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On Some Varieties of Soluble Lie Algebras*

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Abstract: In this paper, we study a class of soluble Lie algebras with variety relations that the commutator of m and n is zero. The aim of the paper is to consider the relationship between the Lie algebra L with the variety relations and the Lie algebra L which satisfies the permutation variety relations for the permutation φ of $\{3, \cdots, k\}$.

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1 Introduction

There are many parallel results between groups and Lie algebras. We can translate some results from groups to Lie algebras. For example, Macdonald[1] discussed some varieties of groups, particularly, some varieties associated with nilpotent groups in 1961, and then Suthathip[2] showed the similar varieties for nilpotent Lie algebras. In this paper, we extend similar varieties in [3] to soluble Lie algebras.

Let L be a Lie algebra, and $x_1, x_2, \cdots, x_n \in L$. The commutator $[x_1, x_2, \cdots, x_n]$ in L is defined by

$$[x_1, x_2] = [x_1, x_2]$$

and

$$[x_1, x_2, \cdots, x_{n-1}, x_n] = [[x_1, x_2, \cdots, x_{n-1}], x_n], \qquad n \geq 2. \tag{1.1}$$

Moreover, we define

$$[x_1, x_2, \cdots, x_m;\ y_1, y_2, \cdots, y_n] = [[x_1, x_2, \cdots, x_m],\ [y_1, y_2, \cdots, y_n]]$$

for any integers m and n. We say that the Lie algebra L is variety $[m, n] = 0$ if it satisfies

$$[[x_1, x_2, \cdots, x_m],\ [y_1, y_2, \cdots, y_n]] = 0, \qquad x_i, y_j \in L.$$

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If a Lie algebra L satisfies $[x_1, x_2, \cdots, x_k] = [x_1, x_2, x_{\varphi(3)}, \cdots, x_{\varphi(k)}]$, where φ is a permutation of $\{3, \cdots, k\}$, then we call that L satisfies $C(k, \varphi)$. If L satisfies $C(k, \varphi)$ for all permutations φ of $\{3, \cdots, k\}$, then we call that L satisfies $C(k)$.

The main result of this paper is that L satisfies $C(n+2)$ $(n \geq 2)$ if and only if L satisfies the law $[n-k, 2+k] = 0$ for all $k = 0, 1, \cdots, n-2$. Then it is easy to see that [3,2]=0 is equivalent to $C(5)$. Furthermore, $[n, 2] = 0$ $(n \geq 3)$ implies $C(2n-1)$. However, the law $[m, n] = 0$ does not imply any nontrivial law $C(k, \varphi)$ for $m, n \geq 3$.

2 The Lie Algebra with Varieties $[m, n] = 0$

Now we want to introduce some properties of the Lie algebra with variety $[m, n] = 0$. Denote by (x) a subalgebra generated by x.

Definition 2.1 *Let L be a Lie algebra. We define the sequence $\{L^n\}_{n\geq 1}$ by*

$$L^1 = L, \qquad L^{n+1} = [L, L^n], \qquad n \geq 1.$$

If $L^{m+1} = 0$, $L^m \neq 0$ for some m, then we say that L has nilpotent class precisely m.

Lemma 2.1[4] *Let A be an associative algebra. Then the following identities hold*:

(1) $(\mathrm{ad}c)^m(a) = \sum\limits_{0\leq j\leq m} (-1)^{m-j} \binom{m}{j} c^j a c^{m-j}$ for all $a, c \in A$;

(2) $[ab, c] = [a, c]b + a[b, c]$ for all $a, b, c \in A$.

Lemma 2.2[3] *If L satisfies $[n, m] = 0$, then $[n+p, m+q] = 0$ for any nonnegative numbers p and q.*

Lemma 2.3[5] *If L satisfies $C(k, \varphi_1)$ and $C(k, \varphi_2)$, then L satisfies $C(k, \varphi)$ for any φ in the group generated by φ_1 and φ_2.*

Lemma 2.4[5] *If L satisfies $C(k)$, then L satisfies $C(m)$ for all $m \geq k$.*

Lemma 2.5 *Let L be a Lie algebra. Then $[a, [x, y]] = 0$ if and only if $[a, x, y] = [a, y, x]$ for any $a, x, y \in L$.*

Proof. It is easily checked by Jacobian identity.

