East Asian Journal on Applied Mathematics
doi: 10.4208/eajam.070422.220922

Vol. **13**, No. 1, pp. 194-212
February 2023

Error Estimates of Finite Difference Methods for the Fractional Poisson Equation with Extended Nonhomogeneous Boundary Conditions

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Received 7 April 2022; Accepted (in revised version) 22 September 2022.

Abstract. Two efficient finite difference methods for the fractional Poisson equation involving the integral fractional Laplacian with extended nonhomogeneous boundary conditions are developed and analyzed. The first one uses appropriate numerical quadratures to handle extended nonhomogeneous boundary conditions and weighted trapezoidal rule with a splitting parameter to approximate the hypersingular integral in the fractional Laplacian. It is proven that the method converges with the second-order accuracy provided that the exact solution is sufficiently smooth and a splitting parameter is suitably chosen. Secondly, if numerical quadratures fail, we employ a truncated based method. Under specific conditions, the convergence rate of this method is optimal, as error estimates show. Numerical experiments are provided to gauge the performance of the methods proposed.

AMS subject classifications: 35R11, 65M06

Key words: Fractional Poisson equation, finite difference method, nonhomogeneous boundary condition, error estimates, integral fractional Laplacian.

1. Introduction

Fractional partial differential equations (FPDEs) provide an adequate and accurate description of various complex physical phenomena such as anomalous diffusion, memory behavior, long-range interaction and so on, which cannot be modeled properly by classical PDEs [3, 16]. Nowadays, FPDEs have been widely applied in various fields, including quantum mechanics [12], ground-water solute transport [4], stochastic dynamics [15] and finance [9].

A generalization of the classical Laplacian — viz. the fractional Laplacian operator can be defined in different ways [14]. In particular, the hypersingular integral fractional Laplacian operator attracted substantial attention. It has been extensively studied by many

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researchers over the last decades. Duo *et al.* [5] developed a finite difference method and proved that the convergence rate of the method depends on the solution regularity and a splitting parameter. Victor and Ying [21] used singularity subtraction for constructing a simple translation-invariant discretization scheme, which can be efficiently handled by fast Fourier transform. Hao *et al.* [10] studied a centered finite difference scheme for a fractional diffusion equation with the integral fractional Laplacian. Acosta *et al.* [1] investigated the regularity of the fractional Laplace equation and proved the optimal convergence of a linear finite element method on quasi-uniform and graded meshes.

We note that in practice, the integral fractional Laplacian with nonhomogeneous boundary condition is more useful. However, the presence of such boundary conditions leads to new problems such as the development of efficient numerical solvers and treatment of far field boundary conditions. Tang *et al.* [20] employed a rational basis in a spectral method for FPDEs with fractional Laplacian on unbounded domains and established optimal error estimates of the corresponding scheme. Xu *et al.* [24] used spherical means to develop an efficient algorithm for multi-dimensional integral fractional Laplacian. Wu *et al.* [23] proposed an efficient operator factorization method, where far field boundary conditions are approximated by numerical quadratures. Sun *et al.* [19] considered a finite difference method for nonhomogeneous fractional Dirichlet problem with compactly supported boundary conditions. Huang and Oberman [11] developed a finite difference-quadrature method with asymptotic approximations of extended Dirichlet boundary condition.

In this paper, we apply a finite difference method to the one-dimensional fractional Poisson equation with the integral fractional Laplacian

$$\begin{aligned} &(-\Delta)^s u(x) = g(x), && x \in (-L, L), \\ &u(x) = f(x), && x \in \mathbb{R} \setminus (-L, L), \end{aligned} \tag{1.1}$$

where $s \in (0,1)$ and $f(x)$ is a positive function decaying to zero as $x \to \pm\infty$. The fractional Laplacian $(-\Delta)^s$ in (1.1) is defined by

$$(-\Delta)^s u(x) := c_{1,s} \text{ P.V.} \int_{\mathbb{R}} \frac{u(x) - u(x')}{|x - x'|^{1+2s}} dx', \tag{1.2}$$

where P.V. denotes the principal value integral, and $c_{1,s}$ denotes the normalization constant

$$c_{1,s} = \frac{2^{2s} s \Gamma(1/2 + s)}{\pi^{1/2} \Gamma(1 - s)}.$$

Let us also recall that if $u(x)$ belongs to the Schwartz space of rapidly decaying functions, then the fractional Laplacian can also be defined by

$$(-\Delta)^s u(\xi) = \mathscr{F}^{-1}\big(|\xi|^{2s} \mathscr{F}(u)\big) \quad \text{for} \quad s > 0,$$

where $\mathscr{F}$ and $\mathscr{F}^{-1}$ are respectively the Fourier transform and its inverse [1].

Here we focus on the error estimates of the numerical methods under consideration. The far field boundary conditions are always assumed to be decreasing, which differs from many existing results in the literature. The main contribution of this work is twofold:

(1) We employ weighted trapezoidal rule with a splitting parameter to approximate the present hypersingular integral, and develop an efficient finite difference method for the fractional Poisson equation, where the integral corresponding to the boundary condition is evaluated by numerical quadratures.

(2) We develop and analyze simple finite difference approximations by truncating the computational domain. The error estimates are given to demonstrate the competition of discretization and truncation errors, so that the optimal convergence of the scheme can be recovered by selecting a suitable truncation parameter.

The rest of the paper is organized as follows. In Section 2, we develop and analyze a finite difference method for the fractional Poisson equation with the integral fractional Laplacian, and give its error estimates. Section 3 is devoted to a finite difference method by truncating the computational domain. The error estimates given demonstrate the effectiveness of the truncation. The results of numerical experiments presented in Section 4 confirm the theoretical findings. Finally, concluding remarks are made in Section 5.

2. Finite Difference Approximations

We start with finite difference approximations of the integral fractional Laplacian operator and then study the error estimates for the fractional Poisson equation (1.1). Let N be a positive integer and $h = 2L/N$ the spatial step size. Thus we consider the discrete grid $x_i = -L + ih$, $i \in \mathbb{Z}$. In addition, we also consider the mesh $\xi_j = jh$ for $0 \le j \le N$. Setting $\xi = |x - x'|$, we discretize the integral fractional Laplacian operator at the points x_i, $1 \le i \le N-1$ as follows:

$$
\begin{aligned}
(-\Delta)^s u(x_i) &= -c_{1,s} \int_0^\infty \frac{u(x_i - \xi) - 2u(x_i) + u(x_i + \xi)}{\xi^{1+2s}} d\xi \\
&= -c_{1,s} \left(\int_0^{2L} \frac{u(x_i - \xi) - 2u(x_i) + u(x_i + \xi)}{\xi^{1+2s}} d\xi \right. \\
&\qquad \left. + \int_{2L}^\infty \frac{f(x_i - \xi) - 2u(x_i) + f(x_i + \xi)}{\xi^{1+2s}} d\xi \right) \\
&= -c_{1,s} \left(\int_0^{2L} \psi_\gamma(x_i, \xi) \omega_\gamma(\xi) d\xi - 2u(x_i) \int_{2L}^\infty \frac{1}{\xi^{1+2s}} d\xi \right. \\
&\qquad \left. + \int_{2L}^\infty \frac{f(x_i - \xi) + f(x_i + \xi)}{\xi^{1+2s}} d\xi \right) \\
&= -c_{1,s} \left(\sum_{j=1}^N \int_{\xi_{j-1}}^{\xi_j} \psi_\gamma(x_i, \xi) \omega_\gamma(\xi) d\xi - 2u(x_i) \int_{2L}^\infty \frac{1}{\xi^{1+2s}} d\xi + F_i + \epsilon_i \right), \quad (2.1)
\end{aligned}
$$

where $\psi_\gamma(x,\xi)$ and $\omega_\gamma(\xi)$ are defined by

$$\psi_\gamma(x_i,\xi)=\frac{u(x_i-\xi)-2u(x_i)+u(x_i+\xi)}{\xi^\gamma},\quad \omega_\gamma(\xi)=\xi^{\gamma-(1+2s)},\quad \gamma\in(2s,2],$$

γ is a splitting parameter, F_i the approximation of the integral by the numerical quadrature, and ϵ_i the truncation error.

