Analysis in Theory and Applications
Anal. Theory Appl., Vol. **39**, No. 1 (2023), pp. 28-41

DOI: 10.4208/ata.OA-2020-0039
March 2023

Sharp Bound for the Generalized m-Linear n-Dimensional Hardy-Littlewood-Pólya Operator

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Received 1 February 2021; Accepted (in revised version) 27 March 2021

Abstract. In this paper, we calculate the sharp bound for the generalized m-linear n-dimensional Hardy-Littlewood-Pólya operator on power weighted central and noncentral homogeneous Morrey spaces. As an application, the sharp bound for the Hardy-Littlewood-Pólya operator on power weighted central and noncentral homogeneous Morrey spaces is obtained. Finally, we also find the sharp bound for the Hausdorff operator on power weighted central and noncentral homogeneous Morrey spaces, which generalizes the previous results.

Key Words: Sharp bound, n-dimensional Hardy-Littlewood-Pólya operator, power weight, Morrey space, Hausdorff operator.

AMS Subject Classifications: 42B25, 26D15

1 Introduction

As a multilinear generalization of Calderón operator, the m-linear n-dimensional Hardy-Littlewood-Pólya operator is defined by

$$\mathcal{P}(f_1,\cdots,f_m)(x)=\int_{\mathbb{R}^{nm}}\frac{f_1(y_1)\cdots f_m(y_m)}{\max(|x|^n,|y_1|^n,\cdots,|y_n|^n)^m}dy_1\cdots dy_m. \tag{1.1}$$

Computation of the operator norm of integral operators is a challenging work in harmonic analysis. In 2006, Bényi and Oh [3] proved that for $n=1$,

$$\|P(f_1,\cdots,f_m)\|_{L^{p_1}\times\cdots\times L^{p_m}\to L^p}=\sum_{i=1}^m\prod_{j=1,\ j\neq i}^m p_j'.$$

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In fact, they proved sharp bound for certain multilinear integral operators that includes the Hardy-Littlewood-Pólya operator. In 2011, Wu and Fu [9] got the best estimate of the m-linear p-adic Hardy-Littlewood-Pólya operator on Lebesgue spaces with power weights. In 2017, Batbold and Sawano [2] studied one-dimensional m-linear Hilbert-type operators that includes Hardy-Littlewood-Pólya operator on weighted Morrey spaces, and they obtained the sharp bounds.

For the Hardy-Littlewood-Pólya operator over p-adic field, we refer to Fu et al. [5] and Li et. al. [6].

Inspired by [2,3,9], we will investigate a more general operator which includes the Hardy-Littlewood-Pólya operator as a special case and consider its operator norm on two power weighted Morrey spaces and its central version.

In the paper we use the following notation: For any measurable function w over a set E is given by

$$w(E) = \int_E w dx.$$

In what follows, $B(x,R)$ denotes the ball centered at x with radius R. Moreover, $|B(x,R)|$ denotes the Lebesgue measure of $B(x,R)$. Also, $B(0,R)$ denotes a ball of radius R centered at the origin.

We use this notation in the following definition of the weighted and weighted central homogeneous Morrey spaces.

Definition 1.1. *Let $w_1, w_2 : \mathbb{R}^n \to (0,\infty)$ are positive measurable functions, $1 \le q < \infty$ and $-1/q \le \lambda < 0$. The weighted Morrey space $L^{q,\lambda}(\mathbb{R}^n, w_1, w_2)$ is defined by*

$$L^{q,\lambda}(\mathbb{R}^n, w_1, w_2) = \{f \in L^q_{loc} : \|f\|_{L^{q,\lambda}(\mathbb{R}^n,w_1,w_2)} < \infty\},$$

where

$$\|f\|_{L^{q,\lambda}(\mathbb{R}^n,w_1,w_2)} = \sup_{a\in\mathbb{R}^n,\ R>0} w_1(B(a,R))^{-(\lambda+1/q)} \left(\int_{B(a,R)} |f(x)|^q w_2(x) dx\right)^{1/q}.$$