Lemma 2.6 *Let L be a Lie algebra with variety $[n, 2] = 0$ $(n \geq 2)$. If $L/Z(L)$ satisfies $C(n+1)$, then L satisfies $C(n+2)$.*

Proof. By Lemma 2.5, we know that L satisfies $C(n+2, \varphi_1)$ for $\varphi_1 = (n+1, n+2)$. Since $L/Z(L)$ satisfies $C(n+1)$, in particular, it satisfies $C(n+1, \varphi_2)$ for $\varphi_2 = (3, 4, \cdots, n+1)$. Thus, for any $x_1, x_2, \cdots, x_{n+1} \in L$, we have

$$[x_1, x_2, x_3, \cdots, x_{n+1}] - [x_1, x_2, x_{\varphi_2(3)}, \cdots, x_{\varphi_2(n+1)}] \in Z(L),$$

and also

$$[x_1, x_2, \cdots, x_{n+1}, x_{n+2}] = [x_1, x_2, x_{\varphi_2(3)}, \cdots, x_{\varphi_2(n+1)}, x_{\varphi_2(n+2)}]$$

for any $x_{n+2} \in L$. That is, L satisfies $C(n+2, \varphi_2)$. Since $S_n = \langle \varphi_1, \varphi_2 \rangle$, by Lemma 2.3, we know that L satisfies $C(n+2)$.

Lemma 2.7 *If L satisfies $[n,m]=0$ and $[n-1,m+1]=0$, then $L/Z(L)$ satisfies $[n-1,m]=0$. Particularly, if L satisfies $[n,n-1]=0$, then $L/Z(L)$ satisfies $[n-1,n-1]=0$.*

Proof. Let

$$\begin{aligned}\bar{a}&=a+Z(L)=[x_1,x_2,\cdots,x_{n-1}]+Z(L),\\ \bar{b}&=b+Z(L)=[y_1,y_2,\cdots,y_m]+Z(L),\end{aligned}\qquad x_i,y_j\in L.$$

By Jacobian identity, we know that $[a,b,z]=[a,z,b]+[a,[b,z]]$ for any $z\in L$. By the hypothesis, L satisfies $[n,m]=0$ and $[n-1,m+1]=0$. Thus we have $[a,z,b]=0$ and $[a,[b,z]]=0$. So $[a,b]\in Z(L)$ and $[\bar{a},\bar{b}]=[a,b]+Z(L)=\bar{0}$. That is, $L/Z(L)$ satisfies $[n-1,m]=0$.

Lemma 2.8 *If L satisfies $[n,m]=0$ and $L/Z(L)$ satisfies $[n-1,m]=0$, then L satisfies $[m+1,n-1]=0$.*

Proof. Let $a=[x_1,x_2,\cdots,x_{m+1}],\ b=[y_1,y_2,\cdots,y_{n-1}]$ for all $x_i,y_j\in L$. Then

$$\begin{aligned}[a,b]&=-[b,a]\\ &=-[b,[x_1,x_2,\cdots,x_m],x_{m+1}]\\ &=-[b,[x_1,x_2,\cdots,x_m],x_{m+1}]+[b,x_{m+1},[x_1,x_2,\cdots,x_m]].\end{aligned}$$

Since $L/Z(L)$ satisfies $[n-1,m]=0$, we have $[b,[x_1,x_2,\cdots,x_m]]\in Z(L)$. Hence,

$$[b,[x_1,x_2,\cdots,x_m],x_{m+1}]=0.$$

Furthermore, since L satisfies $[n,m]=0$, we have

$$[b,x_{m+1},[x_1,x_2,\cdots,x_m]]=0.$$

Therefore, $[a,b]=0$, that is, L satisfies $[m+1,n-1]=0$.

