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Stability and Convergence of $L1$-Galerkin Spectral Methods for the Nonlinear Time Fractional Cable Equation

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Abstract. A numerical scheme for the nonlinear fractional-order Cable equation with Riemann-Liouville fractional derivatives is constructed. Using finite difference discretizations in the time direction, we obtain a semi-discrete scheme. Applying spectral Galerkin discretizations in space direction to the equations of the semi-discrete systems, we construct a fully discrete method. The stability and errors of the methods are studied. Two numerical examples verify the theoretical results.

AMS subject classifications: 35R11, 65M12, 65M70

Key words: Nonlinear fractional cable equation, spectral method, stability, error estimate.

1. Introduction

Fractional models provide a more detailed and comprehensive description of the memory, heredity, and non-locality. Therefore, fractional calculus is widely used in viscoelasticity and non-Newtonian fluid mechanics [28], fractional heat conduction [23], fractional Brownian models with stochastic volatility [20], and physics [6]. Theory and application of fractional calculus has gradually become a hot new issue [22].

In particular, recently the time fractional diffusion equations (TFDE) and fractional wave equations have been intensively studied both theoretically and numerically. Thus Schneider and Wyss [24, 31] analyzed fractional diffusion wave equations, whereas Sun *et al.* [27] investigated a fully discrete difference scheme for their solution. Liu *et al.* [15] studied the stability and convergence of discrete non-Markovian random walk approximations of TFDE obtained by a finite difference method. Langlands and Henry [7] established an

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implicit numerical scheme based on finite difference approximations for time fractional diffusion equations with the Riemann-Liouville fractional derivative. Sun *et al.* [26] used the Alikhanov's work [1], in order to study the stability and convergence of the discrete scheme for fractional wave equations. A new second-order midpoint approximation formula for the Riemann-Liouville derivative has been suggested in [3], and two implicit numerical methods for the fractional cable equation are considered in [16]. For time-fractional parabolic equations with nonsmooth solutions, Liu *et al.* [10] developed a numerical method based on thee change of variable $s = t^{\beta}$ and established an optimal error estimate of the $L1$ finite difference method. The unconditional stability and convergence of the fast difference scheme for a second-order multi-term time-fractional sub-diffusion problem are proved in [5].

Non-linear fractional equations and numerical methods of their solution are also studied. Thus high order methods for nonlinear fractional ordinary differential equations are developed in [12]. Liu *et al.* [17, 18] considered a finite element method combined with a finite difference scheme for a fourth-order nonlinear time fractional reaction-diffusion problem. Li *et al.* [9] considered $L1$-Galerkin FEMs for time-fractional nonlinear parabolic problems, whereas Duo and Zhang [4] studied numerical methods for the fractional nonlinear Schrödinger equation. The stability and convergence of an implicit numerical method for nonlinear fractional diffusion equations are analyzed in [13, 37], and finite element approximations for the nonlinear fractional Cable equation are discussed in [19, 29]. Li and Yi [8] constructed a discrete scheme for a two-dimensional nonlinear time-fractional subdiffusion equation, and Zhang and Jiang [36] developed an unconditional convergent numerical scheme for a two-dimensional nonlinear time fractional diffusion-wave equation. A compact difference scheme for nonlinear fourth order fractional sub-diffusion wave equation has been proposed in [21], and a linearised three-point combined compact difference method with weighted approximation for nonlinear time fractional Klein-Gordon equations is developed [35].

Spectral methods are important numerical tools in fractional differential equations [25]. Thus the spectral-collocation method for fractional integro-differential equations is studied in [33, 34]. Chen and Yang [32] considered a numerical method for nonlinear Volterra integro-differential equations. Wei and Chen [30] studied a Jacobi spectral approach to second kind multidimensional linear Volterra integral equations. Xu and Li [11] considered finite difference-spectral discretizations for the time fractional diffusion equation. A finite difference-spectral method for the fractional Cable equation is investigated in [14].

The fractional Cable equation is used in modelling of anomalous electro-diffusion in nerve cells. In the present work, we develop a numerical scheme for the nonlinear time fractional cable equation which is based on finite difference approximations in the time direction and a Galerkin spectral method in the space direction. The stability and the errors of the corresponding semi-discrete scheme are proved. Besides, we consider a fully discrete scheme and determine the related errors. Numerical examples verify the theoretical results.

This paper is organized as follows. In Section 2, we use finite difference approximations in the time direction and establish a semi-discrete scheme. After that, we employ Galerkin spectral approximations for the space direction and obtain a fully discrete scheme. Section 3 is devoted to the stability of the semi-discrete problem. The error analysis is presented in

Section 4. Two numerical examples considered in Section 5, verify the theoretical error estimates. A brief conclusion is given in Section 6.

2. Model Problem

In this section, we consider the nonlinear time fractional Cable equation, and construct a numerical discrete scheme.

2.1. Preliminaries

Let $s > 0$ be a real number such that $n-1 \le s < n$ and $f(t)$ be a function defined on the interval $[0, T]$.

The left and right Riemann-Liouville fractional derivatives are respectively defined by

$$
\begin{aligned}
{}_0^R D_t^s f(t) &:= \frac{1}{\Gamma(n-s)} \frac{d^n}{dt^n} \int_0^t \frac{f(\tau)}{(t-\tau)^{s-n+1}} d\tau, \qquad t \in (0,T), \\
{}_t^R D_T^s f(t) &= -\frac{(-1)^n}{\Gamma(n-s)} \frac{d^n}{dt^n} \int_t^T \frac{f(\tau)}{(\tau-t)^{s-n+1}} d\tau, \quad t \in (0,T).
\end{aligned}
$$

Besides, the left and right Caputo fractional derivatives are respectively defined by

$$
\begin{aligned}
{}_0^C D_t^s f(t) &= \frac{1}{\Gamma(n-s)} \int_0^t \frac{f^n(\tau)}{(t-\tau)^{s-n+1}} d\tau, \qquad t \in (0,T), \\
{}_t^C D_T^s f(t) &= \frac{(-1)^n}{\Gamma(n-s)} \int_t^T \frac{f^n(\tau)}{(\tau-t)^{s-n+1}} d\tau, \quad t \in (0,T).
\end{aligned}
$$

Lemma 2.1. *The Caputo and Riemann-Liouville fractional derivatives are connected as follows:*

1) If $\alpha \in [n-1, n)$, $n \in \mathbb{N}$, then

$$
{}_a^R D_t^\alpha f(t) = {}_a^C D_t^\alpha f(t) + \sum_{j=0}^{n-1} \frac{f^j(a)(t-a)^{j-\alpha}}{\Gamma(1+j-\alpha)}.
$$

2) If $0 < \alpha < 1$, then

$$
{}_0^R D_t^\alpha f(t) = {}_0^C D_t^\alpha f(t) + \frac{f(0)}{\Gamma(1-\alpha) t^\alpha}.
$$

2.2. Discrete scheme

Let $\Lambda = (-1, 1)$, $I = (0, T]$ and $\Omega := \Lambda \times I$. Consider the nonlinear time fractional Cable equation

$$
\begin{aligned}
&\partial_t u(x,t) = -\mu {}_0^R D_t^\alpha u(x,t) + {}_0^R D_t^\beta \partial_x^2 u(x,t) - \mathscr{F}(u) + g(x,t), \\
&u(x,0) = u_0(x), \qquad x \in \Lambda, \\
&u(-1,t) = u(1,t) = 0, \quad t \in I,
\end{aligned}
\tag{2.1}
$$

where $0 < \alpha, \beta < 1$. Here and in what follows, we assume that there exists a constant $C > 0$ such that

$$|\mathscr{F}(u)| \leq C|u|, \quad |\mathscr{F}'(u)| \leq C.$$

Now we use a finite difference scheme to discretize the above equation in the time direction and obtain a semi-discrete version of the problem. Given an integer $K > 0$, let $t_k = k\Delta t$, $k = 0, 1, \ldots, K$, where $\Delta t = T/K$ is the time step. Using the Taylor formula, we write

$$u(t) = u(s) + \partial_t u(s)(t-s) + \int_s^t \partial_\tau^2 u(\tau)(t-\tau)d\tau, \quad t, s \in I,$$

and if $0 \leq k \leq K-1$, we set

$$\begin{aligned}
D_t^\alpha u(x, t_{k+1}) :&= \frac{1}{\Gamma(1-\alpha)} \sum_{j=0}^{k} \int_{t_j}^{t_{j+1}} \partial_s u(x,s) \frac{ds}{(t_{k+1}-s)^\alpha} \\
&= \frac{1}{\Gamma(1-\alpha)} \sum_{j=0}^{k} \frac{u(x,t_{j+1}) - u(x,t_j)}{\Delta t} \int_{t_j}^{t_{j+1}} \frac{ds}{(t_{k+1}-s)^\alpha} + r_\alpha^{k+1} \\
&= \frac{1}{\Gamma(2-\alpha)} \sum_{j=0}^{k} \frac{u(x,t_{k+1-j}) - u(x,t_{k-j})}{\Delta t^\alpha} \left[(j+1)^{1-\alpha} - j^{1-\alpha} \right] + r_\alpha^{k+1} \\
&= \frac{1}{\Gamma(2-\alpha)} \sum_{j=0}^{k} a_j \frac{u(x,t_{k+1-j}) - u(x,t_{k-j})}{\Delta t^\alpha} + r_\alpha^{k+1},
\end{aligned}$$

where

$$a_j = (j+1)^{1-\alpha} - j^{1-\alpha}, \quad j = 0, 1, 2, \ldots, k, \tag{2.2}$$

and r_α^{k+1} is the truncation error. Note that

$$r_\alpha^{k+1} = \mathscr{O}\left(\Delta t^{2-\alpha}\right).$$

The proof this fact can be found in [11, 14].

Similar expression for the fractional derivative of order β has the form

$$\begin{aligned}
&D_t^\beta \partial_x^2 u(x, t_{j+1}) \\
&= \frac{1}{\Gamma(2-\beta)} \sum_{j=0}^{k} b_j \frac{\partial_x^2 u(x,t_{k+1-j}) - \partial_x^2 u(x,t_{k-j})}{\Delta t^\beta} + r_\beta^{k+1},
\end{aligned} \tag{2.3}$$

where

$$\begin{aligned}
&b_j = (j+1)^{1-\beta} - j^{1-\beta}, \quad j = 0, 1, 2, \ldots, k, \\
&r_\beta^{k+1} = \mathscr{O}\left(\Delta t^{2-\beta}\right).
\end{aligned} \tag{2.4}$$

The first-order time derivative $\partial_t u$ is discretized as follows:

$$\begin{aligned}
\partial_t u(x,t_1) &= \frac{u(x,t_1)-u(x,t_0)}{\Delta t} + r_1, && r_1 = \mathscr{O}(\Delta t),\\
\partial_t u(x,t_{k+1}) &= \frac{3u(x,t_{k+1})-4u(x,t_k)+u(x,t_{k-1})}{2\Delta t} + r_2, && r_2 = \mathscr{O}(\Delta t^2), \quad k \ge 1.
\end{aligned}$$

Let $\{y^k\}_{k=0}^K$ be a grid function. By L_t^α and L_t^1 we respectively denote fractional and first order differential operators defined by

$$L_t^\alpha y^{k+1} = \frac{1}{\Gamma(2-\alpha)\Delta t^\alpha}\sum_{j=0}^{k} a_j \left(y^{k+1-j} - y^{k-j}\right), \quad k \ge 0, \tag{2.5}$$

$$L_t^1 y^{k+1} = \begin{cases} \dfrac{y^1 - y^0}{\Delta t}, & k = 0,\\[2mm] \dfrac{3y^{k+1} - 4y^k + y^{k-1}}{2\Delta t}, & k \ge 1. \end{cases} \tag{2.6}$$

It follows from (2.2), (2.3) and (2.5) that

$$\begin{aligned}
D_t^\alpha u(x,t_{k+1}) &= L_t^\alpha u(x,t_{k+1}) + r_\alpha,\\
D_t^\beta u(x,t_{k+1}) &= L_t^\beta u(x,t_{k+1}) + r_\beta.
\end{aligned} \tag{2.7}$$