Motivated by the work of [5], we approximate the first term on the right-hand side of (2.1) by the weighted trapezoidal rule — viz.

$$\int_{\xi_0}^{\xi_1}\psi_\gamma(x_i,\xi)\omega_\gamma(\xi)d\xi\approx\begin{cases}\psi_2(x_i,\xi_1)\displaystyle\int_{\xi_0}^{\xi_1}\omega_2(\xi)d\xi, & \text{if}\quad \gamma=2,\\ \dfrac{1}{2}\psi_\gamma(x_i,\xi_1)\displaystyle\int_{\xi_0}^{\xi_1}\omega_\gamma(\xi)d\xi, & \text{if}\quad \gamma\in(2s,2),\end{cases}\tag{2.2}$$

$$\begin{aligned}\int_{\xi_{j-1}}^{\xi_j}\psi_\gamma(x_i,\xi)\omega_\gamma(\xi)d\xi\approx{}&\frac{1}{2}\big(\psi_\gamma(x_i,\xi_{j-1})+\psi_\gamma(x_i,\xi_j)\big)\\ &\times\int_{\xi_{j-1}}^{\xi_j}\omega_\gamma(\xi)d\xi,\quad 2\le j\le N.\end{aligned}\tag{2.3}$$

We denote $u_i=u(x_i)$, and let U_i be the finite difference approximation of u_i. Using (2.1)-(2.3), we write the finite difference approximation of the integral fractional Laplacian operator at the grid point x_i, $1\le i\le N-1$ as

$$\begin{aligned}(-\Delta)^s_{h,\gamma}U_i=-\frac{c_{1,s}}{2}\Bigg(&\left\lfloor\frac{\gamma}{2}\right\rfloor\psi_{\gamma,i}(\xi_1)\int_{\xi_0}^{\xi_1}\omega_\gamma(\xi)d\xi+\sum_{j=1}^{N}\psi_{\gamma,i}(\xi_j)\int_{T_j}\omega_\gamma(\xi)d\xi\\ &-4U_i\int_{2L}^{\infty}\xi^{-(1+2s)}d\xi+2F_i\Bigg),\end{aligned}\tag{2.4}$$

where

$$\begin{aligned}&T_j=(\xi_{j-1},\xi_{j+1})\cap[0,2L],\quad 1\le j\le N,\\ &\psi_{\gamma,i}(\xi_j)=(U_{i-j}-2U_i+U_{i+j})(jh)^{-\gamma},\end{aligned}$$

and $\lfloor\cdot\rfloor$ denotes the floor function.

Denote $\nu=\gamma-2s$ and

$$\begin{aligned}d_0&=\sum_{j=1}^{N-1}\frac{(j+1)^\nu-(j-1)^\nu}{j^\gamma}+\frac{N^\nu-(N-1)^\nu}{N^\gamma}+\left\lfloor\frac{\gamma}{2}\right\rfloor+\frac{\nu}{sN^{2s}},\\ d_1&=-\frac{1}{2}\left(\left\lfloor\frac{\gamma}{2}\right\rfloor+2^\nu\right),\\ d_j&=-\frac{1}{2}\left(\frac{(j+1)^\nu-(j-1)^\nu}{j^\gamma}\right),\quad j=2,\dots,N-1,\\ d_N&=-\frac{1}{2}\left(\frac{N^\nu-(N-1)^\nu}{N^\gamma}\right),\qquad j=N.\end{aligned}\tag{2.5}$$

Consequently, the Eq. (2.4) can be written as

$$(-\Delta)_{h,\gamma}^{s} U_i = C_{\gamma,s}\left(d_0 U_i + \sum_{j=1}^{N} d_j U_{i-j} + \sum_{j=1}^{N} d_j U_{i+j}\right) - c_{1,s} F_i, \quad 1 \le i \le N-1 \tag{2.6}$$

with

$$U_{N-1+j} = f(x_{N-1+j}), \quad U_{1-j} = f(x_{1-j}), \quad 1 \le j \le N$$

and $C_{\gamma,s} = c_{1,s}/(\nu h^{2s})$.

For $i \le 0$ and $i \ge N$ we write $f_i = f(x_i)$ and $g_i = g(x_i)$ when $1 \le i \le N-1$. Thus the finite difference approximation of (1.1) has the form

$$C_{\gamma,s}\left(d_0 U_i + \sum_{j=1}^{N} d_j U_{i-j} + \sum_{j=1}^{N} d_j U_{i+j}\right) - c_{1,s} F_i = g_i, \quad 1 \le i \le N-1. \tag{2.7}$$

Furthermore, introducing the terms

$$\rho_i = C_{\gamma,s}\left(\sum_{j=i}^{N} d_j f_{i-j} + \sum_{j=N-i}^{N} d_j f_{i+j}\right) - c_{1,s} F_i$$

and using the notations

$$\mathbf{U} = (U_1, U_2, \cdots, U_{N-1})^T, \quad \boldsymbol{\rho} = (\rho_1, \rho_2, \cdots, \rho_{N-1})^T, \quad \mathbf{G} = (g_1, g_2, \cdots, g_{N-1})^T,$$

we write the system (2.7) as

$$\mathbf{AU} = \mathbf{G} - \boldsymbol{\rho}, \tag{2.8}$$

where

$$\mathbf{A} = C_{\gamma,s}\begin{pmatrix} d_0 & d_1 & \cdots & d_{N-3} & d_{N-2} \\ d_1 & d_0 & d_1 & \cdots & d_{N-3} \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ d_{N-3} & \cdots & d_1 & d_0 & d_1 \\ d_{N-2} & d_{N-3} & \cdots & d_1 & d_0 \end{pmatrix}. \tag{2.9}$$

It is worth noting that $\mathbf{A}$ is a positive definite Toeplitz matrix. Therefore, the system of linear algebraic equations (2.8) can be efficiently solved by various Krylov subspace methods, where matrix-vector multiplication operations can be efficiently performed by the fast Fourier transform method — cf. [13, 18, 22].

2.1. Error estimates

We denote by E_Δ the local truncation error — i.e.

$$E_\Delta = \left\|(-\Delta)^s \mathbf{u} - (-\Delta)_{h,\gamma}^{s} \mathbf{u}\right\|_\infty.$$

Theorem 2.1. *Let* $\mathbf{u} = [u_1, u_2, \ldots, u_{N-1}]^\top$ *be the exact solution vector to the fractional Poisson equation* (1.1). *If* $u \in C^{\lfloor 2s\rfloor, 2s-\lfloor 2s\rfloor+\varepsilon}(\mathbb{R})$ *with* $0 < \varepsilon \le 1 + \lfloor 2s\rfloor - 2s$, *then for any splitting parameter* $\gamma \in (2s, 2]$, $s \in (0,1)$ *the following estimate holds:*

$$E_\Delta \le Ch^\varepsilon + \epsilon, \tag{2.10}$$

where C *is a positive constant independent of* h *and* $\epsilon := c_{1,s}\max_{1\le i\le N-1}|\epsilon_i|$ *an arbitrarily small positive constant. Besides, if* $u \in C^{2+\lfloor 2s\rfloor, 2s-\lfloor 2s\rfloor+\varepsilon}(\mathbb{R})$ *with* $0 < \varepsilon \le 1 + \lfloor 2s\rfloor - 2s$, *then for* $\gamma = 2$ *or* $1+s$, $s \in (0,1)$ *we have*

$$E_\Delta \le Ch^2 + \epsilon \tag{2.11}$$

with a positive constant C *independent of* h.