3 Some Cases for Small m and n

In this section, we consider the construction of the Lie algebra L with variety $[m,n]=0$ for small m and n.

Theorem 3.1 *L satisfies $[3,2]=0$ if and only if L satisfies $C(5)$.*

Proof. If L satisfies $[3,2]=0$, then by Lemma 2.7, $L/Z(L)$ satisfies $[2,2]=0$. Thus, $L/Z(L)$ satisfies $C(4)$. Thereby, by Lemma 2.6, the result is true.

Conversely, if L satisfies $C(5)$, in particular, L satisfies $C(5,\varphi_1)$ for $\varphi_1=(4,5)$. Using Lemma 2.5, the result follows.

Corollary 3.1 *L satisfies $C(n)$ if and only if L satisfies $C(n,\varphi_i)$ $(i=1,2)$ for $\varphi_1=(4,5,\cdots,n-1)$ and $\varphi_2=(n-1,n)$, where $n\geq 5$.*

Proof. If L satisfies $C(n)$, then it is easy to see that L satisfies $C(n,\varphi_i),\ i=1,2$.

Conversely, let L satisfy $C(n,\varphi_i),\ i=1,2$. We proceed by induction on n. If $n=5$, it is the result in Theorem 3.1. By Lemma 2.3, L satisfies $C(n,\varphi)$ for any φ such that $\varphi(3)=3$.

Since L satisfies $C(n,\varphi_1)$, we have $[x_1,x_2,x_3,\cdots,x_{n-1},x_n]=[x_1,x_2,x_{\varphi_1(3)},\cdots,x_{\varphi_1(n-1)},x_n]$ for any $x_n\in L$. Hence,

$$[x_1,x_2,x_3,\cdots,x_{n-1}]-[x_1,x_2,x_{\varphi_1(3)},\cdots,x_{\varphi_1(n-1)}]\in Z(L).$$

That is, $L/Z(L)$ satisfies $C(n-1,\varphi_1)$. Similarly, we can show that $L/Z(L)$ satisfies $C(n-1,\varphi_3)$ for $\varphi_3=(n-2,n-1)\in\langle\varphi_1,\varphi_2\rangle$. By induction on n, $L/Z(L)$ satisfies $C(n-1)$. Thus, by Lemma 2.6, L satisfies $C(n)$.

Theorem 3.2 *L satisfies $C(n+2)$ $(n\geq 2)$ if and only if L satisfies $[n-k,2+k]=0$ for all $k=0,1,\cdots,n-2$.*

Proof. Induction on n. In the cases of $n=2$ and $n=3$, it has been proved in Theorem 3.1. Now, we assume $n>3$. If L satisfies $C(n+2)$, then L satisfies $[n,2]=0$ by Lemma 2.5. Furthermore, L satisfies $C(n+2,\varphi)$ for any φ which fixes $n+2$. Thus,

$$[x_1,x_2,x_3,\cdots,x_n,x_{n+1},x_{n+2}]=[x_1,x_2,x_{\varphi(3)},\cdots,x_{\varphi(n)},x_{\varphi(n+1)},x_{n+2}]$$

for any $x_{n+2}\in L$, and then

$$[x_1,x_2,x_3,\cdots,x_n,x_{n+1}]-[x_1,x_2,x_{\varphi(3)},\cdots,x_{\varphi(n)},x_{\varphi(n+1)}]\in Z(L).$$

So we know that $L/Z(L)$ satisfies $C(n+1)$. Then, by the hypothesis of induction on n, $L/Z(L)$ satisfies $[n-1-k,2+k]=0$ for any nonnegative integer k such that $n-1-k\geq 2$. Finally, by Lemma 2.8 and $[n,2]=0$, we know that L satisfies $[n-k,2+k]=0$ for all $k=0,1,\cdots,n-2$.