Remark 1.1. When $w_1 = w_2 = 1$, $L^{q,\lambda}(\mathbb{R}^n, w_1, w_2)$ is the classical Morrey spaces $L^{q,\lambda}(\mathbb{R}^n)$ and it was introduced by Morrey [8]. Note that $L^{q,-1/q}(\mathbb{R}^n) = L^q(\mathbb{R}^n)$, $L^{q,0}(\mathbb{R}^n) = L^\infty$ and $L^{q,\lambda}(\mathbb{R}^n) = \{0\}$ with $\lambda > 0$. Based on the above reason, we only consider the case $-1/q < \lambda < 0$.

Definition 1.2. *Let $w_1, w_2 : \mathbb{R}^n \to (0,\infty)$ are positive measurable functions, $1 \le q < \infty$ and $-1/q \le \lambda < 0$. The weighted central homogeneous Morrey space $\dot{M}^{q,\lambda}(\mathbb{R}^n, w_1, w_2)$ is defined by*

$$\dot{M}^{q,\lambda}(\mathbb{R}^n, w_1, w_2) = \{f \in L^q_{loc} : \|f\|_{\dot{M}^{q,\lambda}(\mathbb{R}^n,w_1,w_2)} < \infty\},$$

where

$$\|f\|_{\dot{M}^{q,\lambda}(\mathbb{R}^n,w_1,w_2)} = \sup_{R>0} w_1(B(0,R))^{-(\lambda+1/q)} \left(\int_{B(0,R)} |f(x)|^q w_2(x) dx\right)^{1/q}. \tag{1.2}$$

The weighted inhomogeneous central Morrey space $M^{q,\lambda}(\mathbb{R}^n,w_1,w_2)$ is defined analogously with the exception that the supremum over $R>0$ is restricted to $R\geq 1$ in (1.2).

Obviously, $\dot{M}^{q,\lambda}(\mathbb{R}^n,w_1,w_2)\subset M^{q,\lambda}(\mathbb{R}^n,w_1,w_2)$ for $\lambda\geq -1/q$ and $1<q<\infty$.

Remark 1.2. When $w_1=w_2=1$, $\dot{M}^{q,\lambda}(\mathbb{R}^n,w_1,w_2)$ goes back to the classical homogeneous central Morrey spaces $\dot{M}^{q,\lambda}(\mathbb{R}^n)$, which was introduced by Alvarez, Guzmán-Partida and Lakey [1]. Obviously, $\dot{M}^{q,\lambda}(\mathbb{R}^n)$ reduces to $\{0\}$ for $\lambda<-1/q$, and it is true that $\dot{M}^{q,-1/q}(\mathbb{R}^n)=L^q(\mathbb{R}^n)$.

2 Sharp bound for the generalized Hardy-Littlewood-Pólya operator

In this section, we will study the generalized of m-linear n-dimensional Hardy-Littlewood-Pólya operator. Let now $K:\ \mathbb{R}^n\times\cdots\times\mathbb{R}^n\to(0,\infty)$ be a measurable kernel such that

$$C_m=\int_{\mathbb{R}^{nm}}K(y_1,\cdots,y_m)\prod_{i=1}^{m}|y_i|^{-d(\lambda_i,q_i,\alpha,\frac{q_i\gamma_i}{q})}dy_1\cdots dy_m<\infty, \tag{2.1}$$

where

$$d\Big(\lambda_i,q_i,\alpha,\frac{q_i\gamma_i}{q}\Big)=-n\lambda_i+\frac{1}{q_i}\cdot\frac{q_i\gamma_i}{q}-\alpha\left(\lambda_i+\frac{1}{q_i}\right)$$

and $1<q_1,q_2,\cdots,q_m<\infty$ are some arbitrary fixed indices. The m-linear operator T is then defined by

