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A Joint Density Function in the Renewal Risk Model*

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Abstract: In this paper, we consider a general expression for $\phi(u,\ x,\ y)$, the joint density function of the surplus prior to ruin and the deficit at ruin when the initial surplus is u. In the renewal risk model, this density function is expressed in terms of the corresponding density function when the initial surplus is 0. In the compound Poisson risk process with phase-type claim size, we derive an explicit expression for $\phi(u,\ x,\ y)$. Finally, we give a numerical example to illustrate the application of these results.

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1 Introduction

The renewal risk model $\{U(t)\}_{t\geq 0}$ is defined by

$$U(t) = u + ct - \sum_{i=1}^{N(t)} X_i,$$

where u is the initial surplus, c is the rate of premium income per unit time, $\{X_i\}_{i=1}^{\infty}$ is a sequence of independent and identically distributed (i.i.d.) random variables, where X_i represents the amount of the ith claim, and $\{N(t)\}_{t\geq 0}$ is a counting process with $N(t)$ denoting the number of claims up to time t. In addition, X_i has a density function $\theta(x)$ and a distribution function

$$\Theta(x) = 1 - \bar{\Theta}(x) = P\{X \leq x\},$$

where X is an arbitrary X_i. Let

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$$E(X) = \int_0^\infty x\mathrm{d}\Theta(x) < \infty.$$

The sequence of i.i.d. random variables $\{W_i\}_{i=1}^{\infty}$ represents the claim inter-arrival times, with W_1 being the time until the first claim. W_i has a density function $k(t)$ and a distribution function

$$K(t) = 1 - \bar{K}(t) = P\{W \le t\},$$

where W is an arbitrary W_i. Let

$$E(W) = \int_0^\infty t\mathrm{d}K(t) < \infty.$$

We assume that claim amounts are independent of claim inter-arrival times. Further, we assume that

$$cE(W) > E(X).$$

Define the time of ruin

$$T = \inf\{t : U(t) < 0\},$$

where $T = \infty$ if $U(t) \ge 0$ for all $t > 0$. Denote the ruin probability by

$$\psi(u) = P\{T < \infty \mid U(0) = u\},$$

and the survival probability by

$$\delta(u) = 1 - \psi(u).$$

It is well known that

$$\psi(u) = P\{L > u\} = \sum_{n=1}^{\infty}(1-\rho)\rho^n \bar{F}^{*n}(u), \qquad u \ge 0, \tag{1.1}$$

where $\rho = \psi(0)$, L is the well-known maximal aggregate loss in the renewal risk model, and

$$F(y) = 1 - \bar{F}(y)$$

is the so-called ladder height distribution function, which can be interpreted as either the distribution function of the deficit at ruin when initial surplus $u = 0$ or the distribution function of the amount of a drop in surplus, given that a drop below its initial level occurs.

$$F^{*n}(y) = 1 - \bar{F}^{*n}(y)$$

is the distribution function of the n-fold convolution of $F(y)$ with itself (see [1]).

Let

$$\begin{aligned} \varPhi(u,\ x,\ y) &= \int_0^x \int_0^y \phi(u,\ r,\ s)\mathrm{d}s\mathrm{d}r \\ &= P\{U(T_-) \le x,\ |U(T)| \le y,\ T < \infty \mid U(0) = u\}, \end{aligned}$$

where $U(T_-)$ denotes the surplus prior to ruin, and $U(T)$ denotes the deficit at ruin. $\varPhi(u,\ x,\ y)$ may be interpreted as the probability that ruin occurs from initial surplus u with the deficit at ruin no greater than y and the surplus prior to ruin no greater than x. $\phi(u,\ r,\ s)$ denotes the joint density function. Let

$$h(u,\ x) = \int_0^\infty \phi(u,\ x,\ y)\mathrm{d}y,$$

where $h(u,\ x)$ may be interpreted as the defective density function of the surplus prior to ruin from initial surplus u. Let

