the finite rank operators on $\mathcal{H}$, respectively. For every pair of vectors $\boldsymbol{x}, \boldsymbol{y} \in \mathcal{H}$, $\langle \boldsymbol{x}, \boldsymbol{y}\rangle$ denotes the inner product of $\boldsymbol{x}$ and $\boldsymbol{y}$, and $\boldsymbol{x}\otimes \boldsymbol{y}$ stands for the rank-1 linear operator on $\mathcal{H}$ defined by $(\boldsymbol{x}\otimes \boldsymbol{y})\boldsymbol{z} = \langle \boldsymbol{z}, \boldsymbol{y}\rangle \boldsymbol{x}$ for any $\boldsymbol{z} \in \mathcal{H}$. If $\boldsymbol{x}$ is a unit vector, then $\boldsymbol{x}\otimes \boldsymbol{x}$ is a rank-1 projection. $\sigma(\boldsymbol{A})$ is the spectrum of $\boldsymbol{A}$ for any $\boldsymbol{A} \in \mathcal{B}(\mathcal{H})$. For a subset S of $\mathcal{H}$, $[S]$ denotes the closed subspace of $\mathcal{H}$ spanned by S and $\boldsymbol{P}_M$ denotes the orthogonal projection on M for a closed subspace M of $\mathcal{H}$. We denote by $R(\boldsymbol{T})$ and $N(\boldsymbol{T})$ the range and the kernel of a linear map $\boldsymbol{T}$ between two linear spaces. Throughout this paper, we generally denote by $\boldsymbol{I}$ the identity operator on a Hilbert space.

2 Additive Maps Preserving the Star Partial Order

Let φ be an additive map on $\mathcal{B}(\mathcal{H})$. We say that φ preserves the star partial order if $\varphi(\boldsymbol{A}) \overset{*}{\leq} \varphi(\boldsymbol{B})$ for any $\boldsymbol{A}, \boldsymbol{B} \in \mathcal{B}(\mathcal{H})$ such that $\boldsymbol{A} \overset{*}{\leq} \boldsymbol{B}$. We say that φ preserves the star partial order in both directions if $\varphi(\boldsymbol{A}) \overset{*}{\leq} \varphi(\boldsymbol{B})$ if and only if $\boldsymbol{A} \overset{*}{\leq} \boldsymbol{B}$ for any $\boldsymbol{A}, \boldsymbol{B} \in \mathcal{B}(\mathcal{H})$. We firstly give the following lemma which generalizes Lemma 10 in [10].

Let $\boldsymbol{T} \in \mathcal{B}(\mathcal{H})$. We denote by

$$H_1 = \overline{R(\boldsymbol{T}^*)}, \qquad H_2 = N(\boldsymbol{T}), \qquad K_1 = \overline{R(\boldsymbol{T})}, \qquad K_2 = N(\boldsymbol{T}^*),$$

respectively. Then

$$\mathcal{H} = H_1 \oplus H_2 = K_1 \oplus K_2, \tag{2.1}$$

and

$$\boldsymbol{T} = \begin{pmatrix} \boldsymbol{T}_0 & 0 \\ 0 & 0 \end{pmatrix} \tag{2.2}$$

with respect to the orthogonal decompositions (2.1), where $\boldsymbol{T}_0 \in \mathcal{B}(H_1, K_1)$ is an injective operator with dense range.

Lemma 2.1 *Let $\boldsymbol{T} \in \mathcal{B}(\mathcal{H})$ be a nonzero operator. Then $\boldsymbol{T}$ is of rank-1 if and only if for any operator $\boldsymbol{S}$ with $\boldsymbol{S} \overset{*}{\le} \boldsymbol{T}$, we have $\boldsymbol{S} = \boldsymbol{0}$ or $\boldsymbol{S} = \boldsymbol{T}$.*