$$T(f_1,\cdots,f_m)(x)=\int_{\mathbb{R}^{nm}}K(y_1,\cdots,y_m)f_1(|x|y_1)\cdots f_m(|x|y_m)dy_1\cdots dy_m, \tag{2.2}$$

where $x\in\mathbb{R}^n\backslash\{0\}$ and f_i is a measurable function on $\mathbb{R}^n$ with $i=1,\cdots,m$. Note that T is in fact an integral operator having a homogeneous radial kernel $\widetilde{K}$ of degree $-mn$,

$$T(f_1,\cdots,f_m)(x)=\int_{\mathbb{R}^{nm}}\widetilde{K}(x,y_1,\cdots,y_m)f_1(y_1)\cdots f_m(y_m)dy_1\cdots dy_m, \tag{2.3}$$

where

$$\widetilde{K}(x,y_1,\cdots,y_m)=|x|^{-mn}K(|x|^{-1}y_1,\cdots,|x|^{-1}y_m).$$

By inserting $\widetilde{K}$, (2.1) can be rewritten as

$$C_m=\int_{\mathbb{R}^{nm}}\widetilde{K}(1,y_1,\cdots,y_m)\prod_{i=1}^{m}|y_i|^{-d(\lambda_i,q_i,\alpha,\frac{q_i\gamma_i}{q})}dy_1\cdots dy_m<\infty. \tag{2.4}$$

When $\alpha=0$, $n=1$, $\gamma_j=0$ and $\lambda_i=-1/q_j$ with $j=1,\cdots,m$, Bényi and Oh [3] proved that

$$\|T\|_{L^{q_1}\times\cdots\times L^{p_m}\to L^q}=C_m.$$

Recently, Batbold and Sawano [2] showed that for $\alpha \neq -1$ and $n = 1$, there holds

$$\|T\|_{L^{q_1,\lambda_1}(\mathbb{R},x^\alpha,x^{\frac{q_1\gamma_1}{q}})\times\cdots\times L^{q_m,\lambda_m}(\mathbb{R},x^\alpha,x^{\frac{q_m\gamma_m}{q}})\to L^{q,\lambda}(\mathbb{R},x^\alpha,x^\gamma)} = C_m.$$

For more about sharp constants of multiple integral inequalities with homogeneous kernel, we refer to Hong, Yang et al. [4,7]. In this paper, we will sightly modify the method in [2,3] and extend their results to the n-dimensional setting. Our results can be read as the following two theorems.

Theorem 2.1. *Let $m \in \mathbb{N}$, f_i be radial functions in $L^{q_j,\lambda}(\mathbb{R}^n, |x|^\alpha, |x|^{\frac{q_j\gamma_j}{q}})$, $1 \leq q < \infty$, $-1/q \leq \lambda < 0$, $1 < q_j < \infty$, $1/q = 1/q_1 + \cdots + 1/q_m$, $\gamma = \gamma_1 + \cdots + \gamma_m$ and $-1/q_j \leq \lambda_j < 0$ with $j = 1, \cdots, m$. Then*

$$\|T(f_1,\cdots,f_m)\|_{L^{q,\lambda}(\mathbb{R}^n,|x|^\alpha,|x|^\gamma)} \leq C_m \prod_{j=1}^m \|f_j\|_{L^{q_j,\lambda}(\mathbb{R}^n,|x|^\alpha,|x|^{\frac{q_j\gamma_j}{q}})}, \tag{2.5}$$

where C_m is the constant defined by (2.1) *or* (2.4). *Moreover, if $\alpha \neq -n$, $-1/q_j < \lambda_j < 0$, and $q\lambda = q_j\lambda_j$ with $j = 1, \cdots, m$, then C_m is the sharp constant in* (2.5).