Proof. We note that the local truncation error at x_i, $1 \le i \le N-1$ can be written as

$$\begin{aligned}
&(-\Delta)^s u(x_i) - (-\Delta)^s_{h,\gamma} u(x_i) \\
&= -\frac{c_{1,s}}{2}\Bigg[\int_{\xi_0}^{\xi_1}\big(2\psi_\gamma(x_i,\xi) - K_\gamma\psi_\gamma(x_i,\xi_1)\big)\,\omega_\gamma(\xi)d\xi \\
&\qquad + \sum_{j=2}^{N}\int_{\xi_{j-1}}^{\xi_j}\big(2\psi_\gamma(x_i,\xi) - \big(\psi_\gamma(x_i,\xi_{j-1}) + \psi_\gamma(x_i,\xi_j)\big)\big)\,\omega_\gamma(\xi)d\xi\Bigg] \\
&\quad - c_{1,s}\left(\int_{2L}^{\infty}\frac{f(x_i-\xi)+f(x_i+\xi)}{\xi^{1+2s}}d\xi - F_i\right) \\
&= \kappa_i - c_{1,s}\epsilon_i,
\end{aligned}$$

where

$$K_\gamma = \begin{cases} 1, & \text{if} \quad \gamma \in (2s,2), \\ 2, & \text{if} \quad \gamma = 2. \end{cases}$$

Following arguments in [5, 6], we obtain the estimates $|\kappa_i| \le Ch^k$ with a positive constant C independent of h. Here if $u \in C^{\lfloor 2s\rfloor, 2s-\lfloor 2s\rfloor+\varepsilon}(\mathbb{R})$ with $0 < \varepsilon \le 1 + \lfloor 2s\rfloor - 2s$, $\gamma \in (2s, 2]$, then $k = \varepsilon$. However, if $u \in C^{2+\lfloor 2s\rfloor, 2s-\lfloor 2s\rfloor+\varepsilon}(\mathbb{R})$ and $\gamma = 2$ or $1+s$, then the index can be improved to $k = 2$. Moreover, since $c_{1,s}\epsilon_i$ is a sufficiently small constant, the truncation error of the numerical quadrature can be bounded by ϵ. Combing these results finishes the proof. ☐

Theorem 2.2. *Let* $0 < \varepsilon \le 1 + \lfloor 2s\rfloor - 2s$. *If* $u \in C^{\lfloor 2s\rfloor, 2s-\lfloor 2s\rfloor+\varepsilon}(\mathbb{R})$ *is the solution of* (1.1) *and* $\mathbf{U}$ *the discretization solution of* (2.7), *then for any splitting parameter* $\gamma \in (2s, 2]$, $s \in (0,1)$ *the solution error* $E_{\mathbf{u}} = \|\mathbf{u} - \mathbf{U}\|_\infty$ *satisfies the estimate*

$$E_{\mathbf{u}} \le Ch^\varepsilon + \epsilon \tag{2.12}$$

with C a positive constant independent of h. If $u \in C^{2+\lfloor 2s\rfloor, 2s-\lfloor 2s\rfloor+\varepsilon}(\mathbb{R})$ with $0<\varepsilon\le 1+\lfloor 2s\rfloor - 2s$, then for $\gamma=2$ or $1+s$, $s\in(0,1)$, we have

$$E_{\mathbf{u}} \le Ch^2 + \epsilon \tag{2.13}$$

with a positive constant C independent of h.

Proof. Denoting the discretization error by $e_{h,i} = u_i - U_i$ and $\mathbf{E} = (e_{h,1}, e_{h,2}, \cdots, e_{h,N-1})^T$, we have

$$\mathbf{AE} = \mathbf{\Theta},$$

where $\mathbf{A}$ is the matrix defined in (2.9), and $\|\mathbf{\Theta}\|_\infty \le Ch^k + \epsilon$ holds with a positive constant C independent of h, in which k has been defined in the proof of Theorem 2.1.

It should be remarked that the entries of $\mathbf{A}$ satisfy the relations

$$\begin{aligned} &C_{\gamma,s} d_0 > 0, \quad C_{\gamma,s} d_i < 0 \quad \text{for} \quad i \ne 0, \\ &\inf_{i=1,\dots,N-1} \sum_{j=1}^{N-1} C_{\gamma,s} d_{|i-j|} > c_{1,s} \int_{2L}^{\infty} \xi^{-(1+2s)} d\xi > 0. \end{aligned}$$

Using [19, Theorem 4.1] leads to the estimates (2.12) and (2.13). □

3. Truncated Finite Difference Approximation

As is shown in Theorems 2.1 and 2.2, the accuracy of the aforementioned finite difference method relies on the evaluation of the integral in (2.1). However, if $f(x)$ is too complicated so that the numerical quadratures fail to work, then the accuracy of the numerical scheme significantly decreases. Besides, the boundary measurements used to determine the interior information, are very important to inverse problems, such as the fractional Calderón problem where the classical Laplacian is replaced by integral fractional Laplacian [8]. It is difficult to develop numerical methods for such problems and obtain the corresponding error estimates due to little prior knowledge of the boundary data.

Here we construct efficient finite difference approximations based on the truncation of the computational domain. The corresponding error estimates include truncation and discretization errors, and can be balanced by adjusting a truncation parameter. The method is easily implemented and flexible in choosing the truncation parameter. The theoretical results obtained can be used in numerical methods for inverse problems involving the integral fractional Laplacian.

Denoting the truncation parameter by R, we consider the following truncated fractional Poisson equation to account for the effect of the numerical quadratures on f by writing

$$\begin{aligned} &(-\Delta)^s u_R(x) = g(x), && x \in (-L, L), \\ &u_R(x) = f(x), && x \in (-R, R) \setminus (-L, L), \\ &u_R(x) = 0, && x \in \mathbb{R} \setminus (-R, R). \end{aligned} \tag{3.1}$$

Similarly, we introduce the finite difference approximation for the fractional Poisson equation (3.1). Setting $M=\lfloor\frac{R-L}{h}\rfloor+1, R>L$, we discretize the integral fractional Laplacian at the grid points x_i, $1\le i\le N-1$ as

$$
\begin{aligned}
(-\Delta)^s u_R(x_i) &= -c_{1,s}\int_0^\infty \frac{u_R(x_i-\xi)-2u_R(x_i)+u_R(x_i+\xi)}{\xi^{1+2s}}d\xi \\
&= -c_{1,s}\left(\int_0^{2L} \frac{u_R(x_i-\xi)-2u_R(x_i)+u_R(x_i+\xi)}{\xi^{1+2s}}d\xi \right.\\
&\qquad \left. +\int_{2L}^\infty \frac{u_R(x_i-\xi)-2u_R(x_i)+u_R(x_i+\xi)}{\xi^{1+2s}}d\xi\right) \\
&= -c_{1,s}\left(\int_0^{2L} \frac{u_R(x_i-\xi)-2u_R(x_i)+u_R(x_i+\xi)}{\xi^{1+2s}}d\xi \right.\\
&\qquad +\int_{2L}^{L+R} \frac{f(x_i-\xi)-2u_R(x_i)+f(x_i+\xi)}{\xi^{1+2s}}d\xi-2u_R(x_i)\\
&\qquad \left.\times\int_{L+R}^{+\infty}\frac{1}{\xi^{1+2s}}d\xi\right) \\
&= -c_{1,s}\left(\int_0^{2L}\psi_{\gamma,R}(x_i,\xi)\omega_\gamma(\xi)d\xi-2u_R(x_i)\int_{2L}^\infty\frac{1}{\xi^{1+2s}}d\xi \right.\\
&\qquad \left.+\int_{2L}^{L+R}\frac{f(x_i-\xi)+f(x_i+\xi)}{\xi^{1+2s}}d\xi\right) \\
&= -c_{1,s}\left(\sum_{j=1}^N\int_{\xi_{j-1}}^{\xi_j}\psi_{\gamma,R}(x_i,\xi)\omega_\gamma(\xi)d\xi-2u_R(x_i)\int_{2L}^\infty\frac{1}{\xi^{1+2s}}d\xi+F_{R,i}+\epsilon_{R,i}\right),
\end{aligned} \tag{3.2}
$$

where

$$
\psi_{\gamma,R}(x,\xi)=\frac{u_R(x-\xi)-2u_R(x)+u_R(x+\xi)}{\xi^\gamma},\quad \omega_\gamma(\xi)=\xi^{\gamma-(1+2s)},\quad \gamma\in(2s,2],
$$