Using the operators (2.6) and (2.7), we write the problem (2.1) as

$$\begin{aligned}
L_t^1 u^{k+1} = &-\mu L_t^\alpha u^{k+1} + L_t^\beta \partial_x^2 u^{k+1}\\
&- \frac{\mu}{\Gamma(1-\alpha)(k+1)^\alpha \Delta t^\alpha} u(x,0) + \frac{1}{\Gamma(1-\beta)(k+1)^\beta \Delta t^\beta}\partial_x^2 u(x,0)\\
&- \mathscr{F}\left(u^{k+1}\right) + g^{k+1} + r^{k+1} - \mu r_\alpha^{k+1} + r_\beta^{k+1}, \quad k \ge 0,
\end{aligned}$$

where $r^{k+1} = \mathscr{O}(\Delta t)$ for $k = 0$ and $r^{k+1} = \mathscr{O}(\Delta t^2)$ for $k \ge 1$. Consequently, the finite difference scheme for the time discretization of (2.1) has the form

$$\begin{aligned}
L_t^1 u^{k+1} = &-\mu L_t^\alpha u^{k+1} + L_t^\beta \partial_x^2 u^{k+1}\\
&- \frac{\mu}{\Gamma(1-\alpha)(k+1)^\alpha \Delta t^\alpha} u^0 + \frac{1}{\Gamma(1-\beta)(k+1)^\beta \Delta t^\beta}\partial_x^2 u^0\\
&- \mathscr{F}(u^{k+1}) + g^{k+1}, \quad k \ge 0.
\end{aligned} \tag{2.8}$$

We write ${}_e u^k$ for the exact solution $u(x,t_k)$ of the problem (2.1) at the point (x,t_k) and u^k for the semi-discrete solution $u^k(x)$ obtained by the finite difference scheme. The truncation error of the semi-discrete scheme (2.8) is denoted by R^{k+1}, i.e.

$$R^{k+1} = r^{k+1} - \mu r_\alpha^{k+1} + r_\beta^{k+1}, \quad k \ge 0.$$

The semi-discrete scheme (2.8) can be written as

$$\frac{u^1-u^0}{\Delta t} = \frac{-\mu}{\Gamma(2-\alpha)\Delta t^\alpha}\left(u^1-u^0\right)+\frac{1}{\Gamma(2-\beta)\Delta t^\beta}\left(\partial_x^2 u^1-\partial_x^2 u^0\right) - \frac{\mu}{\Gamma(1-\alpha)\Delta t^\alpha}u^0+\frac{1}{\Gamma(1-\beta)\Delta t^\beta}\partial_x^2 u^0-\mathscr{F}(u^1)+g^1, \tag{2.9}$$

$$\begin{aligned}&\frac{3u^{k+1}-4u^k+u^{k-1}}{2\Delta t}\\ =&\frac{-\mu}{\Gamma(2-\alpha)\Delta t^\alpha}\sum_{j=0}^{k}a_j\left(u^{k+1-j}-u^{k-j}\right)+\frac{1}{\Gamma(2-\beta)\Delta t^\beta}\sum_{j=0}^{k}b_j\left(\partial_x^2 u^{k+1-j}-\partial_x^2 u^{k-j}\right)\\ &-\frac{\mu}{\Gamma(1-\alpha)(k+1)^\alpha\Delta t^\alpha}u^0+\frac{1}{\Gamma(1-\beta)(k+1)^\beta\Delta t^\beta}\partial_x^2 u^0-\mathscr{F}\left(u^{k+1}\right)+g^{k+1},\end{aligned} \tag{2.10}$$

and is supplemented by the following boundary and initial conditions:

$$\begin{aligned}&u^{k+1}(1)=u^{k+1}(-1)=0, && k\geq 0,\\ &u^0(x)=u_0, && x\in\Lambda.\end{aligned} \tag{2.11}$$

Consider the weak form of the semi-discrete problem (2.9)-(2.11). For this, we define the function spaces

$$\begin{aligned}&H^1(\Lambda):=\left\{v\in L^2(\Lambda),\partial_x v\in L^2(\Lambda)\right\},\\ &H_0^1(\Lambda):=\left\{v\in H^1(\Lambda),v|_{\partial\Lambda}=0\right\},\\ &H^m(\Lambda):=\left\{v\in L^2(\Lambda),\partial_x^k v\in L^2(\Lambda),\ 0\leq k\leq m,\ k\in N^+\right\}\end{aligned}$$

and the norms on $L^2(\Lambda)$ and $H^1(\Lambda)$ by

$$\|v\|_0=(v,v)^{\frac{1}{2}},\quad \|v\|_1=(v,v)_1^{\frac{1}{2}}.$$

For simplicity, we multiply the Eqs. (2.9) and (2.10) by Δt. Using the notation

$$\tilde{\alpha}=\frac{\mu\Delta t}{\Gamma(2-\alpha)\Delta t^\alpha},\qquad \tilde{\beta}=\frac{\Delta t}{\Gamma(2-\beta)\Delta t^\beta}, \tag{2.12}$$

$$\tilde{\alpha}_{k+1}=\frac{\mu\Delta t}{\Gamma(1-\alpha)(k+1)^\alpha\Delta t^\alpha},\quad \tilde{\beta}_{k+1}=\frac{\Delta t}{\Gamma(1-\beta)(k+1)^\beta\Delta t^\beta}, \tag{2.13}$$

we write the equations

$$\begin{aligned}\left(u^1-u^0,v\right)=&-\tilde{\alpha}\left(u^1-u^0,v\right)-\tilde{\beta}\left(\partial_x u^0-\partial_x u^1,\partial_x v\right)\\ &-\tilde{\alpha}_1\left(u^0,v\right)-\tilde{\beta}_1\left(\partial_x u^0,\partial_x v\right)-\Delta t\left(\mathscr{F}(u^1),v\right)+\Delta t\left(g^1,v\right),\end{aligned} \tag{2.14}$$

$$\begin{aligned}&2\left(3u^{k+1}-4u^k+u^{k-1},v\right)\\ =&-4\tilde{\alpha}\left[a_0\left(u^{k+1},v\right)-\sum_{j=0}^{k-1}\left(a_j-a_{j+1}\right)\left(u^{k-j},v\right)-a_k\left(u^0,v\right)\right]\end{aligned}$$

$$
\begin{aligned}
&-4\tilde{\beta}\left[b_0\left(\partial_x u^{k+1},\partial_x v\right)-\sum_{j=0}^{k-1}\left(b_j-b_{j+1}\right)\left(\partial_x u^{k-j},\partial_x v\right)-b_k\left(\partial_x u^0,\partial_x v\right)\right]\\
&-4\tilde{\alpha}_{k+1}\left(u^0,v\right)-4\tilde{\beta}_{k+1}\left(\partial_x u^0,\partial_x v\right)-4\Delta t\left(\mathscr{F}(u^{k+1}),v\right)\\
&+4\Delta t\left(g^{k+1},v\right),\quad k\geq 1,
\end{aligned}
\tag{2.15}
$$

which should be valid for all $v\in H_0^1(\Lambda)$.

Using the semi-discrete problem, we apply the Galerkin spectral method to discretize the space derivative. Let $P_N(\Lambda)$ denote the set of all polynomials $p(x)$ of the degree at most N and let

$$
P_N^0(\Lambda)=H_0^1(\Lambda)\cap P_N(\Lambda).
$$

Consider the spectral Galerkin method for (2.14) and (2.15). More precisely, we want to find $u_N^{k+1}\in P_N^0(\Lambda)$ such that for all $v_N\in P_N^0(\Lambda)$ the following equations hold:

$$
\begin{aligned}
\left(u_N^1-u_N^0,v_N\right)=&-\tilde{\alpha}\left(u_N^1-u_N^0,v_N\right)-\tilde{\beta}\left(\partial_x u_N^0-\partial_x u_N^1,\partial_x v_N\right)-\tilde{\alpha}_1\left(u_N^0,v_N\right)\\
&-\tilde{\beta}_1\left(\partial_x u_N^0,\partial_x v_N\right)-\Delta t\left(\mathscr{F}\left(u_N^1\right),v_N\right)+\Delta t\left(g^1,v_N\right),
\end{aligned}
\tag{2.16}
$$

$$
\begin{aligned}
&2\left(3u_N^{k+1}-4u_N^k+u_N^{k-1},v_N\right)\\
=&-4\tilde{\alpha}\left[a_0\left(u_N^{k+1},v_N\right)-\sum_{j=0}^{k-1}\left(a_j-a_{j+1}\right)\left(u_N^{k-j},v_N\right)-a_k\left(u_N^0,v_N\right)\right]\\
&-4\tilde{\beta}\left[b_0\left(\partial_x u_N^{k+1},\partial_x v_N\right)-\sum_{j=0}^{k-1}\left(b_j-b_{j+1}\right)\left(\partial_x u_N^{k-j},\partial_x v_N\right)-b_k\left(\partial_x u_N^0,\partial_x v_N\right)\right]\\
&-4\tilde{\alpha}_{k+1}\left(u_N^0,v_N\right)-4\tilde{\beta}_{k+1}\left(\partial_x u_N^0,\partial_x v_N\right)-4\Delta t\left(\mathscr{F}\left(u_N^{k+1}\right),v_N\right)\\
&+4\Delta t\left(g^{k+1},v_N\right),\quad k\geq 1.
\end{aligned}
$$

The above scheme can be also written as

$$
\begin{aligned}
&2\left(3u_N^{k+1}-4u_N^k+u_N^{k-1},v_N\right)\\
=&-4\tilde{\alpha}\left(u_N^{k+1},v_N\right)+4\tilde{\alpha}\sum_{j=0}^{k-1}\left(a_j-a_{j+1}\right)\left(u_N^{k-j},v_N\right)+4\tilde{\alpha}a_k\left(u_N^0,v_N\right)\\
&-4\tilde{\beta}\left(\partial_x u_N^{k+1},\partial_x v_N\right)+4\tilde{\beta}\sum_{j=0}^{k-1}\left(b_j-b_{j+1}\right)\left(\partial_x u_N^{k-j},\partial_x v_N\right)\\
&+4\tilde{\beta}b_k\left(\partial_x u_N^0,\partial_x v_N\right)-4\tilde{\alpha}_{k+1}\left(u_N^0,v_N\right)-4\tilde{\beta}_{k+1}\left(\partial_x u_N^0,\partial_x v_N\right)\\
&-4\Delta t\left(\mathscr{F}\left(u_N^{k+1}\right),v_N\right)+4\Delta t\left(g^{k+1},v_N\right),
\end{aligned}
\tag{2.17}
$$

and we arrive at fully discrete scheme.

3. Stability Analysis

Let us prove the stability of the semi-discrete problem. For this, we have to recall auxiliary results.

Lemma 3.1 (cf. Lin *et al.* [14]). *The coefficients a_j and b_j in the Eqs.* (2.2), (2.4) *satisfy the relations*

$$\begin{aligned}
&a_j > 0, \quad b_j > 0, && j = 0, 1, \ldots, K, \\
&1 = a_0 > a_1 > \cdots > a_j, && a_j \to 0 \quad as \quad j \to \infty, \\
&1 = b_0 > b_1 > \cdots > b_j, && b_j \to 0 \quad as \quad j \to \infty.
\end{aligned}$$

Lemma 3.2 (cf. Lin *et al.* [14]). *The coefficients $\tilde{\alpha}, \tilde{\beta}$ and $\tilde{\alpha}_{k+1}, \tilde{\beta}_{k+1}$ in the Eqs.* (2.12), (2.13) *satisfy the relations*

$$\begin{aligned}
&\tilde{\alpha} a_{k+1} \leq \tilde{\alpha}_{k+1} \leq \tilde{\alpha} a_k, \\
&\tilde{\beta} b_{k+1} \leq \tilde{\beta}_{k+1} \leq \tilde{\beta} b_k.
\end{aligned}$$

Lemma 3.3 (cf. Lin *et al.* [14]). *The second-order backward Euler difference scheme satisfies the equation energy norm*

$$\begin{aligned}
&2\left(3u^{k+1} - 4u^k + u^{k-1}, u^{k+1}\right) \\
&= \|u^{k+1}\|_0^2 - \|u^k\|_0^2 + \|2u^{k+1} - u^k\|_0^2 - \|2u^k - u^{k-1}\|_0^2 \\
&\quad + \|u^{k+1} - 2u^k + u^{k+1}\|_0^2,
\end{aligned}$$

where $\|\cdot\|_0$ is the energy norm.