$$g(u,\ y) = \int_0^\infty \phi(u,\ x,\ y)\mathrm{d}x,$$

where $g(u,\ y)$ may be interpreted as the defective density function of the deficit at ruin from initial surplus u. Define the proper density function of the deficit at ruin when initial surplus $u = 0$ by

$$f(y) = \frac{g(0,\ y)}{\psi(0)}.$$

Clearly, we have

$$f(y) = \frac{\mathrm{d}}{\mathrm{d}y}F(y).$$

The Sparre Andersen risk model is a well recognized risk model. As it was commented by Gerber and Shiu[2], although the model was proposed almost half a century ago, it remains an important area of research in actuarial science. A large number of researchers have studied this model on a variety of topics. Albrecher *et al.*[3] considered the threshold dividend strategies in the renewal risk model. Borovkov and Dickson[4] gave the distribution of ruin time in the renewal risk model. Yang and Zhang[5] studied the Gerber-Shiu function in a Sparre Andersen model with multi-layer dividend strategy. Landriault and Willmot[6] considered discounted penalty function in the renewal risk model with general inter-claim times.

The remainder of this paper is organized as follows. In Section 2, we provide a general solution for $\varPhi(u,\ x,\ y)$, and consequently its joint density function $\phi(u,\ x,\ y)$. In Section 3, we consider a simplifications in compound Poisson process with phase-type claim amount. In Section 4, we give a numerical example to illustrate the application of these results.

2 An Expression for $\phi(u,\ x,\ y)$

In this section, we derive the explicit expression of $\phi(u,\ x,\ y)$.

First, we consider the case when $u \geq x$. In order for the surplus immediately prior to ruin to be less than or equal to x, the surplus cannot fall below 0 on the first occasion that it drops below its initial level u. Hence it follows that

$$\begin{aligned}\varPhi(u,\ x,\ y) &= \int_0^u g(0,\ z)\varPhi(u-z,\ x,\ y)\mathrm{d}z \\ &= \psi(0)\int_0^u f(z)\varPhi(u-z,\ x,\ y)\mathrm{d}z.\end{aligned}$$

Taking partial derivatives with respect to x and y yields

$$\phi(u,\ x,\ y) = \psi(0)\int_0^u f(z)\phi(u-z,\ x,\ y)\mathrm{d}z. \tag{2.1}$$

Secondly, in the case when $0 \leq u < x$, it is possible for ruin to occur at the time the surplus first falls below its initial level u, and for the surplus prior to ruin to be less than or equal to x, and for the deficit at ruin to be less than or equal to y. The probability of this event is

$$J(u,\ x,\ y) = \int_0^{x-u} \int_u^{u+y} \phi(0,\ r,\ s)\mathrm{d}s\mathrm{d}r$$

as the event is equivalent to ruin occurring from initial surplus 0 with a surplus immediately prior to ruin less than or equal to $x-u$ and a deficit at ruin between u and $u+y$. Hence, for $0 \le u < x$, we have

$$\Phi(u,\ x,\ y) = \psi(0)\int_0^u f(z)\Phi(u-z,\ x,\ y)\mathrm{d}z + J(u,\ x,\ y),$$

and

$$\phi(u,\ x,\ y) = \psi(0)\int_0^u f(z)\phi(u-z,\ x,\ y)\mathrm{d}z + \phi(0,\ x-u,\ u+y). \tag{2.2}$$

Therefore, for $u \ge 0$, from (2.1) and (2.2), we have

$$\phi(u,\ x,\ y) = \psi(0)\int_0^u f(z)\phi(u-z,\ x,\ y)\mathrm{d}z + \beta(u,\ x,\ y), \tag{2.3}$$

where

$$\beta(u,\ x,\ y) = I(u<x)\phi(0,\ x-u,\ y+u),$$

with

$$I(A) = \begin{cases} 1, & \text{if } A \text{ occurs;} \\ 0, & \text{otherwise.} \end{cases}$$