$F_{R,i}$ refers to the integral approximated by numerical quadrature, and $\epsilon_{R,i}$ is a sufficiently small truncation error. Similar to the approach in Section 2, we approximate the corresponding integrals as

$$
\int_{\xi_0}^{\xi_1}\psi_{\gamma,R}(x_i,\xi)\omega_\gamma(\xi)d\xi\approx\begin{cases}\psi_{2,R}(x_i,\xi_1)\displaystyle\int_{\xi_0}^{\xi_1}\omega_2(\xi)d\xi, & \text{if}\quad \gamma=2,\\ \dfrac{1}{2}\psi_{\gamma,R}(x_i,\xi_1)\displaystyle\int_{\xi_0}^{\xi_1}\omega_\gamma(\xi)d\xi, & \text{if}\quad \gamma\in(2s,2),\end{cases} \tag{3.3}
$$

$$
\int_{\xi_{j-1}}^{\xi_j}\psi_{\gamma,R}(x_i,\xi)\omega_\gamma(\xi)d\xi\approx\frac{1}{2}\big(\psi_{\gamma,R}(x_i,\xi_{j-1})+\psi_{\gamma,R}(x_i,\xi_j)\big)
$$

$$\times \int_{\xi_{j-1}}^{\xi_j} \omega_\gamma(\xi) d\xi, \quad 2 \le j \le N. \tag{3.4}$$

For simplicity, we denote $u_{R,i} = u_R(x_i)$ and let $U_{R,i}$ be the finite difference approximation of $u_{R,i}$. Combining (3.2)-(3.4) leads to the following finite difference approximation of the integral fractional Laplacian at the grid point x_i, $1 \le i \le N-1$:

$$(-\Delta)^s_{h,\gamma} U_{R,i} = -\frac{c_{1,s}}{2} \Bigg(\left\lfloor \frac{\gamma}{2} \right\rfloor \psi_{\gamma,R,i}(\xi_1) \int_{\xi_0}^{\xi_1} \omega_\gamma(\xi) d\xi + \sum_{j=1}^{N} \psi_{\gamma,R,i}(\xi_j) \int_{T_j} \omega_\gamma(\xi) d\xi$$
$$-4U_{R,i} \int_{2L}^{\infty} \xi^{-(1+2s)} d\xi + 2F_{R,i} \Bigg),$$

where

$$\psi_{\gamma,R,i}(\xi_j) := (U_{R,i-j} - 2U_{R,i} + U_{R,i+j})(jh)^{-\gamma}.$$

The finite difference approximation of (3.1) can be also represented as

$$\begin{aligned} & C_{\gamma,s} \left(d_0 U_{R,i} + \sum_{j=1}^{N} d_j U_{R,i-j} + \sum_{j=1}^{N} d_j U_{R,i+j} \right) - c_{1,s} F_{R,i} = g_i, \quad 1 \le i \le N-1, \\ & U_{R,i} = f_i, \quad -M+1 \le i \le 0 \quad \text{or} \quad N \le i \le N+2M-1, \\ & U_{R,i} = 0, \qquad i \le -M \quad \text{or} \quad i \ge N+2M, \end{aligned} \tag{3.5}$$

if we recall the notations from (2.5). On the other hand, introducing the terms

$$\rho_{R,i} = C_{\gamma,s} \left(\sum_{j=i}^{\min(N,i+M-1)} d_j f_{i-j} + \sum_{j=N-i}^{\min(N,N+M-1-i)} d_j f_{i+j} \right) - c_{1,s} F_{R,i},$$

and the vectors

$$\mathbf{U}_R = (U_{R,1}, U_{R,2}, \cdots, U_{R,N-1})^T, \quad \boldsymbol{\rho}_R = (\rho_{R,1}, \rho_{R,2}, \cdots, \rho_{R,N-1})^T,$$

we write the Eqs.(3.5) in the matrix-vector form

$$\mathbf{A}\mathbf{U}_R = \mathbf{G} - \boldsymbol{\rho}_R, \tag{3.6}$$

where $\mathbf{A}$ is defined in (2.9). As was already mentioned, $\mathbf{A}$ is a positive definite Toeplitz matrix, so the linear algebraic system (3.6) can be efficiently solved by many Krylov subspace methods using the fast Fourier transform.

3.1. Error estimates

Theorem 3.1. *Let $\mathbf{u}_R = [u_{R,1}, u_{R,2}, \ldots, u_{R,N-1}]^\top$ be the exact solution vector to the fractional Poisson equation* (3.1). *If $u \in C^{\lfloor 2s \rfloor, 2s - \lfloor 2s \rfloor + \varepsilon}(\mathbb{R})$, $0 < \varepsilon \le 1 + \lfloor 2s \rfloor - 2s$ is the solution of* (1.1), *then for any splitting parameter $\gamma \in (2s, 2]$, $s \in (0,1)$, the local truncation error*

$$E_{\Delta,R} = \left\| (-\Delta)^s \mathbf{u} - (-\Delta)^s_{h,\gamma} \mathbf{u}_R \right\|_\infty$$

can be estimated as follows:

$$E_{\Delta,R} \le C\left(h^{\varepsilon} + R^{-2s}\max\{f(-R), f(R)\}\right) + 2\epsilon + \epsilon_R \tag{3.7}$$

with a positive constant C independent of h and R and an arbitrarily small constant $\epsilon_R = c_{1,s}\max_{1\le i\le N-1}|\epsilon_{R,i}|$. If $u \in C^{2+\lfloor 2s\rfloor, 2s-\lfloor 2s\rfloor+\varepsilon}(\mathbb{R})$ with $0 < \varepsilon \le 1 + \lfloor 2s\rfloor - 2s$, then for $\gamma = 2$ or $1+s$, $s \in (0,1)$, the local truncation error satisfies

$$E_{\Delta,R} \le C\left(h^{2} + R^{-2s}\max\{f(-R), f(R)\}\right) + 2\epsilon + \epsilon_R \tag{3.8}$$

with a positive constant C independent of h and R.