Proof. The necessity is clear. Conversely, suppose rank$\boldsymbol{T} \ge 2$. Let $\boldsymbol{T}$ have the matrix form (2.2). Let $\boldsymbol{T}_0 = \boldsymbol{U}\boldsymbol{A}$ be the polar decomposition of $\boldsymbol{T}_0$. Then $\boldsymbol{A} \in \mathcal{B}(H_1)$ is an injective positive operator and $\boldsymbol{U} \in \mathcal{B}(H_1, K_1)$ is a unitary operator. Let $\boldsymbol{A} = \displaystyle\int_{\sigma(\boldsymbol{A})} \lambda \mathrm{d}E_\lambda$ be the spectral decomposition of $\boldsymbol{A}$. If $\sigma(\boldsymbol{A}) = \{\lambda\}$ for some positive constant λ, then $\boldsymbol{T} = \lambda \boldsymbol{W}$, where $\boldsymbol{W}$ is a partial isometry with rank at least 2. It is easy to know that there is a rank-1 operator $\boldsymbol{T}_1$ such that $\boldsymbol{T}_1 \overset{*}{\le} \boldsymbol{T}$. This is a contradiction. Now we assume that $\sigma(\boldsymbol{A})$ is not a singleton. Let $\varDelta \subseteq \sigma(\boldsymbol{A})$ be a Borel subset such that both $\boldsymbol{E}(\varDelta)$ and $(\boldsymbol{I} - \boldsymbol{E}(\varDelta))$ are nonzero and $H_1 = \boldsymbol{E}(\varDelta)H_1 \oplus (\boldsymbol{I} - \boldsymbol{E}(\varDelta))H_1$. Then

$$\mathcal{H} = \boldsymbol{E}(\varDelta)H_1 \oplus (\boldsymbol{I} - \boldsymbol{E}(\varDelta))H_1 \oplus H_2 = \boldsymbol{U}\boldsymbol{E}(\varDelta)H_1 \oplus \boldsymbol{U}(\boldsymbol{I} - \boldsymbol{E}(\varDelta))H_1 \oplus K_2. \tag{2.3}$$

Put $\boldsymbol{U}_1 = \boldsymbol{U}|_{\boldsymbol{E}(\varDelta)H_1}$, $\boldsymbol{A}_1 = \boldsymbol{E}(\varDelta)\boldsymbol{A}$, $\boldsymbol{U}_2 = \boldsymbol{U}|_{(\boldsymbol{I}-\boldsymbol{E}(\varDelta))H_1}$ and $\boldsymbol{A}_2 = (\boldsymbol{I} - \boldsymbol{E}(\varDelta))\boldsymbol{A}$ on H_1, respectively. Then

$$\boldsymbol{T} = \begin{pmatrix} \boldsymbol{U}_1\boldsymbol{A}_1 & 0 & 0 \\ 0 & \boldsymbol{U}_2\boldsymbol{A}_2 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

according to (2.3). Let

$$\boldsymbol{T}_\varDelta = \begin{pmatrix} \boldsymbol{U}_1\boldsymbol{A}_1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

according to (2.3) again. It follows that $\boldsymbol{T}_\varDelta \overset{*}{\le} \boldsymbol{T}$ by Lemma 3 in [4]. Note that $\boldsymbol{T}_\varDelta \ne \boldsymbol{T}$ is a nonzero operator. This is a contradiction too. Thus $\boldsymbol{T}$ is of rank-1. The proof is completed.

Our main result is as follows.

Theorem 2.1 *Let φ be an additive surjective map on $\mathcal{B}(\mathcal{H})$. Then φ preserves the star partial order in both directions if and only if one of the following assertions hold*:

(1) *There exist a nonzero $\alpha \in \mathbb{C}$ and two unitary operators $\boldsymbol{U}$ and $\boldsymbol{V}$ on $\mathcal{H}$ such that $\varphi(\boldsymbol{X}) = \alpha\boldsymbol{U}\boldsymbol{X}\boldsymbol{V}$ or $\varphi(\boldsymbol{X}) = \alpha\boldsymbol{U}\boldsymbol{X}^*\boldsymbol{V}$ for all $\boldsymbol{X} \in \mathcal{B}(\mathcal{H})$;*

(2) *There exist a nonzero $\alpha \in \mathbb{C}$ and two anti-unitary operators $\boldsymbol{U}$ and $\boldsymbol{V}$ on $\mathcal{H}$ such that $\varphi(\boldsymbol{X}) = \alpha\boldsymbol{U}\boldsymbol{X}\boldsymbol{V}$ or $\varphi(\boldsymbol{X}) = \alpha\boldsymbol{U}\boldsymbol{X}^*\boldsymbol{V}$ for all $\boldsymbol{X} \in \mathcal{B}(\mathcal{H})$.*

Proof. The sufficiency is clear. We only need prove the necessity. It is clear that φ is injective. Then φ^{-1} preserves the star partial order too. We complete the proof by several steps.

Step 1. φ preserves rank-n operators in both directions.