Theorem 2.2. *Assume that the real parameters m, α, C_m, γ, γ_j, q, q_j with $j = 1, \cdots, m$ as same as in Theorem* 2.1. *Then we have*

$$\|T(f_1,\cdots,f_m)\|_{\dot{M}^{q,\lambda}(\mathbb{R}^n,|x|^\alpha,|x|^\gamma)} \leq C_m \prod_{j=1}^m \|f_j\|_{\dot{M}^{q_j,\lambda}(\mathbb{R}^n,|x|^\alpha,|x|^{\frac{q_j\gamma_j}{q}})}. \tag{2.6}$$

Furthermore, if $\alpha \neq -n$, $-1/q_j < \lambda_j < 0$ and $q\lambda = q_j\lambda_j$ with $j = 1, \cdots, m$, then the constant C_m in (2.6) *is best possible.*

By letting $\lambda_j \to -1/q_j$, $j = 1, \cdots, m$, we recover the result on Lebesgue spaces, which is of independent interest.

Corollary 2.1. *Let $m \in \mathbb{N}$, $1 \leq q < \infty$, $1/q = 1/q_1 + \cdots + 1/q_m$, $\gamma = \gamma_1 + \cdots + \gamma_m$, $\gamma > -n$, $1 < q_j < \infty$ and $\gamma_j < nq(-1/q_j + 1)$ with $j = 1, \cdots, m$. Assume that the kernel K satisfying*

$$D_m = \int_{\mathbb{R}^{nm}} K(y_1,\cdots,y_m)|y_1|^{-\frac{n}{q_1}-\frac{\gamma_1}{q}} \cdots |y_m|^{-\frac{n}{q_m}-\frac{\gamma_m}{q}} dy_1 \cdots dy_m < \infty. \tag{2.7}$$

Then

$$\|T\|_{L^{q_1}(\mathbb{R}^n,|x|^{\frac{q_1\gamma_1}{q}})\times\cdots\times L^{q_m}(\mathbb{R}^n,|x|^{\frac{q_m\gamma_m}{q}})\to L^q(\mathbb{R}^n,|x|^\gamma)} = D_m.$$

In order to prove Theorems 2.1 and 2.2, we need some definitions and lemmas. The following lemma discovers the scaling properties of the power weighted central and non-central homogeneous Morrey space.

Lemma 2.1. *Let* $1 \le q < \infty$, $-1/q \le \lambda < 0$ *and* $\alpha, \gamma \in \mathbb{R}$. *If* $t > 0$ *and* $f \in L^{q,\lambda}(\mathbb{R}^n, |x|^\alpha, |x|^\gamma)$ *(or* $\dot{M}^{q,\lambda}(\mathbb{R}^n, |x|^\alpha, |x|^\gamma)$*), then we have*

$$\|f(t\cdot)\|_{L^{q,\lambda}(\mathbb{R}^n,|x|^\alpha,|x|^\gamma)} = t^{n\lambda-\frac{\gamma}{q}+\alpha\left(\lambda+\frac{1}{q}\right)}\|f\|_{L^{q,\lambda}(\mathbb{R}^n,|x|^\alpha,|x|^\gamma)}, \tag{2.8a}$$

$$\|f(t\cdot)\|_{\dot{M}^{q,\lambda}(\mathbb{R}^n,|x|^\alpha,|x|^\gamma)} = t^{n\lambda-\frac{\gamma}{q}+\alpha\left(\lambda+\frac{1}{q}\right)}\|f\|_{\dot{M}^{q,\lambda}(\mathbb{R}^n,|x|^\alpha,|x|^\gamma)}. \tag{2.8b}$$

Proof. We only prove the scaling in $L^{q,\lambda}(\mathbb{R}^n, |x|^\alpha, |x|^\gamma)$, and the other one is similar. We compute that