Let

$$\tilde{\phi}(s,\ x,\ y) = \int_0^\infty \mathrm{e}^{-su}\phi(u,\ x,\ y)\mathrm{d}u,$$

$$\tilde{\beta}(s,\ x,\ y) = \int_0^\infty \mathrm{e}^{-su}\beta(u,\ x,\ y)\mathrm{d}u,$$

$$\tilde{f}(s) = \int_0^\infty \mathrm{e}^{-su}f(u)\mathrm{d}u.$$

Taking Laplace transform for (2.3) with respects to u, by basic properties of Laplace transform, we obtain

$$\tilde{\phi}(s,\ x,\ y) = \frac{\tilde{\beta}(s,\ x,\ y)}{1-\psi(0)\tilde{f}(s)}.$$

From (1.1) we obtain

$$E(\mathrm{e}^{-sL}) = \int_0^\infty \mathrm{e}^{-su}\mathrm{d}\delta(u) = \frac{\delta(0)}{1-\psi(0)\tilde{f}(s)}$$

(see [1]), which implies that

$$\tilde{\phi}(s,\ x,\ y) = \frac{E(\mathrm{e}^{-sL})}{\delta(0)}\tilde{\beta}(s,\ x,\ y) = \frac{\int_0^\infty \mathrm{e}^{-su}\mathrm{d}\delta(u)}{\delta(0)}\tilde{\beta}(s,\ x,\ y).$$

Since the product of two transforms is the transform of a convolution, it immediately follows that

$$\phi(u,\ x,\ y) = \frac{1}{\delta(0)}\int_0^u \beta(u-z,\ x,\ y)\mathrm{d}\delta(z)$$

$$= \frac{1}{\delta(0)}\int_{\max\{0,u-x\}}^u \phi(0,\ x-u+z,\ u-z+y)\mathrm{d}\delta(z). \tag{2.4}$$

Hence the above equation provides a means of finding $\phi(u,\ x,\ y)$ provided that we know both $\delta(z)$ and $\phi(0,\ x,\ y)$.

3 Simplifications in the Classical Risk Process with Phase-type Claim

In this section, we derive the explicit expression in compound Poisson process with phase-type claim amount.

Assume that $\{N(t)\}_{t\geq 0}$ is a Poisson process with rate $\lambda > 0$. So the claim inter-arrival time W_i has distribution function

$$K(t) = 1 - \mathrm{e}^{-\lambda t}, \qquad t > 0.$$

First, we introduce the phase-type distributions. Phase-type distributions have become an extremely popular tool for applied probabilists wishing to generalize beyond the exponential while retaining some of its key properties (see [7]–[8]). The phase-type family includes the exponential, mixture of exponentials, Erlangian and Coxian distributions as special cases. The class of phase-type distributions is dense in the space of probability distributions on $[0,\ \infty)$. We can always use phase-type distribution as the approximate distribution. Readers interested in finding a good approximating phase-type distributions may refer to [9]–[10].

Phase-type distributions were first introduced by Neuts[11] in 1975. A shortened treatment can be stated as follows. Consider a Markov process with transient states $\{1,\ 2,\ \cdots,\ m\}$ and absorbing state $m+1$, whose infinitesimal generator $\boldsymbol{Q}$ has the form

$$\boldsymbol{Q} = \begin{pmatrix} \boldsymbol{S} & \boldsymbol{S}^0 \\ \boldsymbol{0} & 0 \end{pmatrix}.$$

The diagonal entries S_{ii} are necessarily negative, other entries are non-negative, and $\boldsymbol{S}^0 = -\boldsymbol{S}\boldsymbol{e}'$ ($\boldsymbol{e}'$ is an $m\times 1$ column vector of ones) represents the rates at which transitions occur from the individual transient states to the absorbing state. Let the process start in state i with probability a_i $(i = 1, 2, \cdots, m, m+1)$, and $\boldsymbol{a} = (a_1,\ a_2,\ \cdots,\ a_m)$ (in many practical problems, $a_{m+1} = 0$). Under these assumptions, the time V until absorption has occurred has distribution function

$$F(x) = 1 - \boldsymbol{a}\exp\{\boldsymbol{S}x\}\boldsymbol{e}', \qquad x \geq 0$$