Let $\boldsymbol{A}$ be a rank-1 operator and $\varphi(\boldsymbol{A})=\boldsymbol{B}$. Suppose that rank$\boldsymbol{B}\geq 2$. Then there is a nonzero $\boldsymbol{B}_1\in\mathcal{B}(\mathcal{H})$ such that $\boldsymbol{B}_1\overset{*}{\leq}\boldsymbol{B}$ and $\boldsymbol{B}_1\neq\boldsymbol{B}$ by Lemma 2.1. Put $\boldsymbol{A}_1=\varphi^{-1}(\boldsymbol{B}_1)$. Then $\boldsymbol{A}_1\overset{*}{\leq}\boldsymbol{A}$ and $\boldsymbol{A}_1\neq\boldsymbol{A}$ is a nonzero operator. This is a contradiction by Lemma 2.1. Thus $\boldsymbol{B}$ is of rank-1. It follows that φ preserves rank-1 operators in both directions. Since a rank-n operator is the sum of n rank-1 operators, it is elementary that φ preserves rank-n operators in both directions.

Step 2. Let $\boldsymbol{f},\boldsymbol{g}\in\mathcal{H}$ and $\varphi(\boldsymbol{f}\otimes\boldsymbol{g})=\boldsymbol{u}\otimes\boldsymbol{v}$. Then

$$\{\varphi(\boldsymbol{x}\otimes\boldsymbol{y}):\boldsymbol{x}\in\{\boldsymbol{f}\}^{\perp},\ \boldsymbol{y}\in\{\boldsymbol{g}\}^{\perp}\}=\{\boldsymbol{\xi}\otimes\boldsymbol{\eta}:\boldsymbol{\xi}\in\{\boldsymbol{u}\}^{\perp},\ \boldsymbol{\eta}\in\{\boldsymbol{v}\}^{\perp}\}.$$

In fact, for any $\boldsymbol{x}\in\{\boldsymbol{f}\}^{\perp}$, $\boldsymbol{y}\in\{\boldsymbol{g}\}^{\perp}$, we have

$$\boldsymbol{f}\otimes\boldsymbol{g}\overset{*}{\leq}\boldsymbol{f}\otimes\boldsymbol{g}+r\boldsymbol{x}\otimes\boldsymbol{y},\qquad r\in\mathbb{Q}.$$

Let $\varphi(\boldsymbol{x}\otimes\boldsymbol{y})=\boldsymbol{\xi}\otimes\boldsymbol{\eta}$. Then

$$\boldsymbol{u}\otimes\boldsymbol{v}\overset{*}{\leq}\boldsymbol{u}\otimes\boldsymbol{v}+r\boldsymbol{\xi}\otimes\boldsymbol{\eta},\qquad r\in\mathbb{Q},$$

which implies that

$$\boldsymbol{\xi}\in\{\boldsymbol{u}\}^{\perp},\qquad\boldsymbol{\eta}\in\{\boldsymbol{v}\}^{\perp}.$$

The converse is the same since φ preserves the star partial order in both directions.

Step 3. For any unit vectors $\boldsymbol{f},\boldsymbol{g}\in\mathcal{H}$, $\|\varphi(\boldsymbol{f}\otimes\boldsymbol{f})\|=\|\varphi(\boldsymbol{g}\otimes\boldsymbol{g})\|$. Moreover, if $\boldsymbol{f}\perp\boldsymbol{g}$, then $\|\varphi(\boldsymbol{f}\otimes\boldsymbol{f})\|=\|\varphi(\boldsymbol{f}\otimes\boldsymbol{g})\|$.

Let $\boldsymbol{f}\perp\boldsymbol{g}$, $\varphi(\boldsymbol{f}\otimes\boldsymbol{f})=\boldsymbol{\xi}_1\otimes\boldsymbol{\eta}_1$ and $\varphi(\boldsymbol{g}\otimes\boldsymbol{g})=\boldsymbol{\xi}_2\otimes\boldsymbol{\eta}_2$. By Step 2, $\boldsymbol{\xi}_1\perp\boldsymbol{\xi}_2$ and $\boldsymbol{\eta}_1\perp\boldsymbol{\eta}_2$. Without loss of generality, we may assume that

$$\|\varphi(\boldsymbol{f}\otimes\boldsymbol{f})\|=\|\boldsymbol{\xi}_1\|=\|\boldsymbol{\eta}_1\|=1.$$

Put $\boldsymbol{U}$ and $\boldsymbol{V}$ be two unitary operators on $\mathcal{H}$ such that

$$\boldsymbol{U}\boldsymbol{\xi}_1=\boldsymbol{f},\qquad\boldsymbol{U}\frac{1}{\|\boldsymbol{\xi}_2\|}\boldsymbol{\xi}_2=\boldsymbol{g},\qquad\boldsymbol{U}\{\boldsymbol{\xi}_1,\,\boldsymbol{\xi}_2\}^{\perp}=\{\boldsymbol{f},\,\boldsymbol{g}\}^{\perp},$$

$$\boldsymbol{V}\boldsymbol{\eta}_1=\boldsymbol{f},\qquad\boldsymbol{V}\frac{1}{\|\boldsymbol{\eta}_2\|}\boldsymbol{\eta}_2=\boldsymbol{g},\qquad\boldsymbol{V}\{\boldsymbol{\eta}_1,\,\boldsymbol{\eta}_2\}^{\perp}=\{\boldsymbol{f},\,\boldsymbol{g}\}^{\perp}.$$