Conversely, let L satisfy $[n-k,2+k]=0$ for all nonnegative integers $k\leq n-2$ and assume by induction that if L satisfies $[n-1-k,2+k]=0$ for all $0\leq k\leq n-3$, then L satisfies $C(n+1)$. Since L satisfies $[n-k,2+k]=[n-k-1,2+k+1]=0$, and $L/Z(L)$ satisfies $[n-k-1,2+k]=0$ by Lemma 2.7, $L/Z(L)$ satisfies $C(n+1)$. Hence, L satisfies $C(n+2)$ by Lemma 2.6.

Remark 3.1 By the anticommutativity of Lie bracket, we know that L satisfies $[n,m]=0$ if and only if L satisfies $[m,n]=0$. Thus, we can replace Theorem 3.2 by the following result: L satisfies $C(n+2)$ if and only if L satisfies $[n,2]=[n-1,3]=\cdots=[n-s,2+s]=0$, where $2s=n-2$ if n is even and $2s=n-3$ if n is odd.

Theorem 3.3 *If L satisfies*

$$[n,2]=[n-1,3]=\cdots=[n-k,2+k]=0 \tag{3.1}$$

for some $k<s$, then L satisfies $C(2n-2k-1)$.

Proof. Let L satisfy (3.1). Then, by Lemma 2.2, L satisfies

$$[2n-2k-3,2]=[2n-2k-4,3]=\cdots=[n-k,n-k-1]=0 \tag{3.2}$$

also. By Theorem 3.2, we know that L satisfies $C(2n-2k-1)$. This completes the proof.

In particular, for $k=0$, we get the following results.

Corollary 3.2 *If L satisfies $[n,2]=0$, then L satisfies $C(2n-1)$ for $n\geq 3$.*

Corollary 3.3 *If L satisfies $C(n,\varphi)$ $(n \geq 5)$ for all φ which leave fixed $3, \cdots, m$ $(m \leq n-2)$, or for any set of generators of the group of permutations of $\{m+1, \cdots, n\}$ $(m \geq 3)$, then L satisfies $C(n+m-3)$.*

Theorem 3.4 *Let L satisfy*

$$[n, 2] = [n-k_1, 2+k_1] = \cdots = [n-k_m, 2+k_m] = [n-s, 2+s] = 0, \tag{3.3}$$

where $0 \leq k_1 \leq k_2 \leq \cdots \leq k_m \leq s$ and s is defined in Remark 3.1. Then L satisfies $C(n+1+t)$, where $t = \max\{k_1, k_2-k_1, \cdots, k_m-k_{m-1}, s-k_m\}$.

Proof. Note that (3.1) implies (3.2), and (3.3) implies that L satisfies $[n+t-1, 2] = [n+t-2, 3] = \cdots = [2, n+t-1] = 0$. Hence, L satisfies $C(n+1+t)$ by Theorem 3.2. This completes the proof.

Next, we comment briefly on some results of the situation for

$$[x_1, x_2, \cdots, x_n] = [x_1, x_{\varphi(2)} \cdots, x_{\varphi(n)}]. \tag{3.4}$$

Theorem 3.5 *L satisfies (3.4) for all permutations φ of $\{2, \cdots, n\}$ if and only if L is a nilpotent of class $\leq n-1$.*

Proof. Use induction on n. For $n = 3$, by Lemma 2.5, $[x_1, x_2, x_3] = [x_1, x_3, x_2]$ if and only if $[x_1, [x_2, x_3]] = 0$. If L satisfies the hypotheses for $n > 3$, then L satisfies $[n-2, 2] = 0$ by Lemma 2.5 and L satisfies (3.4) for any φ which fixes n. Thus, as in the proof of Lemma 2.6, $L/Z(L)$ satisfies (3.4) when n is replaced by $n-1$ for any permutation φ of $\{2, \cdots, n-1\}$. Therefore, by induction, $L/Z(L)$ is a nilpotent of class $\leq n-2$ and L is a nilpotent of class $\leq n-1$.

The proof of the converse is trivial.

