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Complete Convergence of Weighted Sums for Arrays of Rowwise m-negatively Associated Random Variables

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Abstract: In this paper, we discuss the complete convergence of weighted sums for arrays of rowwise m-negatively associated random variables. By applying moment inequality and truncation methods, the sufficient conditions of complete convergence of weighted sums for arrays of rowwise m-negatively associated random variables are established. These results generalize and complement some known conclusions.

Key words: complete convergence, negatively associated, m-negatively associated, weighted sum

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1 Introduction

Let $\{X_n,\ n \geq 1\}$ be a sequence of random variables. Hsu and Robbins[1] introduced the concept of complete convergence of $\{X_n\}$. A sequence $\{X_n,\ n = 1, 2, \cdots\}$ of random variables is said to converge completely to a constant C if

$$\sum_{n=1}^{\infty} P(|X_n - C| > \epsilon) < \infty, \qquad \epsilon > 0.$$

In view of the Borel-Cantelli lemma, this implies that $X_n \to C$ almost surely. The converse is true if $\{X_n,\ n \geq 1\}$ is a sequence of independent random variables.

Definition 1.1 *A finite family of random variables $\{X_i,\ 1 \leq i \leq n\}$ is said to be negatively associated (NA, for short) if for every pair of disjoint subsets A and B of $\{1, 2, \cdots, n\}$*

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and any real nondecreasing coordinate-wise functions f_1 *on* $\mathbf{R}^A$ *and* f_2 *on* $\mathbf{R}^B$

$$\operatorname{cov}(f_1(X_i,\ i \in A),\ f_2(X_i,\ i \in B)) \le 0$$

whenever f_1 *and* f_2 *are such that covariance exists.*

An infinite family of random variables $\{X_i, -\infty < i < \infty\}$ *is NA if every finite subfamily is NA.*

The definition of NA was introduced by Alam and Saxena[2] and was studied by Joag-Dev *et al.* (see [3–4]). As pointed out and proved by Joag-Dev and Proschan[3], a number of well-known multivariate distributions possess the NA property. Negative association has found important and wide applications in multivariate statistical analysis and reliability. Many investigators have discussed applications of negative association to probability, stochastic processes and statistics.

Definition 1.2 *Let* $m \ge 1$ *be a fixed integer. A sequence of random variables* $\{X_i,\ i \ge 1\}$ *is said to be m-negatively associated (m-NA, for short) if for any* $n \ge 2$ *and* $i_1, i_2, \cdots, i_n$ *such that* $|i_k - i_j| \ge m$ *for all* $1 \le k \ne j \le n$, $\{X_{i_1},\ X_{i_2},\ \cdots,\ X_{i_n}\}$ *is NA.*

The m-NA random variables is a natural extension from NA random variables. Actually, the NA sequence is just the 1-NA sequence. Moreover, Hu *et al.*[5] showed that there exists a sequence which is not NA but 2-NA.

Hu *et al.*[6] proved a very general result for complete convergence of rowwise independent arrays of random variables which is stated in Theorem 1.1.

Theorem 1.1[6] *Let* $\{X_{ni},\ 1 \le i \le k_n,\ n \ge 1\}$ *be an array of rowwise independent arrays of random variables. Suppose that for every* $\epsilon > 0$ *and some* $\delta > 0$,

(i) $\sum\limits_{n=1}^{\infty} c_n \sum\limits_{i=1}^{k_n} P\{|X_{ni}| > \epsilon\} < \infty$;

(ii) *there exists a* $j \ge 2$ *such that* $\sum\limits_{n=1}^{\infty} c_n \Big(\sum\limits_{i=1}^{k_n} E|X_{ni}|^2 I(|X_{ni}| \le \delta)\Big)^{j/2} < \infty$;

(iii) $\sum\limits_{i=1}^{k_n} EX_{ni} I(|X_{ni}| \le \delta) \to 0$ *as* $n \to \infty$.

Then

$$\sum_{n=1}^{\infty} c_n P\bigg\{\bigg|\sum_{i=1}^{k_n} X_{ni}\bigg| > \epsilon\bigg\} < \infty, \qquad \epsilon > 0.$$

Hu *et al.*[7] obtained the complete convergence of maximum partial sums for arrays of rowwise NA random variables by using an exponential inequality obtained by Shao[8] and their result is given in Theorem 1.2.