Let $\psi=\boldsymbol{U}\varphi\boldsymbol{V}^*$. Then ψ preserves the star partial order in both directions such that

$$\psi(\boldsymbol{f}\otimes\boldsymbol{f})=\boldsymbol{f}\otimes\boldsymbol{f}$$

and

$$\psi(\boldsymbol{g}\otimes\boldsymbol{g})=\|\boldsymbol{\xi}_2\|\|\boldsymbol{\eta}_2\|\boldsymbol{g}\otimes\boldsymbol{g}=\beta_{22}\boldsymbol{g}\otimes\boldsymbol{g}.$$

Then ψ preserves rank-1 operators in both directions. Let $\psi(\boldsymbol{f}\otimes\boldsymbol{g})=\boldsymbol{\xi}_3\otimes\boldsymbol{\eta}_3$. Note that both $\boldsymbol{f}\otimes\boldsymbol{f}+\boldsymbol{f}\otimes\boldsymbol{g}$ and $\boldsymbol{f}\otimes\boldsymbol{g}+\boldsymbol{g}\otimes\boldsymbol{g}$ are of rank-1. Then either $\boldsymbol{f}$ (resp. $\boldsymbol{g}$) and $\boldsymbol{\xi}_3$ or $\boldsymbol{f}$ (resp. $\boldsymbol{g}$) and $\boldsymbol{\eta}_3$ are linearly dependent. We assume that

$$\boldsymbol{\xi}_3\otimes\boldsymbol{\eta}_3=\beta_{12}\boldsymbol{f}\otimes\boldsymbol{g}.$$

We thus have

$$\psi(\boldsymbol{g}\otimes\boldsymbol{f})=\beta_{21}\boldsymbol{g}\otimes\boldsymbol{f}.$$

Put

$$\boldsymbol{E}(r)=\frac{1}{1+r^2}(\boldsymbol{f}\otimes\boldsymbol{f}+r\boldsymbol{f}\otimes\boldsymbol{g}+r\boldsymbol{g}\otimes\boldsymbol{f}+r^2\boldsymbol{g}\otimes\boldsymbol{g}),\qquad r\in\mathbb{Q}.$$

Then $\boldsymbol{E}(r)$ is a projection and

$$\boldsymbol{E}(r) \leq \boldsymbol{f} \otimes \boldsymbol{f} + \boldsymbol{g} \otimes \boldsymbol{g}.$$

Of course,

$$\boldsymbol{E}(r) \overset{*}{\leq} \boldsymbol{f} \otimes \boldsymbol{f} + \boldsymbol{g} \otimes \boldsymbol{g}.$$

It follows that

$$\frac{1}{1+r^2}(\boldsymbol{f} \otimes \boldsymbol{f} + r\beta_{12}\boldsymbol{f} \otimes \boldsymbol{g} + r\beta_{21}\boldsymbol{g} \otimes \boldsymbol{f} + r^2\beta_{22}\boldsymbol{g} \otimes \boldsymbol{g}) \overset{*}{\leq} \boldsymbol{f} \otimes \boldsymbol{f} + \beta_{22}\boldsymbol{g} \otimes \boldsymbol{g}.$$

Then

$$|\beta_{12}| = |\beta_{21}| = \beta_{22} = 1.$$

Thus

$$\|\varphi(\boldsymbol{f} \otimes \boldsymbol{f})\| = \|\psi(\boldsymbol{f} \otimes \boldsymbol{f})\| = \|\psi(\boldsymbol{g} \otimes \boldsymbol{g})\| = \|\varphi(\boldsymbol{g} \otimes \boldsymbol{g})\| = \|\psi(\boldsymbol{f} \otimes \boldsymbol{g})\| = \|\varphi(\boldsymbol{f} \otimes \boldsymbol{g})\| = 1.$$

If $\dim \mathcal{H} = 2$, then for any unit vector $\boldsymbol{x} \in \mathcal{H}$ we have

$$\boldsymbol{x} \otimes \boldsymbol{x} \overset{*}{\leq} \boldsymbol{f} \otimes \boldsymbol{f} + \boldsymbol{g} \otimes \boldsymbol{g} = \boldsymbol{I}.$$