and density function

$$p(x) = -\boldsymbol{a}\exp\{\boldsymbol{S}x\}\boldsymbol{S}\boldsymbol{e}', \qquad x \geq 0,$$

where the matrix exponential is defined by

$$\exp\{\boldsymbol{S}x\} = \sum_{n=0}^{\infty}\frac{x^n}{n!}\boldsymbol{S}^n.$$

As this distribution is completely determined by $\boldsymbol{a}$ and $\boldsymbol{S}$, we say either that V has a phase-type distribution with representation $(\boldsymbol{a},\ \boldsymbol{S})$, or write $V \sim \mathrm{PH}(\boldsymbol{a},\ \boldsymbol{S})$. Occasionally, we say that $F(x)$ has PH representation $(\boldsymbol{a},\ \boldsymbol{S})$. For a more detailed description of phase-type distributions, see [12].

Several well-known ruin-theoretic results can be summarized as follows (see [13]):

If the i.i.d. claim amount random variables $X_i \sim PH(\boldsymbol{a},\ \boldsymbol{S})$, from Theorem 4.4 in [12] we know that the probability of ultimate ruin in the general renewal risk model with phase-distributed claim amounts is given by

$$\psi(u) = \boldsymbol{a}_+ \exp\{u\boldsymbol{B}\}\boldsymbol{e}',$$

where $\boldsymbol{B} = \boldsymbol{S} + \boldsymbol{S}\boldsymbol{e}'\boldsymbol{a}_+$, and the row vector $\boldsymbol{a}_+$ is the unique solution of a fixed-point problem, i.e., $\boldsymbol{a}_+$ satisfies the equation

$$\boldsymbol{a}_+ = \phi(\boldsymbol{a}_+), \tag{3.1}$$

while

$$\phi(\boldsymbol{a}_+) = \boldsymbol{a}\int_0^\infty \exp\{ct(\boldsymbol{S} - \boldsymbol{S}\boldsymbol{e}'\boldsymbol{a}_+)\}\mathrm{d}K(t).$$

In the classical compound Poisson risk process, the claim inter-arrival times are exponentially distributed with

$$K(t) = 1 - \mathrm{e}^{-\lambda t}, \qquad t \geq 0.$$

Note that

$$\begin{aligned}
&\int_0^\infty \exp\{ct(\boldsymbol{S} - \boldsymbol{S}\boldsymbol{e}'\boldsymbol{a}_+)\}\mathrm{d}K(t)\\
=&\int_0^\infty \exp\{ct(\boldsymbol{S} - \boldsymbol{S}\boldsymbol{e}'\boldsymbol{a}_+)\}\lambda\mathrm{e}^{-\lambda t}\mathrm{d}t\\
=&\lambda\int_0^\infty \exp\{t(-\lambda\boldsymbol{I}_m + c\boldsymbol{S} - c\boldsymbol{S}\boldsymbol{e}'\boldsymbol{a}_+)\}\mathrm{d}y\\
=&\lambda(\lambda\boldsymbol{I}_m - c\boldsymbol{S} + c\boldsymbol{S}\boldsymbol{e}'\boldsymbol{a}_+)^{-1},
\end{aligned} \tag{3.2}$$

where $\boldsymbol{I}_m$ represents the $m \times m$ identity matrix. Therefore, substituting (3.2) into (3.1), we obtain the following equation:

$$\lambda\boldsymbol{a}_+ - c\boldsymbol{a}_+\boldsymbol{S} + c\boldsymbol{a}_+\boldsymbol{S}\boldsymbol{e}'\boldsymbol{a}_+ - \lambda\boldsymbol{a} = \boldsymbol{0}. \tag{3.3}$$