$$\begin{aligned}
&\|f(t\cdot)\|_{L^{q,\lambda}(\mathbb{R}^n,|x|^\alpha,|x|^\gamma)} \\
=&\sup_{a\in\mathbb{R}^n,R>0}\left(\int_{B(a,R)}|x|^\alpha dx\right)^{-\lambda-\frac{1}{q}}\left(\int_{B(a,R)}|f(tx)|^q|x|^\gamma dx\right)^{\frac{1}{q}} \\
=&t^{-\frac{n}{q}-\frac{\gamma}{q}}\sup_{a\in\mathbb{R}^n,R>0}\left(\int_{B(a,R)}|x|^\alpha dx\right)^{-\lambda-\frac{1}{q}}\left(\int_{B(ta,tR)}|f(x)|^q|x|^\gamma dx\right)^{\frac{1}{q}} \\
=&t^{n\lambda-\frac{\gamma}{q}+\alpha\left(\lambda+\frac{1}{q}\right)}\sup_{a\in\mathbb{R}^n,R>0}\left(\int_{B(ta,tR)}|x|^\alpha dx\right)^{-\lambda-\frac{1}{q}}\left(\int_{B(ta,tR)}|f(x)|^q|x|^\gamma dx\right)^{\frac{1}{q}} \\
=&t^{n\lambda-\frac{\gamma}{q}+\alpha\left(\lambda+\frac{1}{q}\right)}\|f\|_{L^{q,\lambda}(\mathbb{R}^n,|x|^\alpha,|x|^\gamma)}.
\end{aligned}$$

This finishes the proof of Lemma 2.1. □

For convenience of this paper, we define the dilation index in (2.8a) and (2.8b) by

$$d(\lambda, q, \alpha, \gamma) = -n\lambda + \frac{\gamma}{q} - \alpha\left(\lambda + \frac{1}{q}\right).$$

Unlike the weighted Lebesgue space $L^q(|x|^\gamma)$, the weighted Morrey space $L^{q,\lambda}(\mathbb{R}^n, |x|^\alpha, |x|^\gamma)$ contains $|x|^{-d(\lambda,q,\alpha,\gamma)}$ and the weighted central homogeneous Morrey space $\dot{M}^{q,\lambda}(\mathbb{R}^n, |x|^\alpha, |x|^\gamma)$ contains $|x|^{-d(\lambda,q,\alpha,\gamma)}$ for $\alpha \ne -n$. More precisely,

Lemma 2.2. *Let* $1 \le q < \infty$, $-1/q < \lambda < 0$ *and* $\alpha, \gamma \in \mathbb{R}$ *with* $\alpha \ne -n$. *Then we have*

$$|x|^{-d(\lambda,q,\alpha,\gamma)} \in L^{q,\lambda}(\mathbb{R}^n, |x|^\alpha, |x|^\gamma) \quad \textit{and} \quad |x|^{-d(\lambda,q,\alpha,\gamma)} \in \dot{M}^{q,\lambda}(\mathbb{R}^n, |x|^\alpha, |x|^\gamma).$$

Moreover,

$$\||\cdot|^{-d(\lambda,q,\alpha,\gamma)}\|_{L^{q,\lambda}(\mathbb{R}^n,|x|^\alpha,|x|^\gamma)} \le \max\{|\mathbb{S}^{n-1}|^{-\lambda}n^\lambda, |\mathbb{S}^{n-1}|^{-\lambda}(q\lambda+1)^{-\frac{1}{q}}|n+\alpha|^\lambda\}, \tag{2.9a}$$

$$\||\cdot|^{-d(\lambda,q,\alpha,\gamma)}\|_{\dot{M}^{q,\lambda}(\mathbb{R}^n,|x|^\alpha,|x|^\gamma)} \le |\mathbb{S}^{n-1}|^{-\lambda}(q\lambda+1)^{-\frac{1}{q}}|n+\alpha|^\lambda. \tag{2.9b}$$

Proof. First, we consider (2.9a). Writing out the norm fully, we have

$$
\begin{aligned}
&\| |\cdot|^{-d(\lambda,q,\alpha,\gamma)} \|_{L^{q,\lambda}(\mathbb{R}^n,|x|^\alpha,|x|^\gamma)} \\
=& \sup_{a\in\mathbb{R}^n,\ R>0} \left(\int_{B(a,R)} |x|^\alpha dx \right)^{-\lambda-\frac{1}{q}} \left(\int_{B(a,R)} |x|^{-d(\lambda,q,\alpha,\gamma)q+\gamma} dx \right)^{\frac{1}{q}}.
\end{aligned}
$$