Thus we have $\psi(\boldsymbol{x} \otimes \boldsymbol{x})$ is a projection and

$$\|\varphi(\boldsymbol{x} \otimes \boldsymbol{x})\| = \|\psi(\boldsymbol{x} \otimes \boldsymbol{x})\| = 1 = \|\varphi(\boldsymbol{f} \otimes \boldsymbol{f})\|.$$

Assume that $\dim \mathcal{H} > 2$. For any unit vectors $\boldsymbol{f}$ and $\boldsymbol{g}$, take any unit vector $\boldsymbol{h} \in \{\boldsymbol{f}, \boldsymbol{g}\}^{\perp}$. Then

$$\|\varphi(\boldsymbol{f} \otimes \boldsymbol{f})\| = \|\varphi(\boldsymbol{h} \otimes \boldsymbol{h})\| = \|\varphi(\boldsymbol{g} \otimes \boldsymbol{g})\|.$$

We next assume that $\|\varphi(\boldsymbol{f} \otimes \boldsymbol{f})\| = 1$ for any unit vector $\boldsymbol{f} \in \mathcal{H}$ without loss of generality.

Step 4. Let $\{\boldsymbol{e}_\lambda : \lambda \in \Lambda\}$ be an orthonormal basis of $\mathcal{H}$. Then there are two orthonormal bases $\{\boldsymbol{f}_\lambda : \lambda \in \Lambda\}$ and $\{\boldsymbol{g}_\lambda : \lambda \in \Lambda\}$ such that

$$\varphi(\boldsymbol{e}_\lambda \otimes \boldsymbol{e}_\lambda) = \boldsymbol{f}_\lambda \otimes \boldsymbol{g}_\lambda, \qquad \lambda \in \Lambda. \tag{2.4}$$

If (2.4) holds, then both $\{\boldsymbol{f}_\lambda : \lambda \in \Lambda\}$ and $\{\boldsymbol{g}_\lambda : \lambda \in \Lambda\}$ are orthonormal families of $\mathcal{H}$. If there is a unit vector $\boldsymbol{f} \in \mathcal{H}$ such that $\boldsymbol{f} \perp \boldsymbol{f}_\lambda$ for all $\lambda \in \Lambda$, then $\varphi^{-1}(\boldsymbol{f} \otimes \boldsymbol{g}_{\lambda_0}) = \boldsymbol{x}_0 \otimes \boldsymbol{y}_0$ is a rank-1 operator. By Step 2, $\boldsymbol{e}_\lambda \in \{\boldsymbol{x}_0\}^{\perp}$. This is a contradiction. Thus both $\{\boldsymbol{f}_\lambda : \lambda \in \Lambda\}$ and $\{\boldsymbol{g}_\lambda : \lambda \in \Lambda\}$ are bases of $\mathcal{H}$.

Step 5. φ is linear or conjugate linear on $\mathcal{F}(\mathcal{H})$.

As in Step 4, let $\{\boldsymbol{e}_\lambda : \lambda \in \Lambda\}$ be an orthonormal basis of $\mathcal{H}$. Let $\boldsymbol{U}$ and $\boldsymbol{V}$ be two unitary operators on $\mathcal{H}$ such that $\boldsymbol{U}_1\boldsymbol{f}_\lambda = \boldsymbol{e}_\lambda$ and $\boldsymbol{V}_1\boldsymbol{g}_\lambda = \boldsymbol{e}_\lambda$ for any $\lambda \in \Lambda$. Put

$$\varphi_1(\boldsymbol{X}) = \boldsymbol{U}\varphi(\boldsymbol{X})\boldsymbol{V}^*, \qquad \boldsymbol{X} \in \mathcal{B}(\mathcal{H}).$$

Then φ_1 preserves the star partial order in both directions such that

$$\varphi_1(\boldsymbol{e}_\lambda \otimes \boldsymbol{e}_\lambda) = \boldsymbol{e}_\lambda \otimes \boldsymbol{e}_\lambda, \qquad \lambda \in \Lambda.$$

For any $n \in \mathbf{N}_+$ and $\{\boldsymbol{e}_{\lambda_i} : 1 \leq i \leq n\} \subseteq \{\boldsymbol{e}_\lambda : \lambda \in \Lambda\}$, denote

$$\boldsymbol{P}_n = \sum_{i=1}^{n} \boldsymbol{e}_{\lambda_i} \otimes \boldsymbol{e}_{\lambda_i}.$$