Theorem 1.2[7] *Let* $\{X_{ni},\ 1 \le i \le k_n,\ n \ge 1\}$ *be an array of rowwise NA random variables such that the conditions* (i) *and* (ii) *in Theorem* 1.1 *are satisfied. Then*

$$\sum_{n=1}^{\infty} c_n P\bigg\{\max_{1 \le k \le k_n}\bigg|\sum_{i=1}^{k}(X_{ni} - EX_{ni} I(|X_{ni}| \le \delta))\bigg| > \epsilon\bigg\} < \infty, \qquad \epsilon > 0.$$

Kuczmaszewska[9] investigated complete convergence of weighted sums for arrays of rowwise NA random variables, and proved the following result.

Theorem 1.3[9] *Let $\{X_{ni},\ i\ge 1,\ n\ge 1\}$ be an array of rowwise NA random variables, $\{a_{ni},\ i\ge 1,\ n\ge 1\}$ be an array of real numbers, $\{b_n,\ n\ge 1\}$ be an increasing sequence of positive integers, and $\{c_n, n\ge 1\}$ be a sequence of positive real numbers. If for some $q>2$, $0<t<2$ and any $\epsilon>0$ the following conditions are satisfied*:

(a) $\sum\limits_{n=1}^{\infty} c_n \sum\limits_{i=1}^{b_n} P\{|a_{ni}X_{ni}|\ge \epsilon b_n^{1/t}\}<\infty$;

(b) $\sum\limits_{n=1}^{\infty} c_n b_n^{-q/t} \sum\limits_{i=1}^{b_n} |a_{ni}|^q E|X_{ni}|^q I(|a_{ni}X_{ni}|<\epsilon b_n^{1/t})<\infty$;

(c) $\sum\limits_{n=1}^{\infty} c_n b_n^{-q/t}\left(\sum\limits_{i=1}^{b_n} |a_{ni}|^2 E|X_{ni}|^2 I(|a_{ni}X_{ni}|<\epsilon b_n^{1/t})\right)^{q/2}<\infty$,

then

$$\sum_{n=1}^{\infty} c_n P\left\{\max_{1\le k\le b_n}\left|\sum_{i=1}^{k}(a_{ni}X_{ni}-a_{ni}EX_{ni}I(|a_{ni}X_{ni}|<\epsilon b_n^{1/t}))\right|>\epsilon b_n^{1/t}\right\}<\infty.$$

In this paper, we investigate the complete convergence for arrays of rowwise m-NA random variables which includes many previous results as corollaries. For example, Sung *et al.*[10] and Hu *et al.*[6] investigated independent arrays of random variables and Hu *et al.*[7] investigated rowwise NA arrays of random variables. We point out that in Theorem 2.1 of this paper we not only extends the result of Hu *et al.*[7], but also provide different methods from those used by them.

2 Main Results and Some Lemmas

Now we state our main results. The proof will be given in Section 3. Throughout this paper, C represents a positive constant whose value may different at each appearance. The symbol $I(A)$ denotes the indicator function of A, N denotes the positive integer set and $[x]$ indicates the maximum integer not larger than x. Let $\{b_n,\ n\ge 1\}$ be an increasing sequence of positive integers, $\{c_n,\ n\ge 1\}$ be a sequence of positive real numbers, $\{X_{ni},\ 1\le i\le b_n,\ n\ge 1\}$ be an array of rowwise m-NA random variables, and $\{a_{ni},\ 1\le i\le b_n,\ n\ge 1\}$ be an array of real numbers.

Theorem 2.1 *If for some $t>0$, $\delta>0$ and any $\epsilon>0$, the following conditions are satisfied*:

(i) $\sum\limits_{n=1}^{\infty} c_n \sum\limits_{i=1}^{b_n} P\{|a_{ni}X_{ni}|\ge \epsilon b_n^{1/t}\}<\infty$;

(ii) *there exists some $q\ge 2$ such that*

$$\sum_{n=1}^{\infty} c_n b_n^{-q/t}\left(\sum_{i=1}^{b_n} |a_{ni}|^2 E|X_{ni}|^2 I(|a_{ni}X_{ni}|<\delta b_n^{1/t})\right)^{q/2}<\infty,$$

then

$$\sum_{n=1}^{\infty} c_n P\left\{\max_{1\le k\le b_n}\left|\sum_{i=1}^{k}(a_{ni}X_{ni}-a_{ni}EX_{ni}I(|a_{ni}X_{ni}|<\delta b_n^{1/t}))\right|>\epsilon b_n^{1/t}\right\}<\infty. \tag{2.1}$$

Remark 2.1 Theorem 2.1 improves upon Theorem 1.3 of Kuczmaszewska[9]. Moreover, from Theorem 2.1 we see that the condition (b) in Theorem 1.3 is unnecessary.

Corollary 2.1 extends the main result of Sung *et al.*[10] and can be obtained immediately from Theorem 2.1.