Now we can present a stability result.

Theorem 3.1. *For all sufficiently small Δt, the semi-discrete problem is unconditionally stable and*

$$\begin{aligned}
&\|u^1\|_0^2 + \tilde{\alpha} \sum_{j=0}^{1} a_j \|u^{1-j}\|_0^2 + \tilde{\beta} \sum_{j=0}^{1} b_j \|\partial_x u^{1-j}\|_0^2 \\
&\leq \|u^0\|_0^2 + \tilde{\alpha} a_0 \|u^0\|_0^2 + \tilde{\beta} b_0 \|\partial_x u^0\|_0^2 + C\Delta t \|g^1\|_0^2, \quad k = 0, && (3.1) \\
&S^{k+1} \leq S^k + C\Delta t \|g^{k+1}\|_0^2, \qquad k = 1, \ldots, K-1, && (3.2)
\end{aligned}$$

where

$$S^k = \|u^k\|_0^2 + \|2u^k - u^{k-1}\|_0^2 + 2\tilde{\alpha} \sum_{j=0}^{k} a_j \|u^j\|_0^2 + 2\tilde{\beta} \sum_{j=0}^{k} b_j \|\partial_x u^j\|_0^2, \quad k \geq 1.$$

Proof. Consider first the case $k = 0$. It follows from (2.14) that for all $v \in H_0^1(\Lambda)$ we have

$$\begin{aligned}\left(u^1-u^0,v\right) = &-\tilde{\alpha}\left(u^1-u^0,v\right)-\tilde{\beta}\left(\partial_x u^1-\partial_x u^0,\partial_x v\right)-\tilde{\alpha}_1\left(u^0,v\right)\\ &-\tilde{\beta}_1\left(\partial_x u^0,\partial_x v\right)-\Delta t\left(\mathscr{F}(u^1),v\right)+\Delta t(g^1,v). \end{aligned}\tag{3.3}$$

Choosing $v = u^1$ in (3.3) gives

$$\begin{aligned}\left(u^1,u^1\right) = &\left(u^0,u^1\right)-\tilde{\alpha}\left(u^1,u^1\right)+(\tilde{\alpha}-\tilde{\alpha}_1)\left(u^0,u^1\right)-\tilde{\beta}\left(\partial_x u^1,\partial_x u^1\right)\\ &+\left(\tilde{\beta}-\tilde{\beta}_1\right)\left(\partial_x u^0,\partial_x u^1\right)-\Delta t\left(\mathscr{F}(u^1),u^1\right)+\Delta t\left(g^1,u^1\right).\end{aligned}$$

Since

$$\tilde{\alpha}-\tilde{\alpha}_1=\tilde{\alpha}-\tilde{\alpha}a_1=\tilde{\alpha}(1-a_1),$$

the Cauchy-Schwarz and Young's inequalities, Lemmas 3.1 and 3.2, and the condition imposed on $\mathscr{F}(u)$ yield

$$\begin{aligned}2\|u^1\|_0^2 \le\, &\|u^0\|_0^2+\|u^1\|_0^2-2\tilde{\alpha}\|u^1\|_0^2+(\tilde{\alpha}-\tilde{\alpha}a_1)\|u^0\|_0^2+(\tilde{\alpha}-\tilde{\alpha}a_1)\|u^1\|_0^2\\ &-2\tilde{\beta}\|\partial_x u^1\|_0^2+(\tilde{\beta}-\tilde{\beta}b_1)\|\partial_x u^0\|_0^2+(\tilde{\beta}-\tilde{\beta}b_1)\|\partial_x u^1\|_0^2\\ &-\Delta t\left(\mathscr{F}(u^1),u^1\right)+\Delta t\left(g^1,u^1\right)\\ =\, &\|u^0\|_0^2+\|u^1\|_0^2-\tilde{\alpha}\left(a_0\|u^1\|_0^2+a_1\|u^0\|_0^2\right)\\ &-\tilde{\beta}\left(b_0\|\partial_x u^1\|_0^2+b_1\|\partial_x u^0\|_0^2\right)+\tilde{\alpha}a_0\|u^0\|_0^2+\tilde{\beta}b_0\|\partial_x u^0\|_0^2\\ &-\tilde{\alpha}a_1\|u^1\|_0^2-\tilde{\beta}b_1\|\partial_x u^1\|_0^2+C\Delta t\|u^1\|_0^2+C\Delta t\|g^1\|_0^2.\end{aligned}\tag{3.4}$$

Recalling that $a_0 = b_0 = 1$, we obtain

$$\begin{aligned}&\|u^1\|_0^2+\tilde{\alpha}\left(a_0\|u^1\|_0^2+a_1\|u^0\|_0^2\right)+\tilde{\beta}\left(b_0\|\partial_x u^1\|_0^2+b_1\|\partial_x u^0\|_0^2\right)\\ &\quad+\tilde{\alpha}a_1\|u^1\|_0^2+\tilde{\beta}b_1\|\partial_x u^1\|_0^2\\ &\le \|u^0\|_0^2+\tilde{\alpha}a_0\|u^0\|_0^2+\tilde{\beta}b_0\|\partial_x u^0\|_0^2+C\Delta t\|u^1\|_0^2+C\Delta t\|g^1\|_0^2.\end{aligned}$$

This inequality can we written as

$$\begin{aligned}&(1-C\Delta t)\|u^1\|_0^2+\tilde{\alpha}\sum_{j=0}^{1}a_j\|u^{1-j}\|_0^2+\tilde{\beta}\sum_{j=0}^{1}b_j\|\partial_x u^{1-j}\|_0^2\\ &\quad+\tilde{\alpha}a_1\|u^1\|_0^2+\tilde{\beta}b_1\|\partial_x u^1\|_0^2\\ &\le \|u^0\|_0^2+\tilde{\alpha}a_0\|u^0\|_0^2+\tilde{\beta}b_0\|\partial_x u^0\|_0^2+C\Delta t\|g^1\|_0^2.\end{aligned}$$

It follows that for sufficiently small Δt, the estimate (3.4) yields

$$\begin{aligned}&\|u^1\|_0^2+\tilde{\alpha}\sum_{j=0}^{1}a_j\|u^{1-j}\|_0^2+\tilde{\beta}\sum_{j=0}^{1}b_j\|\partial_x u^{1-j}\|_0^2+\tilde{\alpha}a_1\|u^1\|_0^2+\tilde{\beta}b_1\|\partial_x u^1\|_0^2\\ &\le \|u^0\|_0^2+\tilde{\alpha}a_0\|u^0\|_0^2+\tilde{\beta}b_0\|\partial_x u^0\|_0^2+C\Delta t\|g^1\|_0^2.\end{aligned}\tag{3.5}$$

For $\Delta t > 0$ the last two terms in the left-hand side of (3.5) are positive, and (3.1) is proved.

Assume now that $k \geq 1$ and choose $v = u^{k+1}$ in (2.15). The triangle inequality along with Lemmas 3.2 and 3.3 give

$$\begin{aligned}
&\|u^{k+1}\|_0^2 - \|u^k\|_0^2 + \|2u^{k+1} - u^k\|_0^2 - \|2u^k - u^{k-1}\|_0^2 + \|u^{k+1} - 2u^k + u^{k-1}\|_0^2 \\
\leq &\left[-4\tilde{\alpha} + 2\tilde{\alpha}\sum_{j=0}^{k-1}(a_j - a_{j+1}) + 2\tilde{\alpha}a_k\right]\|u^{k+1}\|_0^2 + 2\tilde{\alpha}\sum_{j=0}^{k-1}(a_j - a_{j+1})\|u^{k-j}\|_0^2 \\
&+ 2(\tilde{\alpha}a_k - \tilde{\alpha}a_{k+1})\|u^0\|_0^2 - 2\tilde{\alpha}a_{k+1}\|u^{k+1}\|_0^2 \\
&+ \left[-4\tilde{\beta} + 2\tilde{\beta}\sum_{j=0}^{k-1}(b_j - b_{j+1}) + 2\tilde{\beta}b_k\right]\left\|\partial_x u^{k+1}\right\|_0^2 \\
&+ 2\tilde{\beta}\sum_{j=0}^{k-1}(b_j - b_{j+1})\left\|\partial_x u^{k-j}\right\|_0^2 + 2\left(\tilde{\beta}b_k - \tilde{\beta}b_{k+1}\right)\left\|\partial_x u^0\right\|_0^2 \\
&- 2\tilde{\beta}_{k+1}\left\|\partial_x u^{k+1}\right\|_0^2 + c_1\Delta t\|u^{k+1}\|_0^2 + C\Delta t\|g^{k+1}\|_0^2. \qquad (3.6)
\end{aligned}$$

Noting that

$$\sum_{j=0}^{k-1}(a_j - a_{j+1}) = 1 - a_k, \quad \sum_{j=0}^{k-1}(b_j - b_{j+1}) = 1 - b_k,$$

and summing the corresponding expressions with $\|u^{k+1}\|_0^2$ and $\|\partial_x u^{k+1}\|_0^2$ in the right-hand side of (3.6), we obtain

$$\begin{aligned}
&\left[-4\tilde{\alpha} + 2\tilde{\alpha}\sum_{j=0}^{k-1}(a_j - a_{j+1}) + 2\tilde{\alpha}a_k\right]\|u^{k+1}\|_0^2 = -2\tilde{\alpha}\|u^{k+1}\|_0^2, \\
&\left[-4\tilde{\beta} + 2\tilde{\beta}\sum_{j=0}^{k-1}(b_j - b_{j+1}) + 2\tilde{\beta}b_k\right]\left\|\partial_x u^{k+1}\right\|_0^2 = -2\tilde{\beta}\left\|\partial_x u^{k+1}\right\|_0^2.
\end{aligned}$$

Therefore, the inequality (3.6) takes the form

$$\begin{aligned}
&\|u^{k+1}\|_0^2 - \|u^k\|_0^2 + \|2u^{k+1} - u^k\|_0^2 - \|2u^k - u^{k-1}\|_0^2 + \|u^{k+1} - 2u^k + u^{k-1}\|_0^2 \\
\leq &-2\tilde{\alpha}\|u^{k+1}\|_0^2 + 2\tilde{\alpha}\sum_{j=0}^{k-1}a_j\|u^{k-j}\|_0^2 - 2\tilde{\alpha}\sum_{j=1}^{k}a_j\|u^{k+1-j}\|_0^2 \\
&- 2\tilde{\alpha}_{k+1}\|u^{k+1}\|_0^2 + 2\tilde{\alpha}a_k\|u^0\|_0^2 - 2\tilde{\alpha}a_{k+1}\|u^0\|_0^2 - 2\tilde{\beta}\left\|\partial_x u^{k+1}\right\|_0^2 \\
&+ 2\tilde{\beta}\sum_{j=0}^{k-1}b_j\left\|\partial_x u^{k-j}\right\|_0^2 - 2\tilde{\beta}\sum_{j=1}^{k}b_j\left\|\partial_x u^{k+1-j}\right\|_0^2 - 2\tilde{\beta}_{k+1}\left\|\partial_x u^{k+1}\right\|_0^2 \\
&+ 2\tilde{\beta}b_k\left\|\partial_x u^0\right\|_0^2 - 2\tilde{\beta}b_{k+1}\left\|\partial_x u^0\right\|_0^2 + C\Delta t\|u^{k+1}\|_0^2 + C\Delta t\|g^{k+1}\|_0^2 \\
= &-2\tilde{\alpha}\sum_{j=0}^{k+1}a_j\|u^{k+1-j}\|_0^2 + 2\tilde{\alpha}\sum_{j=0}^{k}a_j\|u^{k-j}\|_0^2 - 2\tilde{\beta}\sum_{j=0}^{k+1}b_j\|\partial_x u^{k+1-j}\|_0^2
\end{aligned}$$

$$-2\tilde{\alpha}_{k+1}\|u^{k+1}\|_0^2-2\tilde{\beta}_{k+1}\left\|\partial_x u^{k+1}\|_0^2+C\Delta t\|u^{k+1}\|_0^2+C\Delta t\|g^{k+1}\|_0^2.\right. \tag{3.7}$$