Proof. Combining Theorem 2.1 for $1 \le i \le N-1$ gives

$$\begin{aligned}
&\left|(-\Delta)^s u_i - (-\Delta)^s_{h,\gamma} u_{R,i}\right| \\
\le& \left|(-\Delta)^s u_i - (-\Delta)^s_{h,\gamma} u_i\right| + \left|(-\Delta)^s_{h,\gamma} u_i - (-\Delta)^s_{h,\gamma} u_{R,i}\right| \\
\le& C_1 h^k + \epsilon + \left|C_{\gamma,s}\Bigg(\sum_{j=\min(i+M,N+1)}^{N} d_j f_{i-j} + \sum_{j=\min(N+M-i,N+1)}^{N} d_j f_{i+j}\Bigg) - c_{1,s}(F_i - F_{R,i})\right| \\
\le& C_1 h^k + \epsilon + \left|C_{\gamma,s}\Bigg(\sum_{j=\min(i+M,N+1)}^{N} d_j f_{i-j} + \sum_{j=\min(N+M-i,N+1)}^{N} d_j f_{i+j}\Bigg) - c_{1,s}\right. \\
&\left.\times\left(\left(\int_{2L}^{\infty} \frac{f(x_i-\xi)+f(x_i+\xi)}{\xi^{1+2s}}d\xi - \epsilon_i\right) - \left(\int_{2L}^{L+R} \frac{f(x_i-\xi)+f(x_i+\xi)}{\xi^{1+2s}}d\xi - \epsilon_{R,i}\right)\right)\right| \\
\le& C_1 h^k + C_2 R^{-2s}\max\left\{f(-R), f(R)\right\} + \epsilon + |\epsilon_i| + |\epsilon_{R,i}| \\
\le& C\left(h^k + R^{-2s}\max\left\{f(-R), f(R)\right\}\right) + 2\epsilon + \epsilon_R,
\end{aligned} \tag{3.9}$$

where C_1, C_2, C are positive constants independent of h and R, and k is the same as the one in Theorem 2.1. ☐

Remark 3.1. In the proof of Theorem 3.1, we have to estimate the sum in (3.9) for $M < N$. For example, for $i = 1$ the first term in the sum can be estimated as

$$\begin{aligned}
\left|C_{\gamma,s} d_{M+1} f_{-M}\right| &\le f(-R)\left|\frac{c_{1,s}}{2(\gamma-2s)h^{2s}}\left(\frac{(M+2)^{\gamma-2s} - M^{\gamma-2s}}{(M+1)^{\gamma}}\right)\right| \\
&\le C(Mh)^{-2s} f(-R) \le C(R-L)^{-2s} f(-R),
\end{aligned}$$

where C is a positive constant independent of h, L and R. It is obvious that the error could be large if R is close to L. Therefore, we have to choose a sufficiently large R to ensure the optimal convergence rate in Theorem 3.1, e.g. $R \ge 3L$, which is approximately equivalent to $M \ge N$.

Theorem 3.2. *Let $0 < \varepsilon \leq 1 + \lfloor 2s \rfloor - 2s$. If $u \in C^{\lfloor 2s \rfloor, 2s - \lfloor 2s \rfloor + \varepsilon}(\mathbb{R})$ is the solution of* (1.1) *and* $\mathbf{U}_R$ *the discretization solution of* (3.5)*, then for any splitting parameter $\gamma \in (2s, 2]$, $s \in (0,1)$, the solution error $E_{\mathbf{u},R} = \|\mathbf{u} - \mathbf{U}_R\|_\infty$ satisfies the inequality*

$$E_{\mathbf{u},R} \leq C\left(h^\varepsilon + R^{-2s} \max\left\{f(-R), f(R)\right\}\right) + 2\epsilon + \epsilon_R \tag{3.10}$$

with a positive constant C independent of h and R. If $u \in C^{2+\lfloor 2s \rfloor, 2s - \lfloor 2s \rfloor + \varepsilon}(\mathbb{R})$ with $0 < \varepsilon \leq 1 + \lfloor 2s \rfloor - 2s$, then for $\gamma = 2$ or $1 + s$, $s \in (0,1)$, the solution error satisfies

$$E_{\mathbf{u},R} \leq C\left(h^2 + R^{-2s} \max\left\{f(-R), f(R)\right\}\right) + 2\epsilon + \epsilon_R \tag{3.11}$$

with a positive constant C independent of h and R.

Proof. Setting

$$\begin{aligned} &e_i = u_i - U_{R,i}, &&\mathbf{e} = (e_1, e_2, \cdots, e_{N-1})^\top, \\ &e_{h,R,i} = U_i - U_{R,i}, &&\mathbf{E}_R = (e_{h,R,1}, e_{h,R,2}, \cdots, e_{h,R,N-1})^\top \end{aligned}$$

and combining (2.7), (3.5) yields

$$\mathbf{A}\mathbf{E}_R = \mathbf{\Theta}_R,$$

where $\mathbf{A}$ is defined in (2.9), and

$$\|\mathbf{\Theta}_R\|_\infty \leq C R^{-2s} \max\left\{f(-R), f(R)\right\} + \epsilon + \epsilon_R$$

with a positive constant C independent of R.

Analogously, [19, Theorem 4.1] gives

$$\|\mathbf{E}_R\|_\infty \leq C R^{-2s} \max\left\{f(-R), f(R)\right\} + \epsilon + \epsilon_R.$$

Therefore,

$$\begin{aligned} \|\mathbf{e}\|_\infty &= \|\mathbf{E} + \mathbf{E}_R\|_\infty \leq \|\mathbf{E}\|_\infty + \|\mathbf{E}_R\|_\infty \\ &\leq C\left(h^k + R^{-2s} \max\left\{f(-R), f(R)\right\}\right) + 2\epsilon + \epsilon_R \end{aligned}$$

with a positive constant C independent of h and R, and the index k has been defined in Theorem 2.1. □

Remark 3.2. Theorem 3.2 shows that the error can be divided into two parts — viz. the discretization error h^k depending on the smoothness of the solution and the splitting parameter and the truncation error $R^{-2s} \max\{f(-R), f(R)\}$. In order to balance these errors, we can choose $R^{-2s} \max\{f(-R), f(R)\} = \mathcal{O}(h^k)$ thus obtaining $\|\mathbf{e}\|_\infty \leq C h^k$.

4. Numerical Experiments

We carry out numerical experiments to verify the theoretical results in Theorems 2.1, 2.2, 3.1, and 3.2. More exactly, we investigate the influence of the smoothness of the solution and the decay of the boundary condition $f(x)$ on the convergence of methods studied. In what follows, we use two types of hypergeometric functions — viz.

$$ {}_1F_1(a;b;x) = \sum_{k=0}^{\infty} \frac{(a)_k}{(b)_k} \frac{x^k}{k!}, \qquad |x| < 1, \tag{4.1} $$

$$ {}_2F_1(a,b;c;x) = \sum_{k=0}^{\infty} \frac{(a)_k (b)_k}{(c)_k} \frac{x^k}{k!}, \quad |x| < 1, \tag{4.2} $$

where a, b, c with $c \neq 0, -1, -2, \cdots$ are real numbers and

$$ (a)_0 = 1, \quad (a)_k = a(a+1)(a+2)\cdots(a+k-1) = \frac{\Gamma(a+k)}{\Gamma(a)}, \quad k \in \mathbb{N}, $$

cf. [2]. Note that the hypergeometric functions can be extended elsewhere by analytic continuation.

Experiment 4.1 (cf. Dyda [7]). Assume that the Eq. (1.1) has the solution

$$ u(x) = (1 - x^2)_+^{m+r} + c, \quad x \in (-1, 1), \tag{4.3} $$

where c is a positive constant, $r \in (0, 1]$, $m \geq 0$ is an integer, and

$$ (1 - x^2)_+^{m+r} = \max\left\{0, (1 - x^2)^{m+r}\right\}. $$

Then we have $u \in C^{m,r}(\mathbb{R})$, $f(x) = c$ for $x \in \mathbb{R} \setminus (-1, 1)$ and the right-hand side function

$$ g(x) = (-\Delta)^s u(x) = \frac{2^{2s}\Gamma(s + 1/2)\Gamma(m + 1 + r)}{\sqrt{\pi}\Gamma(m + r + 1 - s)} {}_2F_1\left(s + \frac{1}{2}, -(m + r) + s; \frac{1}{2}; |x|^2\right), $$

where ${}_1F_1(a, b, x)$ denotes the hypergeometric function (4.1).