Corollary 2.1 *Under the conditions of Theorem* 2.1, *in addition*, *if the following condition is satisfied*:

$$\max_{1\le k\le b_n} b_n^{-1/t}\sum_{i=1}^{k} a_{ni}EX_{ni}I(|a_{ni}X_{ni}|<\delta b_n^{1/t})\to 0 \qquad \text{as } n\to\infty,$$

then

$$\sum_{n=1}^{\infty} c_n P\Big\{\max_{1\le k\le b_n}\Big|\sum_{i=1}^{k} a_{ni}X_{ni}\Big|>\epsilon b_n^{1/t}\Big\}<\infty.$$

Corollary 2.2 *Let* $EX_{ni}=0$ *for any* $1\le i\le b_n$, $n\ge 1$, *and* $\psi(x)$ *be a real function defined on* $[0,\infty)$ *such that* $\sup\limits_{x\ge\delta}\dfrac{x}{\psi(x)}<\infty$ *and* $\sup\limits_{0\le x<\delta}\dfrac{x^2}{\psi(x)}<\infty$ *for some* $\delta>0$. *Assume that for some* $t>0$ *and any* $\epsilon>0$ *the following conditions are satisfied*:

(a) $\sum\limits_{n=1}^{\infty} c_n\sum\limits_{i=1}^{b_n}P\{|a_{ni}X_{ni}|\ge\epsilon b_n^{1/t}\}<\infty$;

(b) *there exists* $q\ge 2$ *such that* $\sum\limits_{n=1}^{\infty} c_n\Big(\sum\limits_{i=1}^{b_n}E\psi(b_n^{-1/t}|a_{ni}X_{ni}|)\Big)^{q/2}<\infty$;

(c) *if the sequence* $\{c_n,\ n\ge 1\}$ *is not bounded away from zero, that is, if* $\liminf\limits_{n\to\infty} c_n=0$, *and that* $\sum\limits_{i=1}^{b_n}E\psi(b_n^{-1/t}|a_{ni}X_{ni}|)\to 0$ *as* $n\to\infty$.

Then for all $\epsilon>0$,

$$\sum_{n=1}^{\infty} c_n P\Big\{\max_{1\le k\le b_n}\Big|\sum_{i=1}^{k} a_{ni}X_{ni}\Big|>\epsilon b_n^{1/t}\Big\}<\infty.$$

Remark 2.2 It is obvious that if the sequence $\{c_n,\ n\ge 1\}$ is bounded away from zero, that is, if $\liminf\limits_{n\to\infty} c_n>0$, then the assumption (c) is unnecessary, which follows from the assumption (b).

Theorem 2.2 *If for some* $t>0$, $\delta>0$ *and any* $\epsilon>0$, *the following conditions are satisfied*:

(i) $\sum\limits_{n=1}^{\infty} c_n\sum\limits_{i=1}^{b_n}P\{|a_{ni}X_{ni}|\ge\epsilon b_n^{1/t}\}<\infty$;

(ii) *there exists some* $1\le q\le 2$ *such that*

$$\sum_{n=1}^{\infty} c_n b_n^{-q/t}\sum_{i=1}^{b_n}|a_{ni}|^qE|X_{ni}|^qI(|a_{ni}X_{ni}|<\delta b_n^{1/t})<\infty,$$

then

$$\sum_{n=1}^{\infty} c_n P\Big\{\max_{1\le k\le b_n}\Big|\sum_{i=1}^{k}(a_{ni}X_{ni}-a_{ni}EX_{ni}I(|a_{ni}X_{ni}|<\delta b_n^{1/t}))\Big|>\epsilon b_n^{1/t}\Big\}<\infty. \tag{2.2}$$

For the proof of the main results we need to restate a few lemmas for easy reference. The following lemmas play an important role in our main results.

Lemma 2.1[11] *Let $\{X_i,\ 1\le i\le n\}$ be a finite family of NA mean zero random variables with $EX_i^2<\infty$ for every $1\le i\le n$, and set $B_n=\sum\limits_{i=1}^{n}EX_i^2$. Then for all $\epsilon>0$, $a>0$,*

$$P\Big\{\max_{1\le k\le n}\sum_{i=1}^{k}X_i\ge\epsilon\Big\}\le P\Big\{\max_{1\le k\le n}X_k>a\Big\}+\mathrm{e}\cdot\exp\Big\{\frac{\epsilon}{a}-\frac{\epsilon}{a}\ln\Big(1+\frac{\epsilon a}{B_n}\Big)\Big\}.$$