Since $-1/q < \lambda < 0$, there exist $1+q\lambda < t < 1$ such that $t/(1+q\lambda) > 1$. Thus, using Hölder's inequality, we have

$$
\begin{aligned}
&\| |\cdot|^{-d(\lambda,q,\alpha,\gamma)} \|_{L^{q,\lambda}(\mathbb{R}^n,|x|^\alpha,|x|^\gamma)} \\
\le& \sup_{a\in\mathbb{R}^n,R>0} \left(\int_{B(a,R)} |x|^\alpha dx \right)^{-\lambda-\frac{1}{q}} \left(\frac{1}{|B(a,R)|} \int_{B(a,R)} |x|^{\frac{n\lambda t}{\lambda+\frac{1}{q}}+\alpha t} dx \right)^{\frac{q\lambda+1}{qt}} |B(a,R)|^{\frac{1}{q}} \\
\le& \sup_{a\in\mathbb{R}^n,R>0} \left(\int_{B(a,R)} |x|^{\frac{nq\lambda t}{(q\lambda+1)(1-t)}} dx \right)^{\frac{(q\lambda+1)(1-t)}{qt}} |B(a,R)|^{\frac{1}{q}-\frac{q\lambda+1}{qt}}.
\end{aligned}
$$

If $|a| > 2R$, then $|x| > R$. For $-1/q < \lambda < 0$, we have

$$
\begin{aligned}
&\left(\int_{B(a,R)} |x|^{\frac{nq\lambda t}{(q\lambda+1)(1-t)}} dx \right)^{\frac{(q\lambda+1)(1-t)}{qt}} |B(a,R)|^{\frac{1}{q}-\frac{q\lambda+1}{qt}} \\
\le& R^{n\lambda} |B(a,R)|^{\frac{(q\lambda+1)(1-t)}{qt}+\frac{1}{q}-\frac{q\lambda+1}{qt}} \le |\mathbb{S}^{n-1}|^{-\lambda} n^\lambda.
\end{aligned}
$$

If $|a| < R$, then $B(a,R) \subset B(0,3R)$, we have

$$
\begin{aligned}
&\| |\cdot|^{-d(\lambda,q,\alpha,\gamma)} \|_{L^{q,\lambda}(\mathbb{R}^n,|x|^\alpha,|x|^\gamma)} \\
\le& \sup_{R>0} \left(\int_{B(0,3R)} |x|^\alpha dx \right)^{-\lambda-\frac{1}{q}} \left(\int_{B(0,3R)} |x|^{-d(\lambda,q,\alpha,\gamma)q+\gamma} dx \right)^{\frac{1}{q}}.
\end{aligned}
$$

Notice that

$$
-(\alpha+n)\left(\lambda+\frac{1}{q}\right) - d(\lambda,q,\alpha,\gamma) + \frac{\gamma+n}{q} = 0.
$$

Thus, by the scaling argument, we have

$$
\begin{aligned}
&\| |\cdot|^{-d(\lambda,q,\alpha,\gamma)} \|_{L^{q,\lambda}(\mathbb{R}^n,|x|^\alpha,|x|^\gamma)} \\
\le& \left(\int_{B(0,1)} |x|^\alpha dx \right)^{-\lambda-\frac{1}{q}} \left(\int_{B(0,1)} |x|^{\alpha(q\lambda+1)+nq\lambda} dx \right)^{\frac{1}{q}} \\
=& |\mathbb{S}^{n-1}|^{-\lambda} \left(\int_0^1 r^{\alpha+n-1} dr \right)^{-\lambda-\frac{1}{q}} \left(\int_0^1 r^{(\alpha+n)(q\lambda+1)-1} dr \right)^{\frac{1}{q}}.
\end{aligned}
$$