Tables 1 and 2 show how the smoothness of the solution influence the convergence rates of the method under consideration. Note that if $u = (1-x^2)_+^{1+\lfloor 2s \rfloor} + 2^{-2}$ in (4.3), then $u \in C^{0,1}(\mathbb{R})$ for $s \in (0, 0.5)$, and $u \in C^{1,1}(\mathbb{R})$ for $s \in [0.5, 1)$. Choosing splitting parameter $\gamma = 2$, we demonstrate the errors and convergence rates (c.r.) in Table 1. Moreover, if $u = (1 - x^2)_+^{2+2s+0.1} + 2^{-2}$, then $u \in C^{2,2s+0.1}(\mathbb{R})$ for $s \in (0, 0.5)$ and $u \in C^{3,2s-1+0.1}(\mathbb{R})$ for $s \in [0.5, 1)$. We also choose splitting parameter $\gamma = 2$ and present the errors and convergence rates of the method in Table 2. It can be clearly observed that the smoothness of the solutions has significant effect on the convergence. The method has accuracy of order $\mathcal{O}(h^{\varepsilon})$ if $u \in C^{\lfloor 2s \rfloor, 2s - \lfloor 2s \rfloor + \varepsilon}(\mathbb{R})$ and of order $\mathcal{O}(h^2)$ if $u \in C^{2+\lfloor 2s \rfloor, 2s - \lfloor 2s \rfloor + \varepsilon}(\mathbb{R})$. These results are consistent with the conclusions of Theorem 2.1.

As was already mentioned, the splitting parameter γ should be fixed to 2 or $1 + s$ to make sure that the method has the second-order accuracy. In Fig. 1, we use various γ to

Table 1: Experiment 4.1. Errors and convergence rates, $u=(1-x^2)_+^{1+\lfloor 2s\rfloor}+2^{-2}$, $\gamma=2$.

E_Δ	$h=2^{-5}$	$h=2^{-6}$	$h=2^{-7}$	$h=2^{-8}$	$h=2^{-9}$
$s=0.1$	9.489E-4	5.556E-4	3.164E-4	1.810E-4	1.037E-4
	c.r.	0.7722	0.8123	0.8058	0.8036
$s=0.3$	9.247E-3	6.971E-3	5.271E-3	3.990E-3	3.022E-3
	c.r.	0.4076	0.4033	0.4017	0.4009
$s=0.5$	3.834E-4	1.891E-4	9.436E-5	4.719E-5	2.360E-5
	c.r.	1.0197	1.0029	0.9997	0.9997
$s=0.7$	1.234E-3	7.618E-4	4.870E-4	2.949E-4	1.890E-4
	c.r.	0.6959	0.6455	0.7237	0.6418
$s=0.9$	4.738E-3	4.294E-3	3.686E-3	3.157E-3	2.720E-3
	c.r.	0.1420	0.2203	0.2235	0.2149

Table 2: Experiment 4.1. Errors and convergence rates, $u=(1-x^2)_+^{2+2s+0.1}+2^{-2}$, $\gamma=2$.

E_Δ	$h=2^{-5}$	$h=2^{-6}$	$h=2^{-7}$	$h=2^{-8}$	$h=2^{-9}$
$s=0.1$	7.692E-5	1.937E-5	4.853E-6	1.213E-6	3.032E-7
	c.r.	1.9895	1.9969	2.0003	2.0002
$s=0.3$	1.189E-4	3.090E-5	7.872E-6	1.980E-6	4.962E-7
	c.r.	1.9441	1.9728	1.9912	1.9965
$s=0.5$	7.513E-5	9.484E-6	2.049E-6	6.490E-7	1.716E-7
	c.r.	2.9858	2.2106	1.6586	1.9192
$s=0.7$	1.238E-3	4.287E-4	1.242E-4	3.355E-5	8.782E-6
	c.r.	1.5300	1.7873	1.8883	1.9337
$s=0.9$	9.327E-3	2.709E-3	7.102E-4	1.780E-4	4.370E-5
	c.r.	1.7837	1.9315	1.9963	2.0262

check the convergence for $u\in C^{2+\lfloor 2s\rfloor,2s-\lfloor 2s\rfloor+\varepsilon}(\mathbb{R})$. It is clear that convergence rate heavily depends on γ and is equal to 2 if $\gamma=2$ or $1+s$. Otherwise, the numerical scheme has only a sub-optimal convergence. It is consistent with the theoretical results in Theorem 2.1.

Table 3 shows numerical errors and convergence rates of the finite difference approximations for the solutions of the fractional Poisson equation (1.1). We observe that the convergence rate is $\mathscr{O}(h^{\min\{m+r,2\}})$ for $u\in C^{m,r}(\mathbb{R})$, better than the one in Theorem 2.2. This shows that the developed numerical scheme is more accurate.

Experiment 4.2 (cf. Sheng *et al.* [17]). Consider the convergence rates of the method for equations with sufficiently smooth solutions. Choose $u(x)=(1+|x|^2)^{-2}, x\in(-1,1)$. Then $f(x)=(1+|x|^2)^{-2}$ for $x\in\mathbb{R}\setminus(-1,1)$ and

$$g(x)=\frac{2^{2s}\Gamma(s+2)\Gamma(s+1/2)}{\sqrt{\pi}}\,{}_2F_1\left(s+2,s+\frac{1}{2};\frac{1}{2};-|x|^2\right),\quad x\in(-1,1).$$

Table 3: Experiment 4.1. Numerical error $E_{\mathbf{u}}$, $u=(1-x^2)_+^{m+r}+2^{-2}, \gamma=2$.

$E_{\mathbf{u}}$	$h=2^{-5}$	$h=2^{-6}$	$h=2^{-7}$	$h=2^{-8}$	$h=2^{-9}$
			$m=0, \quad r=0.5$		
$s=0.3$	2.312E-2	1.501E-2	1.059E-2	7.477E-3	5.284E-3
	c.r.	0.6232	0.5032	0.5022	0.5008
$s=0.5$	3.451E-2	2.431E-2	1.716E-2	1.212E-2	8.567E-3
	c.r.	0.5055	0.5025	0.5017	0.5005
$s=0.7$	5.036E-2	3.552E-2	2.508E-2	1.772E-2	1.253E-2
	c.r.	0.5036	0.5021	0.5012	0.5000
			$m=0, \quad r=1$		
$s=0.3$	1.359E-3	6.720E-4	3.341E-4	1.665E-4	8.314E-5
	c.r.	1.0160	1.0082	1.0048	1.0019
$s=0.5$	1.651E-3	8.214E-4	4.096E-4	2.045E-4	1.022E-4
	c.r.	1.0072	1.0039	1.0021	1.0007
$s=0.7$	1.381E-3	6.892E-4	3.443E-4	1.720E-4	8.600E-5
	c.r.	1.0027	1.0013	1.0013	1.0000
			$m=1, \quad r=1$		
$s=0.3$	6.174E-5	1.636E-5	4.273E-6	1.095E-6	2.791E-7
	c.r.	1.9160	1.9369	1.9643	1.9721
$s=0.5$	2.175E-5	4.058E-6	8.423E-7	2.057E-7	5.080E-8
	c.r.	2.4222	2.2684	2.0338	2.0176
$s=0.7$	1.337E-4	2.738E-5	5.788E-6	1.262E-6	2.826E-7
	c.r.	2.2878	2.2420	2.1974	2.1589
			$m=3, \quad r=1$		
$s=0.3$	5.981E-5	1.539E-5	3.888E-6	9.760E-7	2.444E-7
	c.r.	1.9584	1.9849	1.9941	1.9976
$s=0.5$	2.015E-5	3.559E-6	7.027E-7	1.522E-7	3.509E-8
	c.r.	2.5012	2.3405	2.2069	2.1168
$s=0.7$	1.411E-4	3.123E-5	7.134E-6	1.672E-6	3.995E-7
	c.r.	2.1757	2.1301	2.0931	2.0653

Table 4: Experiment 4.2. Errors and corresponding convergence rates, $u=(1+|x|^2)^{-2}$, $\gamma=1+s$.