Lemma 2.2[8] *Let $\{X_i,\ 1\le i\le n\}$ be a sequence of NA random variables with mean zero and $E|X_i|^p<\infty$ for every $1\le i\le n$, $1\le p\le 2$. Then*

$$E\max_{1\le k\le n}\Big|\sum_{k=1}^{n}X_k\Big|^p\le 2^{3-p}\sum_{i=1}^{n}E|X_i|^p.$$

Lemma 2.3 *Let $\{X_i,\ i\ge1\}$ be a sequence of m-NA random variables with mean zero and $EX_i^2<\infty$ for every $i\ge1$, and*

$$S_n=\sum_{i=1}^{n}X_i,\quad B_n=\sum_{i=1}^{n}EX_i^2,\qquad n\ge1.$$

Then for all $n\ge m$, $x>0$, $a>0$,

$$P\Big\{\max_{1\le k\le n}S_k\ge x\Big\}\le m\Big[P\Big\{\max_{1\le k\le n}X_k>a\Big\}+\mathrm{e}\cdot\exp\Big\{\frac{x}{ma}\Big(1-\ln\Big(1+\frac{xa}{mB_n}\Big)\Big)\Big\}\Big],\tag{2.3}$$

and

$$P\Big\{\max_{1\le k\le n}|S_k|\ge x\Big\}\le 2m\Big[P\Big\{\max_{1\le k\le n}|X_k|>a\Big\}+\mathrm{e}\cdot\exp\Big\{\frac{x}{ma}\Big(1-\ln\Big(1+\frac{xa}{mB_n}\Big)\Big)\Big\}\Big].\tag{2.4}$$

Proof. From (2.3) we can immediately get (2.4). Hence, to complete the proof, it is enough to show that (2.3) holds.

Given any $1\le k\le n$, let $r=\Big[\dfrac{n}{m}\Big]$. Define

$$Y_i=\begin{cases}X_i, & 1\le i\le n;\\ 0, & i>n,\end{cases}$$

$$T_{mk+j}=\sum_{i=0}^{k}Y_{mi+j},\qquad 1\le j\le m.$$

It is obvious from Definition 1.2 that $\{Y_{mk+j},\ k=0,1,\cdots,r\}$ is a sequence of NA random variables for every $1\le j\le m$, $m\le n$. Since

$$\Big\{\max_{1\le k\le n}S_k\ge x\Big\}\subset\Big\{\max_{0\le k\le r}T_{mk+1}\ge\frac{x}{m}\Big\}\cup\cdots\cup\Big\{\max_{0\le k\le r}T_{mk+m}\ge\frac{x}{m}\Big\},$$

it follows from Lemma 2.1 that

$$\begin{aligned}&P\Big\{\max_{1\le k\le n}S_k\ge x\Big\}\\ \le\ &\sum_{j=1}^{m}P\Big\{\max_{0\le k\le r}T_{mk+j}\ge\frac{x}{m}\Big\}\\ \le\ &\sum_{j=1}^{m}P\big\{\max_{0\le i\le r}Y_{mi+j}>a\big\}+\sum_{j=1}^{m}\mathrm{e}\cdot\exp\left\{\frac{\frac{x}{m}}{a}-\frac{\frac{x}{m}}{a}\ln\Big(1+\frac{ax}{mB_n}\Big)\right\}\end{aligned}$$

$$\le mP\Big\{\max_{1\le k\le n} X_k > a\Big\} + \sum_{j=1}^{m} \mathrm{e}\cdot\exp\Big\{\frac{x}{ma} - \frac{x}{ma}\ln\Big(1+\frac{xa}{mB_n}\Big)\Big\}$$
$$\le m\Big[P\Big\{\max_{1\le k\le n} X_k > a\Big\} + \mathrm{e}\cdot\exp\Big\{\frac{x}{ma} - \frac{x}{ma}\ln\Big(1+\frac{xa}{mB_n}\Big)\Big\}\Big].$$

So, (2.3) holds.

Lemma 2.4 *Let $\{X_i,\ 1\le i\le n\}$ be a sequence of m-NA random variables with mean zero and $E|X_i|^p<\infty$ for every $1\le i\le n$, $1\le p\le 2$. Then*

$$E\max_{1\le k\le n}\Big|\sum_{i=1}^{k} X_i\Big|^p \le m^{p-1}2^{3-p}\sum_{i=1}^{n} E|X_i|^p, \qquad 1\le p\le 2. \tag{2.5}$$