Consequently,

$$\begin{aligned}
&(1-C\Delta t)\|u^{k+1}\|_0^2-\|u^k\|_0^2+\|2u^{k+1}-u^k\|_0^2-\|2u^k-u^{k-1}\|_0^2+\|u^{k+1}-2u^k+u^{k-1}\|_0^2\\
\leq&-2\tilde{\alpha}\sum_{j=0}^{k+1}a_j\|u^{k+1-j}\|_0^2+2\tilde{\alpha}\sum_{j=0}^{k}a_j\|u^{k-j}\|_0^2-2\tilde{\beta}\sum_{j=0}^{k+1}b_j\left\|\partial_x u^{k+1-j}\right\|_0^2\\
&+2\tilde{\beta}\sum_{j=0}^{k}b_j\left\|\partial_x u^{k-j}\right\|_0^2-2\tilde{\alpha}_{k+1}\|u^{k+1}\|_0^2-2\tilde{\beta}_{k+1}\left\|\partial_x u^{k+1}\right\|_0^2++C\Delta t\|g^{k+1}\|_0^2,
\end{aligned}$$

and for sufficiently small Δt, we obtain

$$S^{k+1}\leq S^k+C\Delta t\|g^{k+1}\|_0^2,$$

which completes the proof of (3.2). □

4. Error Analysis

We now discuss the errors of the above semi-discrete and fully discrete methods, starting with the former.

Theorem 4.1. *If $_eu^k$ is the solution of the continuous problem and $\{u^k\}_{k=0}^K$ the solution obtained by the semi-discrete method, then*

$$\left\|_eu^k-u^k\right\|_1\leq CT^{\frac{1+\alpha}{2}}\Delta t^{\min(2-\alpha,2-\beta)},\quad k\geq 1,$$

and C does not depend on T and Δt.

Proof. Let $\{_eu^k=u(x,t_k)\}_{k=0}^K$ be the solution of the following equation:

$$\begin{aligned}
L_t^1u(x,t_{k+1})=&-\mu L_t^\alpha u(x,t_{k+1})+L_t^\beta\partial_x^2u(x,t_{k+1})\\
&-\tilde{\alpha}_{k+1}u(x,0)+\tilde{\beta}_{k+1}\partial_x^2u(x,0)\\
&-\mathscr{F}\big(u(x,t_{k+1})\big)+g^{k+1}+R^{k+1},\quad k\geq 0,
\end{aligned} \tag{4.1}$$

where

$$R^{k+1}=r^{k+1}-\mu r_\alpha^{k+1}+r_\beta^{k+1}.$$

Setting $e^k(x)=_e u^k-u^k(x)$, $k\geq 0$, we subtract (4.1) from (2.8), so that

$$\begin{aligned}
L_t^1e^{k+1}=&-\mu L_t^\alpha e_{k+1}+L_t^\beta\partial_x^2e_{k+1}-\tilde{\alpha}_{k+1}e^0+\tilde{\beta}_{k+1}\partial_x^2e^0\\
&-\big(\mathscr{F}(_eu^{k+1})-\mathscr{F}(u^{k+1})\big)+R^{k+1},\quad k\geq 0.
\end{aligned} \tag{4.2}$$

Following the proof of Theorem 3.1 and using the formula (3.1), we obtain

$$
\begin{aligned}
&\left(e^1,e^1\right)+\tilde{\alpha}\left(e^1,e^1\right)+\tilde{\beta}\left(\partial_x e^1,\partial_x e^1\right)\\
=&\left(e^0,e^1\right)+\tilde{\alpha}\left(e^0,e^1\right)+\tilde{\beta}\left(\partial_x e^0,\partial_x e^1\right)-\tilde{\alpha}_1\left(e^0,e^1\right)\\
&-\tilde{\beta}_1\left(\partial_x e^0,\partial_x e^1\right)-\Delta t\left(\mathscr{F}({}_e u^1)-\mathscr{F}(u^1),e^1\right)+\Delta t\left(R^1,e^1\right),
\end{aligned}
$$

or

$$
\begin{aligned}
&\|e^1\|_0^2+\tilde{\alpha}\sum_{j=0}^{1}a_j\|e^{1-j}\|_0^2+\tilde{\beta}\sum_{j=0}^{1}b_j\left\|\partial_x e^{1-j}\right\|_0^2+\tilde{\alpha}a_1\|e^1\|_0^2+\tilde{\beta}b_1\left\|\partial_x e^1\right\|_0^2\\
\le&\|e^0\|_0^2+\tilde{\alpha}a_0\|e^0\|_0^2+\tilde{\beta}b_0\left\|\partial_x e^0\right\|_0^2+C\Delta t\|R^1\|_0^2+C\Delta t\|e^1\|_0^2.
\end{aligned}
$$

Taking into account that $e^0=0$, we get

$$
\frac{1}{2}\|e^1\|_0^2+\tilde{\alpha}\sum_{j=0}^{1}a_j\|e^j\|_0^2+\tilde{\beta}\sum_{j=0}^{1}b_j\left\|\partial_x e^j\right\|_0^2\le c\Delta t\|R^1\|_0^2.
$$

The relations $a_0=b_0=1$ and $R^1=\mathscr{O}(\Delta t)+\mathscr{O}(\Delta t^{2-\alpha})+\mathscr{O}(\Delta t^{2-\beta})$ yield

$$
\|e^1\|_1\le c\Delta t\|R^1\|_0\le C\Delta t^2,
$$

which completes the proof for $k=0$.

If $k\ge 1$, then

$$
\begin{aligned}
&2\left(3e^{k+1}-4e^k+e^{k-1},v\right)\\
=&-4\tilde{\alpha}\left[\left(e^{k+1},v\right)-\sum_{j=1}^{k}\left(a_j-a_{j+1}\right)\left(e^{k-j},v\right)-a_k\left(e^0,v\right)\right]\\
&-4\tilde{\beta}\left[\left(\partial_x e^{k+1},\partial_x v\right)-\sum_{j=1}^{k-1}\left(b_j-b_{j+1}\right)\left(\partial_x e^{k-j},\partial_x v\right)-b_k\left(\partial_x e^0,\partial_x v\right)\right]\\
&-4\tilde{\alpha}_{k+1}\left(e^0,v\right)+4\tilde{\beta}_{k+1}\left(\partial_x e^0,\partial_x v\right)-4\Delta t\left(\mathscr{F}({}_e u^{k+1})-\mathscr{F}(u^{k+1}),v\right)+4\Delta t\left(R^{k+1},v\right).
\end{aligned}
$$

Since

$$
\left\|\mathscr{F}\left({}_e u^k\right)-\mathscr{F}\left(u^k\right)\right\|_0\le C\left\|{}_e u^k-u^k\right\|_0,
$$

we can proceed similar to (3.7) and use Lemma 3.2. Choosing $v=e^{k+1}$ in the above expression implies

$$
\begin{aligned}
&\|e^{k+1}\|_0^2-\|e^k\|_0^2+\|2e^{k+1}-e^k\|_0^2-\|2e^k-e^{k-1}\|_0^2+\|e^{k+1}-2e^k+e^{k-1}\|_0^2\\
\le&-2\tilde{\alpha}\sum_{j=0}^{k+1}a_j\|e^{k+1-j}\|_0^2+2\tilde{\alpha}\sum_{j=0}^{k}a_j\|e^{k-j}\|_0^2-2\tilde{\beta}\sum_{j=0}^{k+1}b_j\left\|\partial_x e^{k+1-j}\right\|_0^2\\
&+2\tilde{\beta}\sum_{j=0}^{k}b_j\left\|\partial_x e^{k-j}\right\|_0^2-2\tilde{\alpha}_{k+1}\|e^{k+1}\|_0^2-2\tilde{\beta}_{k+1}\left\|\partial_x e^{k+1}\right\|_0^2\\
&+C\Delta t\|e^{k+1}\|_0^2+4C\Delta t\left|\left(R^{k+1},e^{k+1}\right)\right|.
\end{aligned}
$$

Setting

$$\varepsilon^k := \|e^k\|_0^2 + \|2e^k - e^{k-1}\|_0^2 + 2\tilde{\alpha}\sum_{j=0}^{k} a_j \|e^{k-j}\|_0^2 + 2\tilde{\beta}\sum_{j=0}^{k} b_j \left\|\partial_x e^{k-j}\right\|_0^2,$$

and using the triangle inequality, we get

$$\begin{aligned}
\varepsilon^{k+1} &\le \varepsilon^k - 2\tilde{\alpha}_{k+1}\|e^{k+1}\|_0^2 - 2\tilde{\beta}_{k+1}\left\|\partial_x e^{k+1}\right\|_0^2 + C\Delta t\|e^{k+1}\|_0^2 + C\Delta t\left|\left(R^{k+1}, e^{k+1}\right)\right| \\
&\le \varepsilon^k - 2\tilde{\alpha}_{k+1}\|e^{k+1}\|_0^2 - 2\tilde{\beta}_{k+1}\left\|\partial_x e^{k+1}\right\|_0^2 + C\Delta t\|e^{k+1}\|_0^2 \\
&\quad + \frac{2\Delta t^2}{\tilde{\alpha}_{k+1}}\|R^{k+1}\|_0^2 + \frac{\tilde{\alpha}_{k+1}}{2}\|e^{k+1}\|_0^2 \\
&\le \varepsilon^k + \frac{2\Delta t^2}{\tilde{\alpha}_{k+1}}\|R^{k+1}\|_0^2 \le \varepsilon^k + \frac{2\Gamma(1-\alpha)t_{k+1}^{\alpha}}{\mu}\Delta t\|R^{k+1}\|_0^2 \\
&\le \varepsilon^1 + \sum_{j=1}^{k}\frac{2\Gamma(1-\alpha)t_{k+1}^{\alpha}}{\mu}\Delta t\|R^{j+1}\|_0^2 \le \varepsilon^1 + \frac{2\Gamma(1-\alpha)t_{k+1}^{1+\alpha}}{\mu}\max_{2\le j\le k+1}\|R^j\|_0^2.
\end{aligned}$$

Recalling that

$$\|R^j\|_0 = \mathscr{O}\left(\Delta t^2\right) + \mathscr{O}\left(\Delta t^{2-\alpha}\right) + \mathscr{O}\left(\Delta t^{2-\beta}\right), \quad j \ge 2,$$

we write

$$\varepsilon^k \le c_1\Delta t^4 + c_2 T^{1+\alpha}\left(\Delta t^{2-\alpha} + \Delta t^{2-\beta}\right)^2, \quad k = 1, 2, \dots, K.$$

Hence,

$$\|e^k\|_1 \le C T^{\frac{1+\alpha}{2}}\left(\Delta t^{2-\alpha} + \Delta t^{2-\beta}\right), \quad k = 1, 2, \dots, K,$$

where C does not depend on T and Δt. □

In order to evaluate the difference between the semi-discrete and fully-discrete solutions, we need the following result.

Lemma 4.1 (cf. Bernardi & Maday [2], Lin *et al.* [14]). *Let $\Phi_N^1 : H_0^1(\Lambda) \to P_N^0(\Lambda)$ be the H^1-orthogonal projection operator defined by the equations*

$$\begin{aligned}
&6\left(\Phi_N^1\psi, v_N\right) + \tilde{\alpha}\left(\Phi_N^1\psi, v_N\right) + \tilde{\beta}\left(\partial_x\Phi_N^1\psi, \partial_x v_N\right) \\
&= 6\left(\psi, v_N\right) + \tilde{\alpha}\left(\psi, v_N\right) + \tilde{\beta}\left(\partial_x\psi, \partial_x v_N\right)
\end{aligned}$$

valid for all $v_N \in P_N^0(\Lambda)$. If $\psi \in H^m(\Lambda)\cap H_0^1(\Lambda)$, then

$$\left\|\psi - \Phi_N^1\psi\right\|_1 \le cN^{1-m}\|\psi\|_m, \quad m \ge 1,$$

where $\|.\|_1$ is the modified H^1-norm.