Now, we know that the law $[n, 2] = 0$ $(n \geq 3)$ implies $C(2n-1)$. The law $[n, 1] = 0$ means that nilpotence class n implies $C(n+1)$ trivially. However, the law $[m, n] = 0$ $(m, n \geq 3)$ does not imply $C(k, \varphi)$ for any k and any nontrivial φ. It suffices to show this for $[3, 3] = 0$.

Lemma 3.1 *If L satisfies $[3, 3] = 0$ and $C(n, \varphi)$ $(n \geq 4)$, where $\varphi(m) = 3$, $m \neq 3$, then the two-generator subalgebras of L satisfy $C(n+1)$.*

Proof. If $n = 4$, then it is easy to see that L satisfies $C(4)$. Now, let $n \geq 5$, $m = n$ and $H = (x, y)$, $x, y \in L$. Then, we show that H satisfies $[n-1, 2] = 0$. Since L satisfies $[3, 3] = 0$, it suffices to check that $[x_1, x_2, \cdots, x_{n-1}, [x, y]] = 0$. Since L satisfies $C(n, \varphi)$ and $\varphi(n) = 3$, we can assume that $\varphi^l(3) = n$. Thus

$$[x, y, x_3, \cdots, x_{n-1}, [x, y]] = [x, y, [x, y], x_{\varphi^l(4)}, \cdots, x_{\varphi^l(n)}] = 0.$$

If $x_1 = [x, y]$, then by Jacobian identity we have

$$\begin{aligned}[x_1, x_2, \cdots, x_{n-1}, [x, y]] &= [[x, y], x_3, x_2, \cdots, x_{n-1}, [x, y]] \\ &\quad + [[x, y], [x_2, x_3], x_4, \cdots, x_{n-1}, [x, y]].\end{aligned}$$

Since $\varphi(n) = 3$,

$$\begin{aligned}[[x,y],[x_2,x_3],x_4,\cdots,x_{n-1},[x,y]] &= [x,y,[x_2,x_3],x_4,\cdots,x_{n-1},[x,y]]\\ &= [x,y,[x,y],[x_2,x_3],x_4,\cdots,x_{n-1}]\\ &= 0.\end{aligned}$$

We have

$$[x_1,x_2,\cdots,x_{n-1},[x,y]] = [[x,y],x_3,x_2,\cdots,x_{n-1},[x,y]].$$

By the same way, we know that

$$\begin{aligned}[x_1,x_2,\cdots,x_{n-1},[x,y]] &= [[x,y],x_3,\cdots,x_{n-1},[x,y],x_2]\\ &= [[x,y],[x,y],x_3,\cdots,x_{n-1},x_2]\\ &= 0.\end{aligned}$$

Since $[3,3] = 0$, H satisfies $[n-1,2] = 0$. And since L satisfies $[n-2,3] = [n-3,4] = \cdots = [3,n-2] = 0$, by Theorem 3.2, H satisfies $C(n+1)$.

Now we consider the case of $m < n$. We proceed by induction on $n-m$. Suppose that the result is true for the case of $\varphi(m+1) = 3,\ m \neq 2$, then we need to consider the case of $\varphi(m) = 3$ $(m \neq 3)$. By the hypothesis of induction, we know that $H/Z(H)$ satisfies $C(n)$ and $[n-2,2] = 0$. Since H satisfies $[n-2,3] = 0$, by Lemma 2.8 we know that H satisfies $[n-1,2] = 0$. Thus $[3,3] = 0$ implies that H satisfies $C(n+1)$. This completes the proof.

Next, we give a Lie algebra which satisfies $[3,3] = 0$, but the subalgebra (x,y) does not satisfy $C(n+1)$ for any $n \geq 4$.