We conclude that

$$\varphi_1(\boldsymbol{P}_n\mathcal{B}(\mathcal{H})\boldsymbol{P}_n) = \boldsymbol{P}_n\mathcal{B}(\mathcal{H})\boldsymbol{P}_n$$

by the similar way as Step 4 of [10]. In fact, it easily follows that $\varphi_1(\boldsymbol{Q}) = \boldsymbol{Q}$, where $\boldsymbol{Q}$ is the projection onto $\{\boldsymbol{e}_\lambda : \lambda \in S\}$ for any subset $S \subseteq \Lambda$. For any $\boldsymbol{A} \in \boldsymbol{P}_n \mathcal{B}(\mathcal{H}) \boldsymbol{P}_n$, we know that

$$\boldsymbol{A} \overset{*}{\leq} \boldsymbol{A} + r(\boldsymbol{I} - \boldsymbol{P}_n), \qquad r \in \mathbb{Q}.$$

Then

$$\varphi_1(\boldsymbol{A}) \overset{*}{\leq} \varphi_1(\boldsymbol{A}) + r(\boldsymbol{I} - \boldsymbol{P}_n), \qquad r \in \mathbb{Q}.$$

It follows that $\varphi_1(\boldsymbol{A}) \in \boldsymbol{P}_n \mathcal{B}(\mathcal{H}) \boldsymbol{P}_n$ by a simple calculation. $\boldsymbol{P}_n \mathcal{B}(\mathcal{H}) \boldsymbol{P}_n$ can be identified with M_n. So $\varphi_1|_{\boldsymbol{P}_n \mathcal{B}(\mathcal{H}) \boldsymbol{P}_n}$ can be considered as a bijective, additive map on M_n, which preserves the star partial order in both directions. It follows from Theorem 3.1 in [12] that $\varphi_1|_{\boldsymbol{P}_n \mathcal{B}(\mathcal{H}) \boldsymbol{P}_n}$ is linear or conjugate linear. We note that if $\varphi_1|_{\boldsymbol{P}_k \mathcal{B}(\mathcal{H}) \boldsymbol{P}_k}$ is linear (resp. conjugate linear) for some $k \geq 2$, then $\varphi_1|_{\boldsymbol{P}_n \mathcal{B}(\mathcal{H}) \boldsymbol{P}_n}$ is also linear (resp. conjugate linear) for any n. This implies that if $\varphi|_{\boldsymbol{P}_k \mathcal{B}(\mathcal{H}) \boldsymbol{P}_k}$ is linear (resp. conjugate linear) for some $k \geq 2$, then $\varphi|_{\boldsymbol{P}_n \mathcal{B}(\mathcal{H}) \boldsymbol{P}_n}$ is linear (resp. conjugate linear) for any n. We now assume that $\varphi|_{\boldsymbol{P}_k \mathcal{B}(\mathcal{H}) \boldsymbol{P}_k}$ is linear for some $k \geq 2$. Let $\boldsymbol{A}, \boldsymbol{B} \in \mathcal{F}(\mathcal{H})$. Let M be the subspace generated by

$$\{\boldsymbol{e}_{\lambda_i} : 1 \leq i \leq k\} \cup R(\boldsymbol{A}) \cup R(\boldsymbol{A}^*) \cup R(\boldsymbol{B}) \cup R(\boldsymbol{B}^*).$$

Then M is finite dimensional with an orthonormal basis $\{\boldsymbol{h}_j : 1 \leq j \leq m\}$ containing $\{\boldsymbol{e}_{\lambda_i} : 1 \leq i \leq k\}$. It now follows that $\varphi|_{\boldsymbol{P}_M \mathcal{B}(\mathcal{H}) \boldsymbol{P}_M}$ is linear by preceding proof since $\boldsymbol{P}_k \leq \boldsymbol{P}_M$.

Note that $\boldsymbol{A} = \boldsymbol{P}_M \boldsymbol{A} \boldsymbol{P}_M \in \boldsymbol{P}_M \mathcal{B}(\mathcal{H}) \boldsymbol{P}_M$ and $\boldsymbol{B} = \boldsymbol{P}_M \boldsymbol{B} \boldsymbol{P}_M \in \boldsymbol{P}_M \mathcal{B}(\mathcal{H}) \boldsymbol{P}_M$. Then

$$\varphi(\alpha \boldsymbol{A} + \beta \boldsymbol{B}) = \alpha \varphi(\boldsymbol{A}) + \beta \varphi(\boldsymbol{B}), \qquad \alpha, \beta \in \mathbb{C}.$$

Thus φ is linear on $\mathcal{F}(\mathcal{H})$.