Let

$$e_N^k = u^k - u_N^k, \quad \tilde{e}_N^k = \Phi_N^1(u^k) - u_N^k, \quad \bar{e}_N^k = u^k - \Phi_N^1(u^k), \quad k \ge 0.$$

By using the triangle inequality, we have

$$\left\|e_N^k\right\|_1 \le \left\|\tilde{e}_N^k\right\|_1 + \left\|\bar{e}_N^k\right\|_1.$$

Theorem 4.2. *Let $\{u^k\}_{k=1}^K$ be the solution of the semi-discrete problem and $\{u_N^k\}_{k=0}^K$ the solution of the fully-discrete problem. If $u^k \in H^m(\Lambda), k \geq 0, m > 1$, then*

$$\left\|u^k - u_N^k\right\|_1 \leq CT\Delta t^{-1} N^{1-m} \max_{0\leq j\leq k} \|u^j\|_m, \quad k = 0, 1, \ldots, K,$$

where C is a positive constant that does not depend on N and Δt.

Proof. Writing the relation (2.15) as

$$\begin{aligned}
&2\left(3u^{k+1} - 4u^k + u^{k-1}, v\right) \\
=& -4\tilde{\alpha}\left(u^{k+1}, v\right) + 4\tilde{\alpha}\sum_{j=0}^{k-1}\left(a_j - a_{j+1}\right)\left(u^{k-j}, v\right) + 4\left(\tilde{\alpha}a_k - \tilde{\alpha}_{k+1}\right)\left(u^0, v\right) \\
&- 4\tilde{\beta}\left(\partial_x u^{k+1}, \partial_x v\right) + 4\tilde{\beta}\sum_{j=0}^{k-1}\left(b_j - b_{j+1}\right)\left(\partial_x u^{k-j}, \partial_x v\right) + 4\left(\tilde{\beta}b_k - \tilde{\beta}_{k+1}\right)\left(\partial_x u^0, \partial_x v\right) \\
&- 4\Delta t\left(\mathscr{F}(u^{k+1}), v\right) + 4\Delta t\left(g^{k+1}, v\right) \quad \text{for all} \quad v \in H_0^1(\Lambda)
\end{aligned} \tag{4.3}$$

and applying the operator Φ_N^1 to the Eq. (4.3) yields

$$\begin{aligned}
&2\left(\Phi_N^1(3u^{k+1} - 4u^k + u^{k-1}), v_N\right) + 4\tilde{\alpha}\left(\Phi_N^1 u^{k+1}, v_N\right) + 4\tilde{\beta}\left(\partial_x \Phi_N^1 u^{k+1}, \partial_x v_N\right) \\
=& 4\tilde{\alpha}\sum_{j=0}^{k-1}\left(a_j - a_{j+1}\right)\left(\Phi_N^1 u^{k-j}, v_N\right) + 4\left(\tilde{\alpha}a_k - \tilde{\alpha}_{k+1}\right)\left(\Phi_N^1 u^0, v_N\right) \\
&+ 4\tilde{\beta}\sum_{j=0}^{k-1}\left(b_j - b_{j+1}\right)\left(\partial_x \Phi_N^1 u^{k-j}, \partial_x v_N\right) + 4\left(\tilde{\beta}b_k - \tilde{\beta}_{k+1}\right)\left(\partial_x \Phi_N^1 u^0, \partial_x v_N\right) \\
&- 4\Delta t\left(\mathscr{F}(\Phi_N^1 u^{k+1}), v_N\right) + e_\Phi(v_N), \quad \forall v_N \in P_N^0(\Lambda),
\end{aligned} \tag{4.4}$$

where the term $e_\Phi(v_N)$ in (4.4) has the form

$$\begin{aligned}
e_\Phi(v_N) =& \left(8\bar{e}_N^k - 2\bar{e}_N^{k-1}, v_N\right) + 4\tilde{\alpha}\sum_{j=0}^{k-1}\left(a_j - a_{j+1}\right)\left(\bar{e}_N^{k-j}, v_N\right) \\
&+ 4\left(\tilde{\alpha}a_k - \tilde{\alpha}_{k+1}\right)\left(\bar{e}_N^0, v_N\right) + 4\tilde{\beta}\sum_{j=0}^{k-1}\left(b_j - b_{j+1}\right)\left(\partial_x \bar{e}_N^{k-j}, \partial_x v_N\right) \\
&+ 4\left(\tilde{\beta}b_k - \tilde{\beta}_{k+1}\right)\left(\partial_x \bar{e}_N^0, \partial_x v_N\right) - 4\Delta t\left(\mathscr{F}(u^{k+1}) - \mathscr{F}(\Phi_N^1 u^{k+1}), v_N\right).
\end{aligned}$$

It follows from (2.17) and (4.4) that

$$\begin{aligned}
&2\left(3\tilde{e}_N^{k+1} - 4\tilde{e}_N^k + \tilde{e}_N^{k-1}, v_N\right) + 4\tilde{\alpha}\left(\tilde{e}_N^{k+1}, v_N\right) + 4\tilde{\beta}\left(\partial_x \tilde{e}_N^{k+1}, \partial_x v_N\right) \\
=& 4\tilde{\alpha}\sum_{j=0}^{k-1}\left(a_j - a_{j+1}\right)\left(\tilde{e}_N^{k-j}, v_N\right) + 4\left(\tilde{\alpha}a_k - \tilde{\alpha}_{k+1}\right)\left(\tilde{e}_N^0, v_N\right) \\
&+ 4\tilde{\beta}\sum_{j=0}^{k-1}\left(b_j - b_{j+1}\right)\left(\partial_x \tilde{e}_N^{k-j}, \partial_x v_N\right) + 4\left(\tilde{\beta}b_k - \tilde{\beta}_{k+1}\right)\left(\partial_x \tilde{e}_N^0, \partial_x v_N\right)
\end{aligned}$$

$$+e_{\Phi}(v_N)-4\Delta t\left(\mathscr{F}(\Phi_N^1 u^{k+1})-\mathscr{F}(u_N^{k+1}),v_N\right). \tag{4.5}$$

Choosing $v_N=\tilde{e}_N^{k+1}$ in (4.5), we obtain

$$\begin{aligned}
&2\left(3\tilde{e}_N^{k+1}-4\tilde{e}_N^{k}+\tilde{e}_N^{k-1},\tilde{e}_N^{k+1}\right)+4\tilde{\alpha}\left(\tilde{e}_N^{k+1},\tilde{e}_N^{k+1}\right)+4\tilde{\beta}\left(\partial_x\tilde{e}_N^{k+1},\partial_x\tilde{e}_N^{k+1}\right)\\
=&4\tilde{\alpha}\sum_{j=0}^{k-1}\left(a_j-a_{j+1}\right)\left(\tilde{e}_N^{k-j},\tilde{e}_N^{k+1}\right)+4\left(\tilde{\alpha}a_k-\tilde{\alpha}_{k+1}\right)\left(\tilde{e}_N^{0},\tilde{e}_N^{k+1}\right)\\
&+4\tilde{\beta}\sum_{j=0}^{k-1}\left(b_j-b_{j+1}\right)\left(\partial_x\tilde{e}_N^{k-j},\partial_x\tilde{e}_N^{k+1}\right)+4\left(\tilde{\beta}b_k-\tilde{\beta}_{k+1}\right)\left(\partial_x\tilde{e}_N^{0},\partial_x\tilde{e}_N^{k+1}\right)\\
&+e_{\Phi}\left(\tilde{e}_N^{k+1}\right)-4\Delta t\left(\mathscr{F}(\Phi_N^1 u^{k+1})-\mathscr{F}(u_N^{k+1}),\tilde{e}_N^{k+1}\right).
\end{aligned} \tag{4.6}$$

The nonlinear term in (4.6) is estimated as follows:

$$\left\|\mathscr{F}\left(\Phi_N^1 u^k\right)-\mathscr{F}\left(u_N^k\right)\right\|_0\le C\left\|\tilde{e}_N^k\right\|_0,$$

and Lemma 3.3 gives

$$\begin{aligned}
&\left\|\tilde{e}_N^{k+1}\right\|_0^2-\left\|\tilde{e}_N^{k}\right\|_0^2+\left\|2\tilde{e}_N^{k+1}-\tilde{e}_N^{k}\right\|_0^2-\left\|2\tilde{e}_N^{k}-\tilde{e}_N^{k-1}\right\|_0^2+\left\|\tilde{e}_N^{k+1}-2\tilde{e}_N^{k}+\tilde{e}_N^{k-1}\right\|_0^2\\
\le&-2\tilde{\alpha}\sum_{j=0}^{k+1}a_j\left\|\tilde{e}_N^{k+1-j}\right\|_0^2+2\tilde{\alpha}\sum_{j=0}^{k}a_j\left\|\tilde{e}_N^{k-j}\right\|_0^2-2\tilde{\beta}\sum_{j=0}^{k+1}b_j\left\|\partial_x\tilde{e}_N^{k+1-j}\right\|_0^2\\
&+2\tilde{\beta}\sum_{j=0}^{k}b_j\left\|\partial_x\tilde{e}_N^{k-j}\right\|_0^2-2\tilde{\alpha}_{k+1}\left\|\tilde{e}_N^{k+1}\right\|_0^2-2\tilde{\beta}_{k+1}\left\|\partial_x\tilde{e}_N^{k+1}\right\|_0^2\\
&+C\Delta t\left\|\tilde{e}_N^{k+1}\right\|_0^2+\left|e_{\Phi}\left(e_N^{k+1}\right)\right|.
\end{aligned} \tag{4.7}$$

Estimating the first term in $|e_{\Phi}(\tilde{e}_N^{k+1})|$, we write

$$\begin{aligned}
&\left|\left(8\bar{e}_N^{k}-2\bar{e}_N^{k-1}+4\tilde{\alpha}\sum_{j=0}^{k-1}\left(a_j-a_{j+1}\right)\bar{e}_N^{k-j}+4\left(\tilde{\alpha}a_k-\tilde{\alpha}_{k+1}\right)\bar{e}_N^{0},\tilde{e}_N^{k+1}\right)\right|\\
\le&\frac{1}{2\tilde{\alpha}_{k+1}}\left\|8\bar{e}_N^{k}-2\bar{e}_N^{k-1}+4\tilde{\alpha}\sum_{j=0}^{k-1}\left(a_j-a_{j+1}\right)\bar{e}_N^{k-j}+4\left(\tilde{\alpha}a_k-\tilde{\alpha}_{k+1}\right)\bar{e}_N^{0}\right\|_0^2+\frac{\tilde{\alpha}_{k+1}}{2}\left\|\tilde{e}_N^{k+1}\right\|_0^2\\
\le&\frac{1}{2\tilde{\alpha}_{k+1}}\left[C+16\tilde{\alpha}^2\sum_{j=0}^{k-1}\left(a_j-a_{j+1}\right)^2+16\left(\tilde{\alpha}a_k-\tilde{\alpha}_{k+1}\right)^2\right]\max_{0\le j\le k}\left\|\bar{e}_N^{j}\right\|_0^2+\frac{\tilde{\alpha}_{k+1}}{2}\left\|\tilde{e}_N^{k+1}\right\|_0^2\\
\le&\frac{C}{2\tilde{\alpha}_{k+1}}\left[1+16\tilde{\alpha}^2\left(1-\tilde{\alpha}_{k+1}\right)^2\right]\max_{0\le j\le k}\left\|\bar{e}_N^{j}\right\|_0^2+\frac{\tilde{\alpha}_{k+1}}{2}\left\|\tilde{e}_N^{k+1}\right\|_0^2\\
\le&\frac{C}{2\tilde{\alpha}_{k+1}}\left(1+16\tilde{\alpha}^2\right)\max_{0\le j\le k}\left\|\bar{e}_N^{j}\right\|_0^2+\frac{\tilde{\alpha}_{k+1}}{2}\left\|\tilde{e}_N^{k+1}\right\|_0^2.
\end{aligned} \tag{4.8}$$