Assume first that $\alpha+n>0$. In this case, we have

$$\||\cdot|^{-d(\lambda,q,\alpha,\gamma)}\|_{L^{q,\lambda}(\mathbb{R}^n,|x|^\alpha,|x|^\gamma)} \le |\mathbb{S}^{n-1}|^{-\lambda}(q\lambda+1)^{-\frac{1}{q}}(n+\alpha)^{\lambda}.$$

If $\alpha+n<0$, then we calculate the integral and obtain

$$\begin{aligned}
&\||\cdot|^{-d(\lambda,q,\alpha,\gamma)}\|_{L^{q,\lambda}(\mathbb{R}^n,|x|^\alpha,|x|^\gamma)}\\
\le&|\mathbb{S}^{n-1}|^{-\lambda}\left(\int_0^1 r^{\alpha+n-1}dr\right)^{-\lambda-\frac{1}{q}}\left(\int_0^1 r^{(\alpha+n)(q\lambda+1)-1}dr\right)^{\frac{1}{q}}\\
=&|\mathbb{S}^{n-1}|^{-\lambda}\lim_{\epsilon\to 0}\left(\int_\epsilon^1 r^{\alpha+n-1}dr\right)^{-\lambda-\frac{1}{q}}\left(\int_\epsilon^1 r^{(\alpha+n)(q\lambda+1)-1}dr\right)^{\frac{1}{q}}\\
=&|\mathbb{S}^{n-1}|^{-\lambda}(q\lambda+1)^{-\frac{1}{q}}|n+\alpha|^{\lambda}\lim_{\epsilon\to 0}(\epsilon^{\alpha+n}-1)^{-\lambda-\frac{1}{q}}(\epsilon^{(\alpha+n)(q\lambda+1)}-1)^{\frac{1}{q}}.
\end{aligned}$$

The relation

$$t^{ab}-1-(t^a-1)^b=b\int_{t^a-1}^{t^a}s^{b-1}ds-1\le b\int_0^1 s^{b-1}ds-1=0$$

for $a>0$, $0<b<1$ and $t\ge 1$ implies

$$\epsilon^{(\alpha+n)(q\lambda+1)}-1=(1/\epsilon)^{-(\alpha+n)(q\lambda+1)}-1\le(\epsilon^{\alpha+n}-1)^{q\lambda+1}.$$

Thus,

$$\lim_{\epsilon\to 0}(\epsilon^{\alpha+n}-1)^{-\lambda-\frac{1}{q}}(\epsilon^{(\alpha+n)(q\lambda+1)}-1)^{\frac{1}{q}}\le 1.$$

Noting the definition of $L^{q,\lambda}(\mathbb{R}^n,|x|^\alpha,|x|^\gamma)$, we are done.

Now, we start to consider (2.9b). Since

$$\begin{aligned}
&\||\cdot|^{-d(\lambda,q,\alpha,\gamma)}\|_{\dot{M}^{q,\lambda}(\mathbb{R}^n,|x|^\alpha,|x|^\gamma)}\\
=&\sup_{R>0}\left(\int_{B(0,R)}|x|^\alpha dx\right)^{-\lambda-\frac{1}{q}}\left(\int_{B(0,R)}|x|^{-d(\lambda,q,\alpha,\gamma)q+\gamma}dx\right)^{\frac{1}{q}}.
\end{aligned}$$

Thus, using above argument, inequality (2.9b) holds. □

The proof of Theorem 2.1 and Theorem 2.2 are almost the same. For simplification, we only prove Theorem 2.2.