Let $A(Z,3)$ be the associative algebra of formal power series in the noncommuting variables $x,\ y,\ z$ with integer coefficients. Let $[r_1,r_2] = r_1r_2 - r_2r_1$. Then $A(Z,3)$ can be viewed as a Lie algebra. If the relation $r_1[r_2,r_3] = 0$ is added to $A(Z,3)$ for any monomials $r_i \in A(Z,3)$, whenever the degree (as monomial in $x,\ y,\ z$) of r_1 is ≥ 3, then the result that $A(Z,3)$ satisfies $[3,3] = 0$ follows.

Now, we only need to show that $[[r_1,r_2,r_3],[r_4,r_5,r_6]] = 0$ for any $r_i \in A(Z,3)$. Let $[r_1,r_2] = a$, $[r_4,r_5] = b$. Then we have

$$\begin{aligned}[[r_1,r_2,r_3],[r_4,r_5,r_6]] &= [[a,r_3],[b,r_6]]\\ &= [ar_3 - r_3a, br_6 - r_6b]\\ &= (ar_3 - r_3a)(br_6 - r_6b) - (br_6 - r_6b)(ar_3 - r_3a)\\ &= (ar_3br_6 - ar_3r_6b - r_3abr_6 + r_3ar_6b)\\ &\quad - (br_6ar_3 - br_6r_3a - r_6bar_3 + r_6br_3a).\end{aligned}$$

In the expression $(ar_3br_6 - ar_3r_6b - r_3abr_6 + r_3ar_6b)$, we replace a and b by $r_1r_2 - r_2r_1$ and $[r_4,r_5]$, respectively. And in the expression $(br_6ar_3 - br_6r_3a - r_6bar_3 + r_6br_3a)$, we replace a and b by $[r_1,r_2]$ and $r_4r_5 - r_5r_4$, respectively. Then we have $[[r_1,r_2,r_3],[r_4,r_5,r_6]] = 0$. That is, $A(Z,3)$ satisfies $[3,3] = 0$.

We show that the subalgebra $H = (x, y)$ does not satisfy $C(n+1)$ for $n \geq 4$. By Lemma 2.1, we know that

$$\begin{aligned}
[x, y, \underbrace{y, \cdots, y}_{n-3}, [x, y]] &= (-1)^{n-3}[(\mathrm{ad}y)^{n-3}([x, y]), [x, y]] \\
&= (-1)^{n-3}\Big[\sum_{0\leq j\leq n-3-j} (-1)^{n-3-j}\binom{n-3}{j} y^j [x, y] y^{n-3-j}, [x, y]\Big] \\
&= -[x, y]^2 y^{n-3} \\
&\neq 0.
\end{aligned}$$

By Lemma 2.5, we have

$$[x, y, \underbrace{y, \cdots, y}_{n-3}, x, y] \neq [x, y, \underbrace{y, \cdots, y}_{n-3}, y, x].$$

So (x, y) does not satisfy $C(n + 1)$ for any $n \geq 4$.

Hence, by Lemma 3.1, if $[3, 3] = 0$ implies $C(n, \varphi)$, then $\varphi(3) = 3$. Now we suppose that $[3, 3] = 0$ implies $C(m + n + 3, \varphi)$, where $\varphi(m + n + 3) = m + 3,\ n > 0$. Then, in the Lie algebra $A(Z, 3)$, $[x, y, \underbrace{x, \cdots, x}_{m+n}, y] = [x, y, \underbrace{x, \cdots, x}_{m}, y, \underbrace{x, \cdots, x}_{n}]$. Let $[x, y, \underbrace{x, \cdots, x}_{m}] = T$. Then

$$[T, \underbrace{x, \cdots, x}_{n}, y] = [T, y, \underbrace{x, \cdots, x}_{n}]. \tag{3.5}$$

So

$$[(\mathrm{ad}x)^n(T), y] = (\mathrm{ad}x)^n([T, y]). \tag{3.6}$$

By Lemma 2.1, we know that the equality (3.5) holds if and only if $n = 0$. Hence, we have proved the following remark.

Remark 3.2 The law $[m, n] = 0$ $(m, n \geq 3)$ does not imply $C(k, \varphi)$ for any $k \geq 4$ and nontrivial φ.

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