Proof. Let Y_i, T_{mk+j} and r be as in Lemma 2.3. By using the C_r inequality, it follows from Lemma 2.2 that

$$\begin{aligned}
E\max_{1\le k\le n}\Big|\sum_{i=1}^{k} X_i\Big|^p &\le E\Big(\sum_{j=1}^{m}\max_{0\le i\le r}|T_{mi+j}|\Big)^p\\
&\le m^{p-1}\sum_{j=1}^{m} E\max_{0\le i\le r}|T_{mi+j}|^p\\
&\le m^{p-1}2^{3-p}\sum_{j=1}^{m}\sum_{i=0}^{r} E|Y_{mi+j}|^p\\
&= m^{p-1}2^{3-p}\sum_{i=1}^{n} E|X_i|^p.
\end{aligned}$$

3 Proofs of the Main Results

Proof of Theorem 2.1 Let

$$Y_{ni} = a_{ni}X_{ni}I(|a_{ni}X_{ni}|<\delta b_n^{1/t})) + \delta b_n^{1/t}I(a_{ni}X_{ni}\ge\delta b_n^{1/t}) - \delta b_n^{1/t}I(a_{ni}X_{ni}\le-\delta b_n^{1/t}),$$

where $\delta>0$ and $1\le i\le b_n$, $n\ge1$. By Property 6 in [3], we can conclude that

$$\{Y_{ni}-EY_{ni},\ 1\le i\le b_n,\ n\ge1\}$$

is an array of rowwise m-NA random variables. For $n\ge1$ and $1\le k\le b_n$, let

$$\begin{aligned}
S'_k &= \sum_{i=1}^{k} Y_{ni}, \qquad S_k = \sum_{i=1}^{k} a_{ni}X_{ni}\\
T_k &= \sum_{i=1}^{k} a_{ni}X_{ni}I(|a_{ni}X_{ni}|<\delta b_n^{1/t}),\\
Z_k &= \sum_{i=1}^{k}[\delta b_n^{1/t}I(a_{ni}X_{ni}\ge\delta b_n^{1/t}) - \delta b_n^{1/t}I(a_{ni}X_{ni}\le-\delta b_n^{1/t})].
\end{aligned}$$

Noting that for any $n\ge1$,

$$P\Big\{\max_{1\le i\le b_n}\Big|\sum_{j=1}^{i}(a_{nj}X_{nj}-a_{nj}EX_{nj}I(|a_{nj}X_{nj}|<\delta b_n^{1/t}))\Big| > \epsilon b_n^{1/t}\Big\}$$
$$= P\Big\{\max_{1\le i\le b_n}|S_i-ET_i|>\epsilon b_n^{1/t}\Big\}$$

$$\leq \sum_{i=1}^{b_n} P\{|a_{ni}X_{ni}| \geq \delta b_n^{1/t}\} + P\{\max_{1\leq i\leq b_n} |T_i - ET_i| \geq \epsilon b_n^{1/t}\}, \tag{3.1}$$

and that $T_i = S_i' - Z_i$, we find

$$\Big\{\max_{1\leq i\leq b_n} |T_i - ET_i| \geq \epsilon b_n^{1/t}\Big\} \subset \Big\{\max_{1\leq i\leq b_n} |S_i' - ES_i'| \geq \frac{\epsilon}{2} b_n^{1/t}\Big\} \cup \Big\{\max_{1\leq i\leq b_n} |Z_i - EZ_i| \geq \frac{\epsilon}{2} b_n^{1/t}\Big\}.$$

Therefore, we have

$$\begin{aligned} &P\{\max_{1\leq i\leq b_n} |T_i - ET_i| \geq \epsilon b_n^{1/t}\} \\ &\leq P\Big\{\max_{1\leq i\leq b_n} |S_i' - ES_i'| \geq \frac{\epsilon}{2} b_n^{1/t}\Big\} + P\Big\{\max_{1\leq i\leq b_n} |Z_i - EZ_i| \geq \frac{\epsilon}{2} b_n^{1/t}\Big\}. \end{aligned} \tag{3.2}$$

Using Markov's inequality, we get

$$P\Big\{\max_{1\leq i\leq b_n} |Z_i - EZ_i| \geq \frac{\epsilon}{2} b_n^{1/t}\Big\} \leq C \sum_{i=1}^{b_n} P\{|a_{ni}X_{ni}| \geq \delta b_n^{1/t}\}. \tag{3.3}$$

Combining condition (i) with (3.1)–(3.3) we see that, to complete the proof, it is enough to show that

$$\sum_{n=1}^{\infty} c_n P\Big\{\max_{1\leq i\leq b_n} |S_i' - ES_i'| \geq \epsilon b_n^{1/t}\Big\} < \infty.$$