Based on Corollary 3.1 in [12], we try as the candidate solution

$$\boldsymbol{a}_+ = -\frac{\lambda}{c}\boldsymbol{a}\boldsymbol{S}^{-1}.$$

Then the left-hand side of (3.3) becomes

$$\begin{aligned}
&\lambda\boldsymbol{a}_+ - c\boldsymbol{a}_+\boldsymbol{S} + c\boldsymbol{a}_+\boldsymbol{S}\boldsymbol{e}'\boldsymbol{a}_+ - \lambda\boldsymbol{a}\\
=&-\frac{\lambda^2}{c}\boldsymbol{a}\boldsymbol{S}^{-1} + \lambda\boldsymbol{a} + \frac{\lambda^2}{c}\boldsymbol{a}\boldsymbol{S}^{-1}\boldsymbol{S}\boldsymbol{e}'\boldsymbol{a}\boldsymbol{S}^{-1} - \lambda\boldsymbol{a}\\
=&-\frac{\lambda^2}{c}\boldsymbol{a}\boldsymbol{S}^{-1} + \frac{\lambda^2}{c}\boldsymbol{a}\boldsymbol{S}^{-1}\\
=&\,\boldsymbol{0}.
\end{aligned}$$

Thus the probability of ultimate ruin in the compound Poisson risk process with phase-type distribution claim amounts is given by

$$\psi(u) = -\frac{\lambda}{c}\boldsymbol{a}\boldsymbol{S}^{-1}\exp\{u\boldsymbol{B}\}\boldsymbol{e}', \tag{3.4}$$

where

$$\boldsymbol{B} = \boldsymbol{S} + \frac{\lambda}{c}\boldsymbol{S}\boldsymbol{e}'\boldsymbol{a}\boldsymbol{S}^{-1}.$$

It is well known that in the compound Poisson risk process

$$\phi(0,\ x,\ y) = \frac{\lambda}{c}p(x+y)$$

(see [14]). Thus when $X_i \sim \mathrm{PH}(\boldsymbol{a},\ \boldsymbol{S})$, we have

$$\phi(0,\ x,\ y) = -\frac{\lambda}{c}\boldsymbol{a}\exp\{\boldsymbol{S}(x+y)\}\boldsymbol{S}\boldsymbol{e}', \qquad x \geq 0. \tag{3.5}$$

Substituting (3.4) and (3.5) into (2.4), for $0 \leq u \leq x$, we obtain

$$\begin{aligned}
\phi(u,\ x,\ y) &= \frac{p(x+y)}{\delta(0)}\int_0^u \frac{\lambda}{c}\mathrm{d}\delta(z) \\
&= -\frac{\lambda}{c}\boldsymbol{a}\exp\{\boldsymbol{S}(x+y)\}\boldsymbol{S}\boldsymbol{e}'\frac{1-\psi(u)}{1-\psi(0)} \\
&= -\frac{\lambda}{c}\boldsymbol{a}\exp\{\boldsymbol{S}(x+y)\}\boldsymbol{S}\boldsymbol{e}'\frac{1+\dfrac{\lambda}{c}\boldsymbol{a}\boldsymbol{S}^{-1}\exp\{u\boldsymbol{B}\}\boldsymbol{e}'}{1+\dfrac{\lambda}{c}\boldsymbol{a}\boldsymbol{S}^{-1}\boldsymbol{e}'},
\end{aligned} \tag{3.6}$$

and for $u > x$ we obtain

$$\begin{aligned}
\phi(u,\ x,\ y) &= \frac{p(x+y)}{\delta(0)}\int_{u-x}^u \frac{\lambda}{c}\mathrm{d}\delta(z) \\
&= -\frac{\lambda}{c}\boldsymbol{a}\exp\{\boldsymbol{S}(x+y)\}\boldsymbol{S}\boldsymbol{e}'\frac{\psi(u-x)-\psi(u)}{1-\psi(0)} \\
&= -\frac{\lambda}{c}\boldsymbol{a}\exp\{\boldsymbol{S}(x+y)\}\boldsymbol{S}\boldsymbol{e}'\frac{-\dfrac{\lambda}{c}\boldsymbol{a}\boldsymbol{S}^{-1}\exp\{(u-x)\boldsymbol{B}\}\boldsymbol{e}'+\dfrac{\lambda}{c}\boldsymbol{a}\boldsymbol{S}^{-1}\exp\{u\boldsymbol{B}\}\boldsymbol{e}'}{1+\dfrac{\lambda}{c}\boldsymbol{a}\boldsymbol{S}^{-1}\boldsymbol{e}'}.
\end{aligned} \tag{3.7}$$

4 Example

In this section, we illustrate the application of the results of the previous section with an example. We comment that the computation of matrix exponentials is a simple task with the aid of software. The results in this section can be readily obtained using packages such as Mathematica.