Analogously, for the second term in $|e_{\Phi}(\tilde{e}_N^{k+1})|$ we obtain

$$\left|\left(4\tilde{\beta}\sum_{j=0}^{k-1}\left(b_j-b_{j+1}\right)\partial_x\bar{e}_N^{k-j}+4\left(\tilde{\beta}b_k-\tilde{\beta}_{k+1}\right)\partial_x\bar{e}_N^{0},\partial_x\tilde{e}_N^{k+1}\right)\right|$$

$$\leq \frac{1}{2\tilde{\beta}_{k+1}}\left\|\tilde{\beta}\sum_{j=0}^{k-1}\left(b_j-b_{j+1}\right)\partial_x\bar{e}_N^{k-j}+\left(\tilde{\beta}\,b_k-\tilde{\beta}_{k+1}\right)\partial_x\bar{e}_N^0\right\|_0^2+2\tilde{\beta}_{k+1}\left\|\partial_x\tilde{e}_N^{k+1}\right\|_0^2$$

$$\leq \frac{C}{\tilde{\beta}_{k+1}}\tilde{\beta}^2\max_{0\leq j\leq k}\left\|\partial_x\bar{e}_N^j\right\|_0^2+\frac{\tilde{\beta}_{k+1}}{2}\left\|\partial_x\tilde{e}_N^{k+1}\right\|_0^2. \tag{4.9}$$

Combining (4.7)-(4.9) leads to the estimate

$$\begin{aligned}
&\left\|\tilde{e}_N^{k+1}\right\|_0^2-\left\|\tilde{e}_N^k\right\|_0^2+\left\|2\tilde{e}_N^{k+1}-\tilde{e}_N^k\right\|_0^2-\left\|2\tilde{e}_N^k-\tilde{e}_N^{k-1}\right\|_0^2+\left\|\tilde{e}_N^{k+1}-2\tilde{e}_N^k+\tilde{e}_N^{k-1}\right\|_0^2\\
&\leq -2\tilde{\alpha}\sum_{j=0}^{k+1}a_j\left\|\tilde{e}_N^{k+1-j}\right\|_0^2+2\tilde{\alpha}\sum_{j=0}^{k}a_j\left\|\tilde{e}_N^{k-j}\right\|_0^2-2\tilde{\beta}\sum_{j=0}^{k+1}b_j\left\|\partial_x\tilde{e}_N^{k+1-j}\right\|_0^2+2\tilde{\beta}\sum_{j=0}^{k}b_j\left\|\partial_x\tilde{e}_N^{k-j}\right\|_0^2\\
&\quad+\frac{C}{\tilde{\alpha}_{k+1}}\left(1+16\tilde{\alpha}^2\right)\max_{0\leq j\leq k}\left\|\bar{e}_N^j\right\|_0^2+\frac{C}{\tilde{\beta}_{k+1}}\tilde{\beta}^2\max_{0\leq j\leq k}\left\|\partial_x\bar{e}_N^j\right\|_0^2
\end{aligned} \tag{4.10}$$

with $\tilde{\alpha}_{k+1}$ and $\tilde{\beta}_{k+1}$ such that

$$\frac{1}{\tilde{\alpha}_{k+1}}\leq CT^{\alpha}\Delta t^{-1},\quad \frac{1}{\tilde{\beta}_{k+1}}\leq CT^{\beta}\Delta t^{-1}.$$

Therefore, we get

$$\begin{aligned}
&\frac{C}{4\tilde{\alpha}_{k+1}}\left(1+16\tilde{\alpha}^2\right)\max_{0\leq j\leq k}\left\|\bar{e}_N^j\right\|_0^2+\frac{C}{\tilde{\beta}_{k+1}}\tilde{\beta}^2\max_{0\leq j\leq k}\left\|\partial_x\bar{e}_N^j\right\|_0^2\\
&\leq CT\Delta t^{-1}\max_{0\leq j\leq k}\left\|\bar{e}_N^j\right\|_1^2.
\end{aligned}$$

Setting

$$\varepsilon_N^k:=\left\|\tilde{e}_N^k\right\|_0^2+\left\|2\tilde{e}_N^k-\tilde{e}_N^{k-1}\right\|_0^2+2\tilde{\alpha}\sum_{j=0}^{k}a_j\left\|\tilde{e}_N^{k-j}\right\|_0^2+2\tilde{\beta}\sum_{j=0}^{k}b_j\left\|\partial_x\tilde{e}_N^{k-j}\right\|_0^2,$$

we write (4.10) in the form

$$\varepsilon_N^{k+1}\leq\varepsilon_N^k+CT\Delta t^{-1}\max_{0\leq j\leq k}\left\|\bar{e}_N^j\right\|_1^2,\quad k=0,1,\ldots,K.$$

Thus

$$\begin{aligned}
&\varepsilon_N^k\leq\varepsilon_N^0+cTk\Delta t^{-1}\max_{0\leq j\leq k}\left\|\bar{e}_N^j\right\|_1^2\leq CT^2\Delta t^{-2}\max_{0\leq j\leq k}\left\|\bar{e}_N^j\right\|_1^2, && k=0,1,\ldots,K,\\
&\left\|\tilde{e}_N^k\right\|_1^2\leq CT^2\Delta t^{-2}\max_{0\leq j\leq k}\left\|\bar{e}_N^j\right\|_1^2, && k=0,1,\ldots,K.
\end{aligned}$$

Finally, using the triangle inequality, we get

$$\left\|e_N^k\right\|_1=\left\|\tilde{e}_N^k+\bar{e}_N^k\right\|_1\leq\left\|\tilde{e}_N^k\right\|_1+\left\|\bar{e}_N^k\right\|_1\leq CT\Delta t^{-1}\max_{0\leq j\leq k}\left\|\bar{e}_N^j\right\|_1,\quad k=0,1,\ldots,K,$$

and the corresponding estimates from Lemma 4.1 give

$$\left\|e_N^k\right\|_1 \leq CT\Delta t^{-1}N^{1-m}\max_{0\leq j\leq k}\|u^j\|_m, \quad k=0,1,\ldots,K.$$

The proof is complete. □

Writing

$$E_N^k ={}_e u^k - u_N^k, \quad \tilde{E}_N^k = \Phi_N^1\left({}_eu^k\right) - u_N^k, \quad \bar{E}_N^k ={}_e u^k - \Phi_N^1\left({}_eu^k\right), \quad k\geq 0,$$

and using the triangle inequality yields

$$\left\|E_N^k\right\|_1 \leq \left\|\tilde{E}_N^k\right\|_1 + \left\|\bar{E}_N^k\right\|_1.$$

Theorem 4.3. *Let ${}_eu^k$ be the solution of the continuous problem* (2.1) *and $\{u_N^k\}_{k=1}^K$ the solution of the corresponding fully discrete problem. If ${}_eu^k \in H^m(\Lambda)$ for all $t\in[0,T], m>1$, then*

$$\left\|{}_eu^k - u_N^k\right\|_1 \leq CT\left(\Delta t^{\min(2-\alpha,2-\beta)} + \Delta t^{-1}N^{1-m}\|{}_eu\|_{L^\infty(H^m)}\right), \quad k=0,1,\ldots,K,$$

where

$$\|{}_eu\|_{L^\infty(H^m)} := \max_{0\leq j\leq k}\left\|{}_eu^j\right\|_m$$

and C does not depend on T and Δt.

Proof. Let $\{{}_eu^{k+1}\}_{k=0}^K$ be the solution of the equation

$$\begin{aligned}
&2\left(3{}_eu^{k+1} - 4{}_eu^k +{}_e u^{k-1}, v\right) \\
&= -4\tilde{\alpha}\left({}_eu^{k+1}, v\right) + 4\tilde{\alpha}\sum_{j=0}^{k-1}\left(a_j - a_{j+1}\right)\left({}_eu^{k-j}, v\right) + 4\left(\tilde{\alpha}a_k - \tilde{\alpha}_{k+1}\right)\left({}_eu^0, v\right) \\
&\quad - 4\tilde{\beta}\left(\partial_x({}_eu^{k+1}), \partial_x v\right) + 4\tilde{\beta}\sum_{j=0}^{k-1}\left(b_j - b_{j+1}\right)\left(\partial_x({}_eu^{k-j}), \partial_x v\right) \\
&\quad + 4\left(\tilde{\beta}b_k - \tilde{\beta}_{k+1}\right)\left(\partial_x({}_eu^0), \partial_x v\right) - 4\Delta t\left(\mathscr{F}({}_eu^{k+1}), v\right) \\
&\quad + 4\Delta t\left(g^{k+1}, v\right) + 4\Delta t\left(R^{k+1}, v\right), \quad \forall v\in H_0^1(\Lambda),
\end{aligned} \tag{4.11}$$

where $R^{k+1} = r^{k+1} - \mu r_\alpha^{k+1} + r_\beta^{k+1}$. Applying the operator Φ_N^1 to (4.11) gives

$$\begin{aligned}
&2\left(\Phi_N^1(3{}_eu^{k+1} - 4{}_eu^k +{}_e u^{k-1}), v_N\right) + 4\tilde{\alpha}\left(\Phi_N^1({}_eu^{k+1}), v_N\right) + 4\tilde{\beta}\left(\partial_x\Phi_N^1({}_eu^{k+1}), \partial_x v_N\right) \\
&= 4\tilde{\alpha}\sum_{j=0}^{k-1}\left(a_j - a_{j+1}\right)\left(\Phi_N^1({}_eu^{k-j}), v_N\right) + 4\left(\tilde{\alpha}a_k - \tilde{\alpha}_{k+1}\right)\left(\Phi_N^1({}_eu^0), v_N\right) \\
&\quad + 4\tilde{\beta}\sum_{j=0}^{k-1}\left(b_j - b_{j+1}\right)\left(\partial_x\Phi_N^1({}_eu^{k-j}), \partial_x v_N\right) + 4\left(\tilde{\beta}b_k - \tilde{\beta}_{k+1}\right)\left(\partial_x\Phi_N^1({}_eu^0), \partial_x v_N\right) \\
&\quad - 4\Delta t\left(\mathscr{F}(\Phi_N^1({}_eu^{k+1})), v_N\right) + E_\Phi(v_N), \quad \forall v_N\in P_N^0(\Lambda),
\end{aligned} \tag{4.12}$$

where

$$
\begin{aligned}
E_\Phi(v_N) = &\left(8\bar{E}_N^k - 2\bar{E}_N^{k-1}, v_N\right) + 4\tilde{\alpha}\sum_{j=0}^{k-1}\left(a_j - a_{j+1}\right)\left(\bar{E}_N^{k-j}, v_N\right) \\
&+ 4\left(\tilde{\alpha}a_k - \tilde{\alpha}_{k+1}\right)\left(\bar{E}_N^0, v_N\right) + 4\tilde{\beta}\sum_{j=0}^{k-1}\left(b_j - b_{j+1}\right)\left(\partial_x\bar{E}_N^{k-j}, \partial_x v_N\right) \\
&+ 4\left(\tilde{\beta}\,b_k - \tilde{\beta}_{k+1}\right)\left(\partial_x\bar{E}_N^0, \partial_x v_N\right) - 4\Delta t\left(\mathscr{F}({}_e u^{k+1}) - \mathscr{F}(\Phi_N^1({}_e u^{k+1})), v_N\right).
\end{aligned}
$$