Set

$$B_n = \sum_{i=1}^{b_n} \mathrm{var}(Y_{ni}), \qquad n \geq 1.$$

For any $\epsilon > 0$ and $a > 0$, set

$$\begin{aligned} A = \Big\{n \Big| &\sum_{i=1}^{b_n} P\Big\{|a_{ni}X_{ni}| \geq \min\Big\{\delta,\ \frac{a}{3}\Big\} b_n^{1/t}\Big\} < \min\Big\{1,\ \frac{a}{3\delta}\Big\} \\ &\text{and } b_n^{-2/t} \sum_{i=1}^{b_n} |a_{ni}|^2 E|X_{ni}|^2 I(|a_{ni}X_{ni}| < \delta b_n^{1/t}) < \Big(\frac{a}{3}\Big)^2\Big\}, \\ B = N - A&, \end{aligned}$$

where $N = \{1,\ 2,\ 3,\ \cdots\}$. Note

$$\begin{aligned} &\sum_{n\in B} c_n P\Big\{\max_{1\leq i\leq b_n} |S_i' - ES_i'| \geq \epsilon b_n^{1/t}\Big\} \\ &\leq \sum_{n\in B} c_n \\ &\leq \frac{1}{\min\Big\{1,\ \frac{a}{3\delta}\Big\}} \sum_{n=1}^{\infty} c_n \sum_{i=1}^{b_n} P\Big\{|a_{ni}X_{ni}| \geq \min\Big\{\delta,\ \frac{a}{3}\Big\} b_n^{1/t}\Big\} \\ &\quad + \Big(\frac{3}{a}\Big)^q \sum_{n=1}^{\infty} c_n b_n^{-q/t} \Big(\sum_{i=1}^{b_n} |a_{ni}|^2 E|X_{ni}|^2 I(|a_{ni}X_{ni}| < \delta b_n^{1/t})\Big)^{q/2} \\ &< \infty. \end{aligned}$$

Hence it suffices to prove that

$$\sum_{n\in A} c_n P\Big\{\max_{1\leq i\leq b_n} |S_i' - ES_i'| \geq \epsilon b_n^{1/t}\Big\} < \infty.$$

By Lemma 2.3, we have

$$\begin{aligned}
&\sum_{n\in A} c_n P\Big\{\max_{1\le i\le b_n} |S_i' - ES_i'| \ge \epsilon b_n^{1/t}\Big\}\\
\le\ & \sum_{n\in A} c_n \bigg[2mP\Big\{\max_{1\le i\le b_n} |Y_{ni} - EY_{ni}| \ge a b_n^{1/t}\Big\} + 2m\mathrm{e}\cdot\exp\Big(1+\frac{\epsilon}{ma}\Big)\Big(1+\frac{\epsilon a b_n^{2/t}}{mB_n}\Big)^{-\frac{\epsilon}{ma}}\bigg]\\
\le\ & 2m\sum_{n\in A} c_n P\Big\{\max_{1\le i\le b_n} |Y_{ni} - EY_{ni}| \ge a b_n^{1/t}\Big\} + 2m\mathrm{e}\cdot\exp\Big(1+\frac{\epsilon}{ma}\Big)\sum_{n\in A} c_n\Big(\frac{mB_n}{\epsilon a b_n^{2/t}}\Big)^{\frac{\epsilon}{ma}}.
\end{aligned} \tag{3.4}$$

Note that for any $n \in A$,

$$\begin{aligned}
&\max_{1\le i\le b_n} |EY_{ni}| \le \max_{1\le i\le b_n} E|Y_{ni}|\\
\le\ & \max_{1\le i\le b_n} E|a_{ni}X_{ni}|I(|a_{ni}X_{ni}| < \delta b_n^{1/t}) + \delta b_n^{1/t}\sum_{i=1}^{b_n} P\{|a_{ni}X_{ni}| \ge \delta b_n^{1/t}\}\\
\le\ & b_n^{1/t}\bigg[\Big(b_n^{-2/t}\sum_{i=1}^{b_n} Ea_{ni}^2X_{ni}^2 I(|a_{ni}X_{ni}| < \delta b_n^{1/t})\Big)^{1/2} + \delta\sum_{i=1}^{b_n} P\{|a_{ni}X_{ni}| \ge \delta b_n^{1/t}\}\bigg]\\
\le\ & \frac{2a}{3} b_n^{1/t}.
\end{aligned}$$