E_{Δ}	$h=2^{-5}$	$h=2^{-6}$	$h=2^{-7}$	$h=2^{-8}$	$h=2^{-9}$
$s=0.25$	1.068E-4	2.661E-5	6.644E-6	1.660E-6	4.150E-7
	c.r.	2.0049	2.0018	2.0009	2.0000
$s=0.5$	3.259E-4	7.958E-5	1.966E-5	4.884E-6	1.217E-6
	c.r.	2.0340	2.0171	2.0091	2.0047
$s=0.75$	1.050E-3	2.431E-4	5.734E-5	1.373E-5	3.325E-6
	c.r.	2.1108	2.0839	2.0622	2.0459

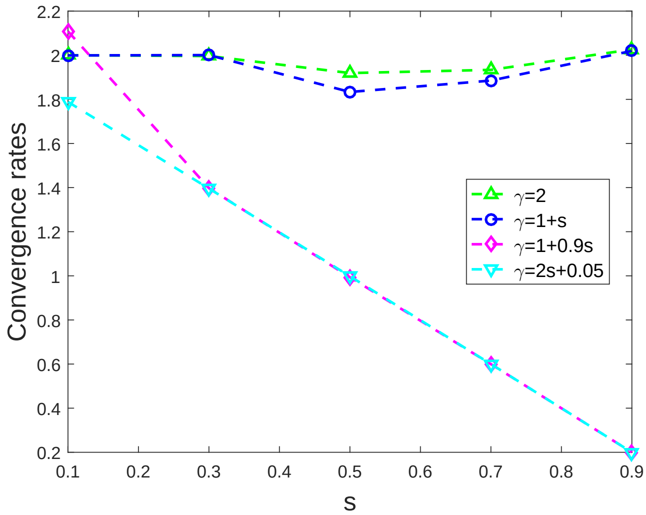

Figure 1: Convergence of the finite difference methods, $u \in C^{2+\lfloor 2s \rfloor, 2s-\lfloor 2s \rfloor+\varepsilon}(\mathbb{R})$, γ changes.

In Table 4, we present the errors and corresponding convergence rates of the finite difference approximations for $\gamma = 1+s$. Note that if the solution is sufficiently smooth, the numerical scheme has the second-order accuracy — i.e. the error estimates in Theorem 2.1 are optimal.

Theorem 3.2 shows that the convergence rates of the methods depend on s, R, γ and on the solution smoothness. More exactly, the solution error has the second-order accuracy if $u \in C^{2+\lfloor 2s \rfloor, 2s-\lfloor 2s \rfloor+\varepsilon}(\mathbb{R})$ with $0 < \varepsilon \leq 1+\lfloor 2s \rfloor - 2s$, $\gamma = 1+s$, and the truncation error is comparable to the discretization error. Next choose the approximate truncation as R to make sure that $R^{-2s} \max\{f(-R), f(R)\} = h^2$ and demonstrate in Table 5 the numerical errors and

Table 5: Experiment 4.2. Numerical error $E_{\mathbf{u},R}$, $u = (1+|x|^2)^{-2}$, $\gamma = 1+s$.

$s = 0.25$	h	2^{-5}	2^{-6}	2^{-7}	2^{-8}	2^{-9}	2^{-10}
	R	4.5701	6.2793	8.5889	11.7200	15.9722	21.7522
	$E_{\mathbf{u},R}$	4.924E-5	1.407E-5	3.961E-6	1.088E-6	2.928E-7	7.764E-8
		c.r.	1.8072	1.8287	1.8642	1.8937	1.9150
$s = 0.5$	h	2^{-5}	2^{-6}	2^{-7}	2^{-8}	2^{-9}	2^{-10}
	R	3.8994	5.2020	6.9069	9.1460	12.0927	15.9750
	$E_{\mathbf{u},R}$	5.419E-5	1.308E-5	3.756E-6	1.056E-6	2.906E-7	7.861E-8
		c.r.	2.0507	1.8001	1.8306	1.8615	1.8862
$s = 0.75$	h	2^{-5}	2^{-6}	2^{-7}	2^{-8}	2^{-9}	2^{-10}
	R	3.4229	4.4569	5.7756	7.4630	9.6271	12.4060
	$E_{\mathbf{u},R}$	1.271E-4	2.778E-5	6.137E-6	1.365E-6	3.047E-7	6.849E-8
		c.r.	2.1938	2.1784	2.1686	2.1634	2.1534

convergence rates for the finite difference approximations of the truncated fractional Poisson equation (3.1). Note that the approximations are of second-order accuracy, consistent with the theoretical results.

Experiment 4.3. Consider the convergence of the finite difference approximations for the truncated fractional Poisson equation (3.1). Let $u(x)=(1+x^2)_+^{1+\lfloor 2s\rfloor}+2^{-4}$ with $x\in(-1,1)$, $s=0.3,0.5,0.7$, and $\gamma=2$, and $f(x)$ and $g(x)$ can be determined from $u(x)$. To verify the theoretical results of Theorem 3.1, we always choose R to satisfy the condition

$$R^{-2s}\max\big\{f(-R),f(R)\big\}=h^{\varepsilon}$$

for different h and ε. Recall that ε is given in Theorem 3.1.

Table 6 displays numerical errors $E_{\Delta,R}$ for different h and R. It should be noted that the convergence rate of the approximations is ε provided that the discretization error is comparable to the truncation error — e.g. Theorem 3.1 shows that for $s=0.5$ and $h=2^{-10}$, if $R=64$ then we immediately get $R^{-2s}\max\{f(-R),f(R)\}=h^{\varepsilon}$, and the convergence rate is equal to $\varepsilon=1$. Consequently, the conclusions of Theorem 3.1 are confirmed.

Table 6: Experiment 4.3. Errors and convergence rates, $u=(1+x^2)_+^{1+\lfloor 2s\rfloor}+2^{-4}$, $\gamma=2$.

$s=0.5$	h	2^{-5}	2^{-6}	2^{-7}	2^{-8}	2^{-9}	2^{-10}
	R	2	4	8	16	32	64
	$E_{\Delta,R}$	2.252E-2	7.956E-3	4.421E-3	2.340E-3	1.206E-3	6.121E-4
	c.r.		1.5011	0.8477	0.9179	0.9563	0.9784
$s=0.3$	h	2^{-14}	2^{-15}	2^{-16}	2^{-17}	2^{-18}	2^{-19}
	R	6.3496	10.0794	16	25.3984	40.3175	64
	$E_{\Delta,R}$	1.524E-2	1.190E-2	9.192E-3	7.054E-3	5.389E-3	4.105E-3
	c.r.		0.3569	0.3725	0.3819	0.3884	0.3926
$s=0.7$	h	2^{-14}	2^{-15}	2^{-16}	2^{-17}	2^{-18}	2^{-19}
	R	8.8327	11.8880	16	21.5344	28.9832	39.0084
	$E_{\Delta,R}$	1.187E-3	8.122E-4	5.509E-4	3.711E-4	2.488E-4	1.665E-4
	c.r.		0.5474	0.5600	0.5700	0.5768	0.5795

Experiment 4.4 (cf. Sheng *et al.* [17]). Let $u(x)=e^{-|x|^2}, x\in(-1,1)$, then we have [17]

$$g(x)=(-\Delta)^s u=\frac{2^{2s}\Gamma(s+1/2)}{\Gamma(1/2)}\,{}_1F_1\left(s+\frac{1}{2};\frac{1}{2};-|x|^2\right),\quad x\in(-1,1),$$

and $f(x)=e^{-|x|^2}$ for $x\in\mathbb{R}\setminus(-1,1)$.