If $\varphi|_{\boldsymbol{P}_k \mathcal{B}(\mathcal{H}) \boldsymbol{P}_k}$ is conjugate linear for some $k \geq 2$, then φ is conjugate linear on $\mathcal{F}(\mathcal{H})$.

We now next assume that φ is linear on $\mathcal{F}(\mathcal{H})$. Then φ is a rank preserving linear bijection on $\mathcal{F}(\mathcal{H})$. It follows from Theorem 2.1.6 in [13] that the following statements hold.

(1) There exist two linear maps $\boldsymbol{A}$ and $\boldsymbol{C}$ on $\mathcal{H}$ such that for all $\boldsymbol{x}, \boldsymbol{y} \in \mathcal{H}$,

$$\varphi(\boldsymbol{x} \otimes \boldsymbol{y}) = \boldsymbol{A}\boldsymbol{x} \otimes \boldsymbol{C}\boldsymbol{y};$$

(2) There exist two conjugate linear maps $\boldsymbol{A}$ and $\boldsymbol{C}$ on $\mathcal{H}$, such that for all $\boldsymbol{x}, \boldsymbol{y} \in \mathcal{H}$,

$$\varphi(\boldsymbol{x} \otimes \boldsymbol{y}) = \boldsymbol{A}\boldsymbol{y} \otimes \boldsymbol{C}\boldsymbol{x}.$$

Note that both $\boldsymbol{A}$ and $\boldsymbol{C}$ are invertible since φ is bijective on $\mathcal{F}(\mathcal{H})$. Assume that (1) holds. Then for any unit vectors $\boldsymbol{e}, \boldsymbol{f} \in \mathcal{H}$ such that $\langle \boldsymbol{e}, \boldsymbol{f} \rangle = 0$, we have that

$$\langle \boldsymbol{A}\boldsymbol{e}, \boldsymbol{A}\boldsymbol{f} \rangle = 0$$

by Step 2. Note that $(\boldsymbol{e} + \boldsymbol{f}) \perp (\boldsymbol{e} - \boldsymbol{f})$. It follows that

$$\|\boldsymbol{A}\boldsymbol{e}\| = \|\boldsymbol{A}\boldsymbol{f}\|.$$

If $\dim \mathcal{H} = 2$, then for any unit vector $\boldsymbol{x} \in \mathcal{H}$, we have $\boldsymbol{x} = \alpha \boldsymbol{e} + \beta \boldsymbol{f}$ for some constants $\alpha, \beta \in \mathbb{C}$ with $|\alpha|^2 + |\beta|^2 = 1$. We easily have that

$$\|\boldsymbol{A}\boldsymbol{x}\| = \|\boldsymbol{A}\boldsymbol{e}\| = \|\boldsymbol{A}\boldsymbol{f}\|$$

by an elementary calculus. If $\dim \mathcal{H} > 2$, then for any unit vectors $\boldsymbol{x}, \boldsymbol{y} \in \mathcal{H}$, there is a unit vector $\boldsymbol{z} \in \{\boldsymbol{x}, \boldsymbol{y}\}^\perp$. It now follows that

$$\|\boldsymbol{A}\boldsymbol{x}\| = \|\boldsymbol{A}\boldsymbol{z}\| = \|\boldsymbol{A}\boldsymbol{y}\|.$$

Thus $\boldsymbol{U} = \dfrac{\boldsymbol{A}}{\|\boldsymbol{A}\|}$ is a unitary operator. We similarly have that $\boldsymbol{V} = \dfrac{\boldsymbol{C}^*}{\|\boldsymbol{C}\|}$ is also a unitary operator. Put $\alpha = \|\boldsymbol{A}\|\|\boldsymbol{C}\|$. Then we have that

$$\varphi(\boldsymbol{F}) = \alpha \boldsymbol{U}\boldsymbol{F}\boldsymbol{V}, \qquad \boldsymbol{F} \in \mathcal{F}(\mathcal{H}).$$

Put

$$\phi(\boldsymbol{X}) = \alpha^{-1}\boldsymbol{U}^*\varphi(\boldsymbol{X})\boldsymbol{V}^*, \qquad \boldsymbol{X} \in \mathcal{B}(\mathcal{H}).$$

Then ϕ is an additive bijection on $\mathcal{B}(\mathcal{H})$ preserving the star partial order in both directions such that

$$\phi(\boldsymbol{F}) = \boldsymbol{F}, \qquad \boldsymbol{F} \in \mathcal{F}(\mathcal{H}).$$