Thus, for any $n \in A$, we have

$$\begin{aligned}
&\sum_{n\in A} c_n P\Big\{\max_{1\le i\le b_n} |Y_{ni} - EY_{ni}| \ge a b_n^{1/t}\Big\}\\
\le\ & \sum_{n\in A} c_n P\Big\{\max_{1\le i\le b_n} |Y_{ni}| \ge \frac{a}{3} b_n^{1/t}\Big\}\\
\le\ & \sum_{n=1}^{\infty} c_n \sum_{i=1}^{b_n} P\Big\{|a_{ni}X_{ni}| \ge \min\Big\{\delta,\ \frac{a}{3}\Big\} b_n^{1/t}\Big\}\\
<\ & \infty.
\end{aligned} \tag{3.5}$$

Therefore, by (3.4) and (3.5), the proof will be completed if we show that

$$\sum_{n\in A} c_n\Big(\frac{mB_n}{\epsilon a b_n^{2/t}}\Big)^{\frac{\epsilon}{ma}} < \infty.$$

Choose $a = \dfrac{2\epsilon}{mq}$. Noting that

$$\sum_{i=1}^{b_n} P\{|a_{ni}X_{ni}| \ge \delta b_n^{1/t}\} \le 1 \qquad \text{for } n \in A,\ \frac{q}{2} \ge 1,$$

we have

$$\begin{aligned}
&\sum_{n\in A} c_n\Big(\frac{mB_n}{\epsilon a b_n^{2/t}}\Big)^{\frac{\epsilon}{ma}}\\
\le\ & C\sum_{n\in A} c_n b_n^{-q/t} B_n^{q/2}\\
\le\ & C\sum_{n\in A} c_n b_n^{-q/t}\bigg[\sum_{i=1}^{b_n} a_{ni}^2 EX_{ni}^2 I(|a_{ni}X_{ni}| < \delta b_n^{1/t})
\end{aligned}$$

$$
\begin{aligned}
&\qquad + \delta^2 b_n^{-2/t} \sum_{i=1}^{b_n} P\{|a_{ni}X_{ni}| \geq \delta b_n^{1/t}\}\Big]^{q/2} \\
&\leq C\Big[\sum_{n\in A} c_n b_n^{-q/t}\Big(\sum_{i=1}^{b_n} a_{ni}^2 EX_{ni}^2 I(|a_{ni}X_{ni}| < \delta b_n^{1/t})\Big)^{q/2} \\
&\qquad + \sum_{n\in A} c_n \Big(\sum_{i=1}^{b_n} P\{|a_{ni}X_{ni}| \geq \delta b_n^{1/t}\}\Big)^{q/2}\Big] \\
&\leq C\Big[\sum_{n=1}^{\infty} c_n b_n^{-q/t}\Big(\sum_{i=1}^{b_n} a_{ni}^2 EX_{ni}^2 I(|a_{ni}X_{ni}| < \delta b_n^{1/t})\Big)^{q/2} \\
&\qquad + \sum_{n=1}^{\infty} c_n \sum_{i=1}^{b_n} P\{|a_{ni}X_{ni}| \geq \delta b_n^{1/t}\}\Big] \\
&< \infty.
\end{aligned}
$$

Therefore (2.1) holds.

Proof of Corollary 2.2 Note that

$$
\begin{aligned}
&\sum_{i=1}^{b_n} Ea_{ni}^2 X_{ni}^2 I(|a_{ni}X_{ni}| < \delta b_n^{1/t}) \\
\leq &\sum_{i=1}^{b_n} E\frac{a_{ni}^2 X_{ni}^2}{\psi(b_n^{-1/t}|a_{ni}X_{ni}|)} I(|a_{ni}X_{ni}| < \delta b_n^{1/t})\psi(b_n^{-1/t}|a_{ni}X_{ni}|) \\
\leq &\Big(\sup_{0\leq x<\delta} \frac{x^2}{\psi(x)}\Big) b_n^{2/t} \sum_{i=1}^{b_n} E\psi(b_n^{-1/t}|a_{ni}X_{ni}|). \qquad (3.6)
\end{aligned}
$$

Since $EX_{ni} = 0$, it follows that

$$
\begin{aligned}
&b_n^{-1/t} \sum_{i=1}^{b_n} |Ea_{ni}X_{ni} I(|a_{ni}X_{ni}| < \delta b_n^{1/t})| \\
= &b_n^{-1/t} \sum_{i=1}^{b_n} |Ea_{ni}X_{ni} I(|a_{ni}X_{ni}| \geq \delta b_n^{1/t})| \\
\leq &\sum_{i=1}^{b_n} E\frac{b_n^{-1/t}|a_{ni}X_{ni}|}{\psi(b_n^{-1/t}|a_{ni}X_{ni}|)} I(|a_{ni}X_{ni}| \geq \delta b_n^{1/t})\psi(b_n^{-1/t}|a_{ni}X_{ni}|) \\
\leq &\Big(\sup_{x\geq\delta} \frac{x}{\psi(x)}\Big) \sum_{i=1}^{b_n} E\psi(b_n^{-1/t}|a_{ni}X_{ni}|) \to 0. \qquad (3.7)
\end{aligned}
$$

Thus, by (3.6), (3.7), (a), (b) and (c), we see that the conditions of Corollary 2.1 are satisfied. So, by Corollary 2.1 we complete the proof of Corollary 2.2.