We consider that individual claim amount $X_i \sim \mathrm{PH}(\boldsymbol{a},\ \boldsymbol{S})$ with $\boldsymbol{a} = \left(\frac{1}{2},\ \frac{1}{2}\right)$ and $\boldsymbol{S} = \begin{pmatrix} -3 & 0 \\ 0 & -7 \end{pmatrix}$. In this case, the distribution is an equal mixture of two exponentials at rates 3 and 7, respectively, where $\{N(t)\}_{t\geq 0}$ is a Poisson process with rate $\lambda = 1$ and the rate of premium income per unit time $c = \frac{1}{3}$. From (3.4), we have

$$B = S + \frac{\lambda}{c} S e' a S^{-1} = \begin{pmatrix} \dfrac{-3}{2} & \dfrac{9}{14} \\ \dfrac{7}{2} & \dfrac{-11}{2} \end{pmatrix}.$$

The matrix exponential $\exp\{uB\}$ can be calculated as

$$\exp\{uB\} = \begin{pmatrix} \dfrac{9}{10}\mathrm{e}^{-u} + \dfrac{1}{10}\mathrm{e}^{-6u} & \dfrac{9}{70}\mathrm{e}^{-u} - \dfrac{9}{70}\mathrm{e}^{-6u} \\ \dfrac{7}{10}\mathrm{e}^{-u} - \dfrac{7}{10}\mathrm{e}^{-6u} & \dfrac{1}{10}\mathrm{e}^{-u} + \dfrac{9}{10}\mathrm{e}^{-6u} \end{pmatrix}.$$

From (3.5)–(3.7), we have

$$\phi(0,\ x,\ y) = \frac{3}{2}(7\mathrm{e}^{-7(x+y)} + 3\mathrm{e}^{-3(x+y)}),$$

$$\phi(u,\ x,\ y) = \frac{3}{20}(7\mathrm{e}^{-7(x+y)} + 3\mathrm{e}^{-3(x+y)})(35 - \mathrm{e}^{-6u}(1 + 24\mathrm{e}^{5u})), \qquad 0 \le u \le x,$$

and

$$\phi(u,\ x,\ y) = \frac{3}{20}(7\mathrm{e}^{-7(x+y)} + 3\mathrm{e}^{-3(x+y)})(\mathrm{e}^{-6u+x}(\mathrm{e}^{5x} + 24\mathrm{e}^{5u}) - \mathrm{e}^{-6u}(1 + 24\mathrm{e}^{5u})), \quad u > x.$$

Thus, we have the defective density function of the surplus prior to ruin $h(u,x)$ and the defective density functions of the deficit at ruin $g(u,y)$, namely,

$$\begin{aligned} h(u,x) &= \int_0^\infty \phi(u,\ x,\ y)\mathrm{d}y \\ &= \begin{cases} \dfrac{3\mathrm{e}^{-6u-7x}(-1 - 24\mathrm{e}^{5u} + 35\mathrm{e}^{6u})(1 + \mathrm{e}^{4x})}{20}, & 0 \le u \le x; \\ \dfrac{3\mathrm{e}^{-6u-7x}(-1 - \mathrm{e}^{4x} + \mathrm{e}^{6x} + \mathrm{e}^{10x} + 24\mathrm{e}^{5u}(-1 + \mathrm{e}^{x})(1 + \mathrm{e}^{4x}))}{20}, & u > x, \end{cases} \end{aligned}$$

and

$$g(u,\ y) = \int_0^\infty \phi(u,\ x,\ y)\mathrm{d}x = \frac{3\mathrm{e}^{-6u-7y}(3 + 2\mathrm{e}^{5u} - \mathrm{e}^{4y} + 6\mathrm{e}^{5u+4y})}{10}.$$

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