It follows from (2.17) and (4.12) that

$$
\begin{aligned}
&2\left(3\tilde{E}_N^{k+1} - 4\tilde{E}_N^k + \tilde{E}_N^{k-1}, v_N\right) + 4\tilde{\alpha}\left(\tilde{E}_N^{k+1}, v_N\right) + 4\tilde{\beta}\left(\partial_x\tilde{E}_N^{k+1}, \partial_x v_N\right) \\
= &4\tilde{\alpha}\sum_{j=0}^{k-1}\left(a_j - a_{j+1}\right)\left(\tilde{E}_N^{k-j}, v_N\right) + 4\left(\tilde{\alpha}a_k - \tilde{\alpha}_{k+1}\right)\left(\tilde{E}_N^0, v_N\right) \\
&+ 4\tilde{\beta}\sum_{j=0}^{k-1}\left(b_j - b_{j+1}\right)\left(\partial_x\tilde{E}_N^{k-j}, \partial_x v_N\right) + 4\left(\tilde{\beta}\,b_k - \tilde{\beta}_{k+1}\right)\left(\partial_x\tilde{E}_N^0, \partial_x v_N\right) \\
&+ E_\Phi(v_N) - 4\Delta t\left(\mathscr{F}(\Phi_N^1({}_e u^{k+1})) - \mathscr{F}(u_N^{k+1}), v_N\right) + 4\Delta t\left(R^{k+1}, v_N\right).
\end{aligned}
\tag{4.13}
$$

Choosing $v_N = \tilde{E}_N^{k+1}$ in (4.13) gives

$$
\begin{aligned}
&2\left(3\tilde{E}_N^{k+1} - 4\tilde{E}_N^k + \tilde{E}_N^{k-1}, \tilde{E}_N^{k+1}\right) + 4\tilde{\alpha}\left(\tilde{E}_N^{k+1}, \tilde{E}_N^{k+1}\right) + 4\tilde{\beta}\left(\partial_x\tilde{E}_N^{k+1}, \partial_x\tilde{E}_N^{k+1}\right) \\
= &4\tilde{\alpha}\sum_{j=0}^{k-1}\left(a_j - a_{j+1}\right)\left(\tilde{E}_N^{k-j}, \tilde{E}_N^{k+1}\right) + 4\left(\tilde{\alpha}a_k - \tilde{\alpha}_{k+1}\right)\left(\tilde{E}_N^0, \tilde{E}_N^{k+1}\right) \\
&+ 4\tilde{\beta}\sum_{j=0}^{k-1}\left(b_j - b_{j+1}\right)\left(\partial_x\tilde{E}_N^{k-j}, \partial_x\tilde{E}_N^{k+1}\right) + 4\left(\tilde{\beta}\,b_k - \tilde{\beta}_{k+1}\right)\left(\partial_x\tilde{E}_N^0, \partial_x\tilde{E}_N^{k+1}\right) \\
&+ E_\Phi\left(\tilde{E}_N^{k+1}\right) - 4\Delta t\left(\mathscr{F}(\Phi_N^1({}_e u^{k+1})) - \mathscr{F}(u_N^{k+1}), \tilde{E}_N^{k+1}\right) + 4\Delta t\left(R^{k+1}, \tilde{E}_N^{k+1}\right).
\end{aligned}
\tag{4.14}
$$

Since the nonlinear term in (4.14) is estimated as

$$
\left\|\mathscr{F}\left(\Phi_N^1 u^k\right) - \mathscr{F}\left(u_N^k\right)\right\|_0 \le C\left\|\tilde{E}_N^k\right\|_0,
$$

Lemma 3.3 implies that

$$
\begin{aligned}
&\left\|\tilde{E}_N^{k+1}\right\|_0^2 - \left\|\tilde{E}_N^k\right\|_0^2 + \left\|2\tilde{E}_N^{k+1} - \tilde{E}_N^k\right\|_0^2 - \left\|2\tilde{E}_N^k - \tilde{E}_N^{k-1}\right\|_0^2 + \left\|\tilde{E}_N^{k+1} - 2\tilde{E}_N^k + \tilde{E}_N^{k-1}\right\|_0^2 \\
\le &-2\tilde{\alpha}\sum_{j=0}^{k+1}a_j\left\|\tilde{E}_N^{k+1-j}\right\|_0^2 + 2\tilde{\alpha}\sum_{j=0}^{k}a_j\left\|\tilde{E}_N^{k-j}\right\|_0^2 - 2\tilde{\beta}\sum_{j=0}^{k+1}b_j\left\|\partial_x\tilde{E}_N^{k+1-j}\right\|_0^2 \\
&+ 2\tilde{\beta}\sum_{j=0}^{k}b_j\left\|\partial_x\tilde{E}_N^{k-j}\right\|_0^2 - 2\tilde{\alpha}_{k+1}\left\|\tilde{E}_N^{k+1}\right\|_0^2 - 2\tilde{\beta}_{k+1}\left\|\partial_x\tilde{E}_N^{k+1}\right\|_0^2 \\
&+ C\Delta t\left\|\tilde{E}_N^{k+1}\right\|_0^2 + \left|E_\Phi\left(E_N^{k+1}\right)\right| + C\left|\left(R^{k+1}, \tilde{E}_N^{k+1}\right)\right|.
\end{aligned}
\tag{4.15}
$$

Now we estimate the first term in $|E_\Phi(\tilde{E}_N^{k+1})|$ as

$$
\begin{aligned}
&\left|\left(8\bar{E}_N^k - 2\bar{E}_N^{k-1} + 4\tilde{\alpha}\sum_{j=0}^{k-1}\left(a_j - a_{j+1}\right)\bar{E}_N^{k-j} + 4\left(\tilde{\alpha}a_k - \tilde{\alpha}_{k+1}\right)\bar{E}_N^0, \tilde{E}_N^{k+1}\right)\right| \\
\le & \frac{1}{2\tilde{\alpha}_{k+1}}\left\|8\bar{E}_N^k - 2\bar{E}_N^{k-1} + 4\tilde{\alpha}\sum_{j=0}^{k-1}(a_j - a_{j+1})\bar{E}_N^{k-j} + 4(\tilde{\alpha}a_k - \tilde{\alpha}_{k+1})\bar{E}_N^0\right\|_0^2 + \frac{\tilde{\alpha}_{k+1}}{2}\left\|\tilde{E}_N^{k+1}\right\|_0^2 \\
\le & \frac{1}{2\tilde{\alpha}_{k+1}}\left[C + 16\tilde{\alpha}^2\sum_{j=0}^{k-1}(a_j - a_{j+1})^2 + 16(\tilde{\alpha}a_k - \tilde{\alpha}_{k+1})^2\right]\max_{0\le j\le k}\left\|\bar{E}_N^j\right\|_0^2 + \frac{\tilde{\alpha}_{k+1}}{2}\left\|\tilde{E}_N^{k+1}\right\|_0^2 \\
\le & \frac{C}{2\tilde{\alpha}_{k+1}}\left[1 + 16\tilde{\alpha}^2(1-\tilde{\alpha}_{k+1})^2\right]\max_{0\le j\le k}\left\|\bar{E}_N^j\right\|_0^2 + \frac{\tilde{\alpha}_{k+1}}{2}\left\|\tilde{E}_N^{k+1}\right\|_0^2 \\
\le & \frac{C}{2\tilde{\alpha}_{k+1}}\left(1 + 16\tilde{\alpha}^2\right)\max_{0\le j\le k}\left\|\bar{E}_N^j\right\|_0^2 + \frac{\tilde{\alpha}_{k+1}}{2}\left\|\tilde{E}_N^{k+1}\right\|_0^2.
\end{aligned} \tag{4.16}
$$

Analogously, the second term of $|E_\Phi(\tilde{E}_N^{k+1})|$ can be estimated as

$$
\begin{aligned}
&\left|\left(4\tilde{\beta}\sum_{j=0}^{k-1}\left(b_j - b_{j+1}\right)\partial_x\bar{E}_N^{k-j} + 4\left(\tilde{\beta}b_k - \tilde{\beta}_{k+1}\right)\partial_x\bar{E}_N^0, \partial_x\tilde{E}_N^{k+1}\right)\right| \\
\le & \frac{1}{2\tilde{\beta}_{k+1}}\left\|\tilde{\beta}\sum_{j=0}^{k-1}\left(b_j - b_{j+1}\right)\partial_x\bar{E}_N^{k-j} + \left(\tilde{\beta}b_k - \tilde{\beta}_{k+1}\right)\partial_x\bar{E}_N^0\right\|_0^2 + \frac{\tilde{\beta}_{k+1}}{2}\left\|\partial_x\tilde{E}_N^{k+1}\right\|_0^2 \\
\le & \frac{C}{\tilde{\beta}_{k+1}}\tilde{\beta}^2\max_{0\le j\le k}\left\|\partial_x\bar{E}_N^j\right\|_0^2 + \frac{\tilde{\beta}_{k+1}}{2}\left\|\partial_x\tilde{E}_N^{k+1}\right\|_0^2.
\end{aligned} \tag{4.17}
$$

Combining the estimates (4.15)-(4.17), we get

$$
\begin{aligned}
&\left\|\tilde{E}_N^{k+1}\right\|_0^2 - \left\|\tilde{E}_N^k\right\|_0^2 + \left\|2\tilde{E}_N^{k+1} - \tilde{E}_N^k\right\|_0^2 - \left\|2\tilde{E}_N^k - \tilde{E}_N^{k-1}\right\|_0^2 + \left\|\tilde{E}_N^{k+1} - 2\tilde{E}_N^k + \tilde{E}_N^{k-1}\right\|_0^2 \\
\le & -2\tilde{\alpha}\sum_{j=0}^{k+1}a_j\left\|\tilde{E}_N^{k+1-j}\right\|_0^2 + 2\tilde{\alpha}\sum_{j=0}^{k}a_j\left\|\tilde{E}_N^{k-j}\right\|_0^2 - 2\tilde{\beta}\sum_{j=0}^{k+1}b_j\left\|\partial_x\tilde{E}_N^{k+1-j}\right\|_0^2 \\
&+ 2\tilde{\beta}\sum_{j=0}^{k}b_j\left\|\partial_x\tilde{E}_N^{k-j}\right\|_0^2 + \frac{C}{\tilde{\alpha}_{k+1}}\left(1 + 16\tilde{\alpha}^2\right)\max_{0\le j\le k}\left\|\bar{E}_N^j\right\|_0^2 + \frac{C}{\tilde{\beta}_{k+1}}\tilde{\beta}^2\max_{0\le j\le k}\left\|\partial_x\bar{E}_N^j\right\|_0^2 \\
&+ C\Delta t\left\|\tilde{E}_N^{k+1}\right\|_0^2 + C\Delta t\|R^{k+1}\|_0^2
\end{aligned} \tag{4.18}
$$

with $\tilde{\alpha}_{k+1}$ and $\tilde{\beta}_{k+1}$ such that

$$
\frac{1}{\tilde{\alpha}_{k+1}} \le CT^\alpha\Delta t^{-1}, \quad \frac{1}{\tilde{\beta}_{k+1}} \le CT^\beta\Delta t^{-1}.
$$

It follows

$$
\frac{C}{4\tilde{\alpha}_{k+1}}\left(1 + 16\tilde{\alpha}^2\right)\max_{0\le j\le k}\left\|\bar{E}_N^j\right\|_0^2 + \frac{C}{\tilde{\beta}_{k+1}}\tilde{\beta}^2\max_{0\le j\le k}\left\|\partial_x\bar{E}_N^j\right\|_0^2 \le CT\Delta t^{-1}\max_{0\le j\le k}\left\|\bar{E}_N^j\right\|_1^2.
$$

Setting

$$\eta_N^k := \left\|\tilde{E}_N^k\right\|_0^2 + \left\|2\tilde{E}_N^k - \tilde{E}_N^{k-1}\right\|_0^2 + 2\tilde{\alpha}\sum_{j=0}^{k} a_j \left\|\tilde{E}_N^{k-j}\right\|_0^2 + 2\tilde{\beta}\sum_{j=0}^{k} b_j \left\|\partial_x \tilde{E}_N^{k-j}\right\|_0^2,$$

we write (4.18) in the form

$$\eta_N^{k+1} \le \eta_N^k + C\Delta t\|R^{k+1}\|_0^2 + CT\Delta t^{-1}\max_{0\le j\le k}\left\|\bar{E}_N^j\right\|_1^2, \quad k=0,1,\ldots,K.$$