Proof of Theorem 2.2 Let Y_{ni}, S'_i be as in the proof of Theorem 2.1. From the proof of Theorem 2.1, we need only to prove that

$$
\sum_{n=1}^{\infty} c_n P\Big\{\max_{1\leq i\leq b_n} |S'_i - ES'_i| \geq \epsilon b_n^{1/t}\Big\} < \infty \qquad (3.8)
$$

holds.

In fact, using the C_r inequality, for any $r > 0$, we can estimate

$$E|Y_{ni} - EY_{ni}|^r \le C(E|a_{ni}X_{ni}|^r I(|a_{ni}X_{ni}| < \delta b_n^{1/t}) + b_n^{r/t} P\{|a_{ni}X_{ni}| \ge \delta b_n^{1/t}\}).$$

Thus, using Markov's inequality, by the above estimation and (2.5) we obtain

$$\begin{aligned}
&P\Big\{\max_{1\le i\le b_n} |S_i' - ES_i'| \ge \epsilon b_n^{1/t}\Big\} \\
&= P\Big\{\max_{1\le k\le b_n} \Big|\sum_{i=1}^{k}(Y_{ni} - EY_{ni})\Big| \ge \epsilon b_n^{1/t}\Big\} \\
&\le \epsilon^{-q} b_n^{-q/t} E \max_{1\le k\le b_n} \Big|\sum_{i=1}^{k}(Y_{ni} - EY_{ni})\Big|^q \\
&\le C\epsilon^{-q} b_n^{-q/t} \sum_{i=1}^{b_n} E|Y_{ni} - EY_{ni}|^q \\
&\le C\Big[\sum_{i=1}^{b_n} P\{|a_{ni}X_{ni}| \ge \delta b_n^{1/t}\} + b_n^{-q/t} \sum_{i=1}^{b_n} |a_{ni}|^q E|X_{ni}|^q I(|a_{ni}X_{ni}| < \delta b_n^{1/t})\Big]. \qquad (3.9)
\end{aligned}$$

Therefore, from the conditions (i), (ii) and (3.9), we know that (3.8) holds.

References

[1] Hsu P L, Robbins H. Complete convergence and the law of large numbers. *Proc. Nat. Acad. Sci. U.S.A.*, 1947, **33**(2): 25–31.

[2] Alam K, Saxena K M L. Positive dependence in multivariate distributions. *Comm. Statist. Theory Methods*, 1981, **10**(12): 1183–1196.

[3] Joag-Dev K, Proschan F. Negative association of random variables with applications. *Ann. Statist.*, 1983, **11**(1): 286–295.

[4] Block H W, Savits T H, Sharked M. Some concepts of negative dependence. *Ann. Probab.*, 1982, **10**(3): 765–772.

[5] Hu Y J, Ming R X, Yang W Q. Large deviations and moderate deviations for m-negatively associated random variables. *Acta Math. Sci.* (*English Ed.*), 2007, **27**(4): 886–896.

[6] Hu T C, Szynal D, Volodin A. A note on complete convergence for arrays. *Statist. Probab. Lett.*, 1998, **38**(1): 27–31.

[7] Hu T C, Volodin A. A Remark on Complete Convergence for Arays of Rowwise Negatively Associated Variables. in: Proceedings of the 3rd Sino-International Symposium on Probability, Statistics, and Quantitative Management. Taipei, Taiwan, Republic of China: ICAQM/CDMS, 2006: 9–18.

[8] Shao Q M. A comparison theorem on moment inequalities between negatively associated and independent random variables. *J. Theoret. Probab.*, 2000, **13**(2): 343–356.

[9] Kuczmaszewska A. On complete convergence for arrays of rowwise negatively associated random variables. *Statist. Probab. Lett.*, 2009, **79**(1): 116–124.

[10] Sung S H, Volodin A, Hu T C. More on complete convergence for arrays. *Statist. Probab. Lett.*, 2005, **71**(4): 303–311.

[11] Liu L X, Wu R. Inequalities of maximal partial sum and the laws of iterated logarithm for sequence of NA random variables. *Acta Math. Sinica*, 2002, **45**(5): 969–978.