Now let $\boldsymbol{P} \in \mathcal{B}(\mathcal{H})$ be any projection. Then for any finite rank projection $\boldsymbol{Q}$, if $\boldsymbol{Q} \leq \boldsymbol{P}$, we have

$$\lambda\boldsymbol{Q} \overset{*}{\leq} \lambda\boldsymbol{P}, \qquad \lambda \in \mathbb{C}.$$

Then

$$\lambda\boldsymbol{Q} \overset{*}{\leq} \phi(\lambda\boldsymbol{P}).$$

Noting that $\{\lambda\boldsymbol{Q} : \boldsymbol{Q} \leq \boldsymbol{P}\}$ is a $*$-increasing net and $*$-bounded from above such that

$$\lim_{\boldsymbol{Q}\leq\boldsymbol{P}} \lambda\boldsymbol{Q} = \lambda\boldsymbol{P}$$

in strong operator topology, by Proposition 3.5 in [1], we have

$$\lim_{\boldsymbol{Q}\leq\boldsymbol{P}} \lambda\boldsymbol{Q} = \lambda\boldsymbol{P} \overset{*}{\leq} \phi(\lambda\boldsymbol{P}).$$

We note that the $*$-increasing and $*$-bounded sequences are considered in this proposition. However, the proposition still holds if we replaces a sequence by a net. By considering ϕ^{-1}, we have

$$\phi(\lambda\boldsymbol{P}) = \lambda\boldsymbol{P}.$$

Then $\phi(\boldsymbol{X}) = \boldsymbol{X}$ for all $\boldsymbol{X} \in \mathcal{B}(\mathcal{H})$ since $\boldsymbol{X}$ is a linear combination of finitely many projections from Theorem 3 in [14]. Thus

$$\varphi(\boldsymbol{X}) = \alpha\boldsymbol{U}\boldsymbol{X}\boldsymbol{V}, \qquad \boldsymbol{X} \in \mathcal{B}(\mathcal{H}).$$

If (2) holds, then there are two anti-unitary operators $\boldsymbol{U}$ and $\boldsymbol{V}$ such that

$$\varphi(\boldsymbol{X}) = \alpha\boldsymbol{U}\boldsymbol{X}^*\boldsymbol{V}, \qquad \boldsymbol{X} \in \mathcal{B}(\mathcal{H}).$$

If φ is conjugate linear on $\mathcal{F}(\mathcal{H})$, then we similarly have two unitary operators $\boldsymbol{U}$ and $\boldsymbol{V}$ on $\mathcal{H}$ such that

$$\varphi(\boldsymbol{X}) = \alpha\boldsymbol{U}\boldsymbol{X}^*\boldsymbol{V}, \qquad \boldsymbol{X} \in \mathcal{B}(\mathcal{H})$$

or two anti-unitary operators $\boldsymbol{U}$ and $\boldsymbol{V}$ on $\mathcal{H}$ such that

$$\varphi(\boldsymbol{X}) = \alpha\boldsymbol{U}\boldsymbol{X}\boldsymbol{V}, \qquad \boldsymbol{X} \in \mathcal{B}(\mathcal{H}).$$

The proof is completed.

The following corollary is a generalization of the main result in [10].

Corollary 2.1 *Let φ be an additive surjective map on $\mathcal{K}(\mathcal{H})$. Then φ preserves the star partial order in both directions if and only if one of the following holds*:

(1) *There exist a nonzero $\alpha \in \mathbb{C}$ and two unitary operators $\boldsymbol{U}$ and $\boldsymbol{V}$ on $\mathcal{H}$ such that*

$$\varphi(\boldsymbol{X}) = \alpha \boldsymbol{U}\boldsymbol{X}\boldsymbol{V}, \qquad \boldsymbol{X} \in \mathcal{K}(\mathcal{H})$$

or

$$\varphi(\boldsymbol{X}) = \alpha \boldsymbol{U}\boldsymbol{X}^*\boldsymbol{V}, \qquad \boldsymbol{X} \in \mathcal{K}(\mathcal{H});$$

(2) *There exist a nonzero $\alpha \in \mathbb{C}$ and two anti-unitary operators $\boldsymbol{U}$ and $\boldsymbol{V}$ on $\mathcal{H}$ such that*

$$\varphi(\boldsymbol{X}) = \alpha \boldsymbol{U}\boldsymbol{X}\boldsymbol{V}, \qquad \boldsymbol{X} \in \mathcal{K}(\mathcal{H})$$

or

$$\varphi(\boldsymbol{X}) = \alpha \boldsymbol{U}\boldsymbol{X}^*\boldsymbol{V}, \qquad \boldsymbol{X} \in \mathcal{K}(\mathcal{H}).$$

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