Thus

$$\eta_N^{k+1} \le \eta_N^0 + Ct_k \max_{0\le j\le k}\|R^j\|_0^2 + cT^2\Delta t^{-2}\max_{0\le j\le k}\left\|\bar{E}_N^j\right\|_1^2, \quad k=0,1,\ldots,K,$$

or

$$\left\|\tilde{E}_N^k\right\|_1^2 \le CT^2\left(\Delta t^{2-\alpha} + \Delta t^{2-\beta}\right)^2 + CT^2\Delta t^{-2}\max_{0\le j\le k}\left\|\bar{E}_N^j\right\|_1^2, \quad \forall k=0,1,\ldots,K.$$

Using again Lemma 4.1, we write

$$\left\|\bar{E}_N^k\right\|_1 \le CN^{1-m}\left\|{}_e u^j\right\|_m, \quad k=0,1,\ldots,K,$$

which leads to the estimates.

$$\left\|E_N^k\right\|_1 \le CT\left(\Delta t^{\min(2-\alpha,2-\beta)} + \Delta t^{-1}N^{1-m}\|{}_e u\|_{L^\infty(H^m)}\right), \quad k=0,1,\ldots,K.$$

The proof is complete. □

5. Numerical Experiments

Let $L_N(x)$ be the Legendre polynomial of the degree N. The Gauss-Lobatto-Legendre (GLL) point $x_i, i=0,1,\ldots N$ are the roots of the polynomial $(1-x^2)L_N'(x)$. Besides, let $\omega_i, i=0,1,\ldots N$ be the weights of the Gauss-Lobatto-Legendre quadratures, so that

$$\int_{-1}^{1}\varphi(x)dx = \sum_{i=0}^{N}\varphi(x_i)\omega_i$$

for any $\varphi(x)\in P_{2N-1}(\Lambda)$. Consider the discrete inner product

$$(\phi,\psi)_N = \sum_{i=0}^{N}\phi(x_i)\psi(x_i)\omega_i, \quad \forall\phi,\psi\in C^0(\Lambda)$$

and the following discrete problem: Find $u_N^{k+1}\in P_N^0(\Lambda)$ such that equations

$$\begin{aligned}
&\left(u_N^1, v_N\right)_N + \tilde{\alpha}\left(u_N^1, v_N\right)_N - \tilde{\beta}\left(\partial_x^2 u_N^1, v_N\right)_N \\
&= \left(u_N^0, v_N\right)_N + \tilde{\alpha}\left(u_N^0, v_N\right)_N - \tilde{\beta}\left(\partial_x^2 u_N^0, v_N\right)_N + \Delta t\left(g^1, v_N\right)_N - \Delta t\left(\mathscr{F}(u_N^1), v_N\right)_N, \\
&3\left(u_N^{k+1}, v_N\right)_N + 2\tilde{\alpha}\left(u_N^{k+1}, v_N\right)_N - 2\tilde{\beta}\left(\partial_x^2 u_N^{k+1}, v_N\right)_N
\end{aligned}$$

$$
\begin{aligned}
&= 2\tilde{\alpha}\sum_{j=1}^{k}\left(a_j - a_{j+1}\right)\left(u_N^j, v_N\right)_N + 2\tilde{\alpha}a_k\left(u_N^0, v_N\right)_N - 2\tilde{\beta}\sum_{j=1}^{k}\left(b_j - b_{j+1}\right)\left(\partial_x u_N^j, v_N\right)_N \\
&\quad + 2\tilde{\beta}b_k\left(\partial_x u_N^0, v_N\right)_N + \left(4u_N^k - u_N^{k-1}, v_N\right)_N + 2\Delta t\left(g^{k+1}, v_N\right)_N - 2\Delta t\left(\mathscr{F}(u_N^{k+1}), v_N\right)_N
\end{aligned}
$$

hold for any $v_N \in P_N^0(\Lambda)$.

In order to derive a linear system for each time level, we represent the unknown solution $u_N^{k+1}(x)$ in the form

$$
u_N^{k+1}(x) = \sum_{j=0}^{N} u_j^{k+1} l_j(x),
$$

where $u_j^{k+1} = u_N^{k+1}(x_j)$ are the unknowns of the approximate solution, l_j is the Lagrangian polynomial defined on Λ, as follows:

$$
l_j \in P_N(\Lambda), \quad l_j(x_i) = \delta_{ij}, \quad i, j \in (0, N),
$$

and δ_{ij} is Kronecker-delta. Combined with the homogeneous boundary condition $u_0^{k+1} = u_N^{k+1} = 0$, we use the Lagrange polynomials $l_i(x)$, $i = 1, 2, \ldots, N-1$ as test functions.

We also use the Picard iterations to solve the corresponding nonlinear problems.

5.1. Numerical examples

We now present two numerical examples aimed to confirm the theoretical error estimates for nonlinear problems.

Example 5.1. Let $\mu = 1$ and $\mathscr{F}(u) = u^2$. We consider the problem (2.1) withe the forcing term is

$$
g(x,t) = 2\sin\pi x\left(t + \frac{t^{2-\alpha}}{\Gamma(3-\alpha)} + \frac{\pi^2 t^{2-\beta}}{\Gamma(3-\beta)}\right) + t^4\sin^2\pi x
$$

and the initial value is $u(x,0) = 0$. This problem has the solution $u(x,t) = t^2\sin\pi x$.

Numerical results for the method with different time steps Δt and the equations with fractional derivatives of different orders are shown in Table 1. Numerical results for the method with various parameters N and the equations with fractional derivatives of different orders are presented in Table 2. The exact and discrete solution are displayed in Fig. 1. Note that the numerical solution is a good approximation of the exact solution.

Table 1: Errors, $N = 16$.

α	β	$\Delta t_1 = 1/10$	$\Delta t_2 = 1/100$	$\Delta t_3 = 1/1000$	$Rate(\Delta t_1/\Delta t_2)$	$Rate(\Delta t_2/\Delta t_3)$
		$\Vert_e u^k - u_N^k\Vert_{H^1}$	$\Vert_e u^k - u_N^k\Vert_{H^1}$	$\Vert_e u^k - u_N^k\Vert_{H^1}$		
0.1	0.2	$1.4000E-03$	$3.1804E-05$	$7.1489E-07$	1.6436	1.6482
0.3	0.7	$3.5890E-03$	$2.4000E-04$	$1.4663E-05$	1.1761	1.2140
0.5	0.5	$1.0918E-03$	$4.0308E-04$	$1.4727E-05$	1.4360	1.4373
0.8	0.9	$7.5182E-03$	$9.3264E-04$	$9.1726E-05$	1.1073	1.0289

Table 2: Errors, $\Delta t = 1/1000$.

α	β	$N_1=6$	$N_2=12$	$N_3=18$	$N_4=24$
		$\|\|_e u^k - u_N^k\|\|_{H^1}$	$\|\|_e u^k - u_N^k\|\|_{H^1}$	$\|\|_e u^k - u_N^k\|\|_{H^1}$	$\|\|_e u^k - u_N^k\|\|_{H^1}$
0.1	0.2	$7.2356E-04$	$7.2113E-07$	$7.2048E-07$	$7.2068E-07$
0.3	0.7	$6.8552E-04$	$7.5192E-05$	$7.5247E-05$	$7.5268E-07$
0.5	0.5	$7.2087E-04$	$1.4832E-05$	$1.4842E-05$	$1.4847E-07$
0.8	0.9	$7.8769E-04$	$3.9327E-07$	$3.9357E-07$	$3.9368E-07$

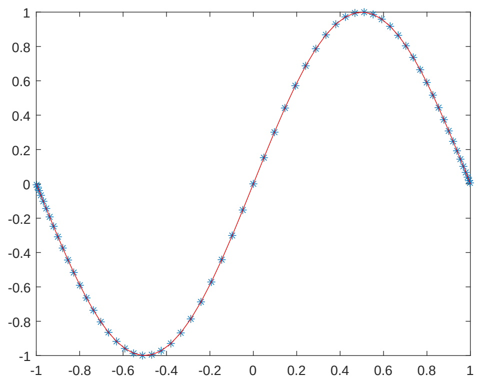

Figure 1: Numerical solution, $N = 64$, $\Delta t = 1/1000$.

Example 5.2. Let $\mu = 0$ and $\mathscr{F}(u) = u^3 - u$. We now consider the problem (2.1) with the forcing term

$$g(x,t) = \left((\alpha+1)t^{\alpha} + \frac{4\pi^2\Gamma(2+\alpha)}{\Gamma(2)} t - t^{1+\alpha} \right) \sin 2\pi x + t^{3+3\alpha} \sin^3 2\pi x$$

and the initial value $u(x,0) = 0$. This problem has the solution $u(x,t) = t^{1+\beta} \sin 2\pi x$.

Numerical results for the method with different time steps Δt and the equations with fractional derivatives of different orders are shown in Table 3. Numerical results for the method with various parameters N and equations with fractional derivatives of different orders are presented in Table 4. The results are consistent with the theoretical analysis.

Table 3: The H^1-error for $N = 16$, $\Delta t_1 = 1/10, \Delta t_2 = 1/100, \Delta t_3 = 1/1000$.

α	β	$\Delta t_1 = 1/10$	$\Delta t_2 = 1/100$	$\Delta t_3 = 1/1000$	$Rate(\Delta t_1/\Delta t_2)$	$Rate(\Delta t_2/\Delta t_3)$
		$\|\|_e u^k - u_N^k\|\|_{H^1}$	$\|\|_e u^k - u_N^k\|\|_{H^1}$	$\|\|_e u^k - u_N^k\|\|_{H^1}$		
0.1	0.2	$8.9151E-03$	$1.3506E-04$	$5.3004E-06$	1.8196	1.4062
0.3	0.7	$1.2242E-02$	$9.1702E-04$	$6.1084E-05$	1.1255	1.1765
0.5	0.5	$1.7177E-02$	$2.2739E-04$	$4.8902E-06$	1.8782	1.6674
0.8	0.9	$4.6487E-04$	$1.0412E-05$	$7.5942E-07$	1.0089	1.0562

Table 4: Error estimate when $\Delta t = 1/1000$.

α	β	$N_1 = 6$	$N_2 = 12$	$N_3 = 18$	$N_4 = 24$
		$\Vert_e u^k - u_N^k \Vert_{H^1}$	$\Vert_e u^k - u_N^k \Vert_{H^1}$	$\Vert_e u^k - u_N^k \Vert_{H^1}$	$\Vert_e u^k - u_N^k \Vert_{H^1}$
0.1	0.2	$2.6742E-01$	$2.6122E-05$	$5.6629E-06$	$5.6619E-06$
0.3	0.7	$2.6879E-01$	$8.6063E-05$	$6.5548E-05$	$6.5531E-05$
0.5	0.5	$2.6673E-01$	$3.0548E-05$	$5.2692E-06$	$5.2675E-06$
0.8	0.9	$2.7210E-01$	$2.0760E-04$	$4.1474E-04$	$4.1464E-04$

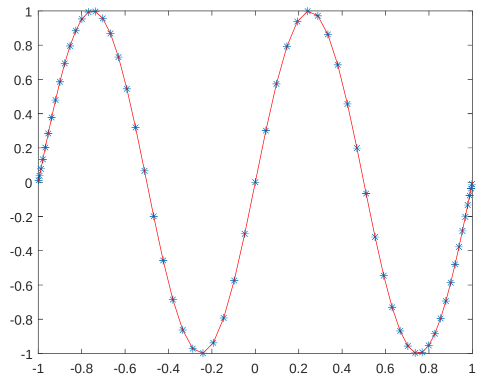

Figure 2: Numerical solution, $N = 64$, $\Delta t = 1/1000$.

Exact and discrete solutions are displayed in Fig. 2. The numerical solution is a good approximation of the exact solution.

6. Conclusion

We construct a numerical scheme for the nonlinear fractional-order Cable equation with Riemann-Liouville fractional derivatives based on finite difference approximations in time and spectral approximations in space. Using finite difference discretizations in the time direction, we obtain a semi-discrete scheme. Applying spectral Galerkin discretizations in space direction to the equations of the semi-discrete systems, we construct a fully discrete method. The stability and errors of the methods are studied. Two numerical examples verify the theoretical results.

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