According to Theorem 3.2, for $u\in C^{2+\lfloor 2s\rfloor,2s-\lfloor 2s\rfloor+\varepsilon}(\mathbb{R})$, $0<\varepsilon\le 1+\lfloor 2s\rfloor-2s$, and $\gamma=2$, the errors have the accuracy of order $\mathscr{O}(h^2+R^{-2s}\max\{f(-R),f(R)\})$, and if the truncation parameter R is chosen so that $R^{-2s}\max\{f(-R),f(R)\}\le h^2$, then we have the second-order convergence. Otherwise the convergence rates are unstable or even vanish

Table 7: Experiment 4.4. Numerical error $E_{\mathbf{u},R}$, $u=e^{-|x|^2}$, $\gamma=2$.

$E_{\mathbf{u},R}$	$h=2^{-5}$	$h=2^{-6}$	$h=2^{-7}$	$h=2^{-8}$	$h=2^{-9}$
$R=2.5$	1.531E-5	1.646E-5	1.708E-5	1.718E-5	1.716E-5
	c.r.	n.c.	n.c.	n.c.	n.c.
$R=3$	1.055E-5	2.687E-6	6.921E-7	1.912E-7	1.021E-7
	c.r.	1.9732	1.9569	1.8559	n.c.
$R=3.5$	1.053E-5	2.664E-6	6.690E-7	1.679E-7	4.240E-8
	c.r.	1.9828	1.9935	1.9944	1.9855

Table 8: Experiment 4.4. Numerical errors and corresponding rates, $u=e^{-|x|^2}$, $\gamma=2$.

$s=0.25$	h	2^{-5}	2^{-6}	2^{-7}	2^{-8}	2^{-9}	2^{-10}
	R	2.5426	2.7936	3.0250	3.2407	3.4436	3.6356
	$E_{\mathbf{u},R}$	1.413E-5	2.881E-6	6.885E-7	1.715E-7	4.272E-8	1.065E-8
		c.r.	2.2941	2.0650	2.0052	2.0052	2.0041
$s=0.5$	h	2^{-5}	2^{-6}	2^{-7}	2^{-8}	2^{-9}	2^{-10}
	R	2.4562	2.7060	2.9371	3.1531	3.3565	3.5491
	$E_{\mathbf{u},R}$	1.020E-5	1.865E-6	5.625E-7	1.382E-7	3.423E-8	8.530E-9
		c.r.	2.4513	1.7293	2.0251	2.0134	2.0046
$s=0.75$	h	2^{-5}	2^{-6}	2^{-7}	2^{-8}	2^{-9}	2^{-10}
	R	2.3738	2.6215	2.8517	3.0674	3.2709	3.4640
	$E_{\mathbf{u},R}$	4.990E-5	1.277E-5	3.094E-6	7.340E-7	1.758E-7	4.258E-8
		c.r.	1.9663	2.0452	2.0756	2.0618	2.0457

(n.c. in Table 7). Table 7 presents numerical errors $E_{\mathbf{u},R}$ and the convergence rates for $s=0.25,\gamma=2$ and a fixed R. We observe that if R is sufficiently large, then the truncation error is smaller than the discretization error and thus the convergence rates can reach to 2, consistent with Theorem 3.2. Besides, we also provide the solution errors and convergence rates for $s=0.25,0.5,0.75$, $\gamma=2$ in Table 8. Note that the parameter R is now chosen to force $R^{-2s}\max\{f(-R),f(R)\}=h^2$. The approximations of the truncated fractional Poisson equation (3.1) again have the second-order accuracy, which supports Theorem 3.2.

Experiment 4.5. Here we consider the convergence of finite difference approximations when the solution is an odd function. Let $u(x)=xe^{-|x|^2}, x\in(-1,1)$, and $f(x)=xe^{-|x|^2}, x\in\mathbb{R}\setminus(-1,1)$. In this case, we do not have an explicit formula for $g(x)$. Instead, formula (2.6) with small step size $h=2^{-15}$ is used in order to get approximate values of $g(x)$. Table 9 shows the solution errors and convergence rates of the method for $s=0.25,0.5,0.75$, $\gamma=2$, and the truncation parameter R satisfying $R^{-2s}\max\{|f(R)|,|f(-R)|\}=h^2$. Note that in the case of truncated fractional Poisson equation (3.1), the method has second-order accuracy, consistent with Theorem 3.2.

Table 9: Experiment 4.5. Numerical errors and corresponding convergence rates, $u = xe^{-|x|^2}$, $\gamma = 2$.

$s = 0.25$	h	2^{-5}	2^{-6}	2^{-7}	2^{-8}	2^{-9}	2^{-10}
	R	2.7263	2.9771	3.2073	3.4213	3.6222	3.8121
	$E_{\mathbf{u},R}$	1.478E-5	3.594E-6	9.063E-7	2.276E-7	5.707E-8	1.429E-8
		c.r.	2.0400	1.9875	1.9935	1.9957	1.9977
$s = 0.5$	h	2^{-5}	2^{-6}	2^{-7}	2^{-8}	2^{-9}	2^{-10}
	R	2.4562	2.7060	2.9371	3.1531	3.3565	3.5491
	$E_{\mathbf{u},R}$	3.542E-6	1.224E-6	2.821E-7	6.575E-8	1.582E-8	3.921E-9
		c.r.	1.5330	2.1173	2.1011	2.0552	2.0125
$s = 0.75$	h	2^{-5}	2^{-6}	2^{-7}	2^{-8}	2^{-9}	2^{-10}
	R	2.5426	2.7936	3.0250	3.2407	3.4436	3.6356
	$E_{\mathbf{u},R}$	3.682E-5	8.832E-6	2.050E-6	4.848E-7	1.207E-7	3.447E-8
		c.r.	2.0597	2.1071	2.0802	2.0060	1.8080

5. Conclusions

We introduce and analyze two finite difference approximations for solving the fractional Poisson equation with extended nonhomogeneous boundary condition. The difficulties caused by such boundary conditions can be overcome by employing appropriate numerical quadratures. Error estimates show that the convergence rates of the local truncation errors and the solution errors are $\mathscr{O}(h^{\varepsilon})+\epsilon$ if $u \in C^{\lfloor 2s\rfloor,2s-\lfloor 2s\rfloor+\varepsilon}(\mathbb{R})$ with $0<\varepsilon\le 1+\lfloor 2s\rfloor-2s$, $\gamma\in(2s,2]$, and $\mathscr{O}(h^2)+\epsilon$ if $u\in C^{2+\lfloor 2s\rfloor,2s-\lfloor 2s\rfloor+\varepsilon}(\mathbb{R})$ with $0<\varepsilon\le 1+\lfloor 2s\rfloor-2s$, $\gamma=2$ or $1+s$.

If there is insufficient prior knowledge about boundary conditions or the numerical quadrature does not work, we employ truncated finite difference approximations. Theoretical results show that if $u \in C^{\lfloor 2s\rfloor,2s-\lfloor 2s\rfloor+\varepsilon}(\mathbb{R})$ with $0<\varepsilon\le 1+\lfloor 2s\rfloor-2s$ and $\gamma\in(2s,2]$, then the convergence rates of local truncation errors and solution errors are of order $\mathscr{O}(h^{\varepsilon}) + R^{-2s}\max\{f(-R), f(R)\} + 2\epsilon + \epsilon_R$. Moreover, for $u \in C^{2+\lfloor 2s\rfloor,2s-\lfloor 2s\rfloor+\varepsilon}(\mathbb{R})$ with $0<\varepsilon\le 1+\lfloor 2s\rfloor-2s$, $\gamma=2$ or $1+s$ the convergence rates can be improved to $\mathscr{O}(h^2)+R^{-2s}\max\{f(-R),f(R)\}+2\epsilon+\epsilon_R$. The optimal convergence of the truncated finite difference approximations can be recovered using an appropriate truncation parameter R so that the total error will be dominated by the discretization error. Numerical experiments support the theoretical results.

Acknowledgement

The authors would like to express their most sincere thanks to the referees for their very helpful comments and suggestions, which greatly improved the quality of this paper.

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