Proof of Theorem 2.2. Set

$$g_j(x)=\frac{1}{w_n}\int_{|\xi|=1}f_j(|x|\xi)d\xi,\quad x\in\mathbb{R}^n,$$

where $w_n = 2\pi^{n/2}/\Gamma(n/2)$ and $j = 1,\cdots,m$. Obviously, g_j $(j = 1,\dots,m)$ are radial functions and $T(g_{f_1},\cdots,g_{f_m})(x)$ is equal to

$$\begin{aligned}
&\int_{\mathbb{R}^{nm}} K(y_1,\cdots,y_m)g_1(|x|y_1)\cdots g_m(|x|y_m)dy_1\cdots dy_m\\
=&\int_{\mathbb{R}^{nm}} K(y_1,\cdots,y_m)\prod_{j=1}^m\left(\frac{1}{w_n}\int_{|\xi_j|=1} f_j(|x||y_j|\xi_j)d\xi_j\right)dy_1\cdots dy_m\\
=&\frac{1}{w_n^m}\int_{|\xi_1|=1}\cdots\int_{|\xi_m|=1}\left(\int_{\mathbb{R}^{nm}} K(y_1,\cdots,y_m)\prod_{j=1}^m f_j(|x||y_j|\xi_j)dy_1\cdots dy_m\right)d\xi_1\cdots d\xi_m\\
=&\int_{\mathbb{R}^{nm}} K(y_1,\cdots,y_m)f_1(|x|y_1)\cdots f_m(|x|y_m)dy_1\cdots dy_m\\
=&T(f_1,\cdots,f_m)(x).
\end{aligned}$$

Using the generalized Minkowski's inequality and Hölder's inequality, for $j = 1,\cdots,m$, we conclude that

$$\begin{aligned}
&\|g_j\|_{\dot{M}^{q_j,\lambda_j}(\mathbb{R}^n,|x|^\alpha,|x|^{\frac{q_j\gamma_j}{q}})}\\
\le&\frac{1}{w_n}\sup_{R>0}\left(\int_{B(0,R)}|x|^\alpha dx\right)^{-\lambda-\frac{1}{q}}\int_{|\xi_j|=1}\left(\int_{B(0,R)}|f_j(|x|\xi_j)|^q|x|^{\frac{q_j\gamma_j}{q}}dx\right)^{\frac{1}{q}}d\xi_j\\
\le&\left(\int_{B(0,R)}|x|^\alpha dx\right)^{-\lambda-\frac{1}{q}}\left(\frac{1}{w_n}\int_{|\xi_j|=1}\int_{B(0,R)}|f_j(|x|\xi_j)|^q|x|^{\frac{q_j\gamma_j}{q}}dxd\xi_j\right)^{\frac{1}{q}}\\
=&\|f_j\|_{\dot{M}^{q_j,\lambda_j}(\mathbb{R}^n,|x|^\alpha,|x|^{\frac{q_j\gamma_j}{q}})}.
\end{aligned}$$

Therefore one has that

$$\frac{\|T(f_1,\cdots,f_m)\|_{\dot{M}^{q,\lambda}(\mathbb{R}^n,|x|^\alpha,|x|^\gamma)}}{\prod_{j=1}^m\|f_j\|_{\dot{M}^{q_j,\lambda_j}(\mathbb{R}^n,|x|^\alpha,|x|^{\frac{q_j\gamma_j}{q}})}}\le\frac{\|T(g_1,\cdots,g_m)\|_{\dot{M}^{q,\lambda}(\mathbb{R}^n,|x|^\alpha,|x|^\gamma)}}{\prod_{j=1}^m\|g_j\|_{\dot{M}^{q_j,\lambda_j}(\mathbb{R}^n,|x|^\alpha,|x|^{\frac{q_j\gamma_j}{q}})}}.$$

This implies the operator T and its restriction to radial functions have the same operator norm in $\dot{M}^{q,\lambda}(\mathbb{R}^n,|x|^\alpha,|x|^\gamma)$. So, without loss of generality, we assume that f_j, $j = 1,\cdots,m$ are radial functions in the rest of the proof.

By Minkowski's inequality and Hölder's inequality, we have

$$\begin{aligned}
&\|T(f_1,\cdots,f_m)\|_{\dot{M}^{q,\lambda}(\mathbb{R}^n,|x|^\alpha,|x|^\gamma)}\\
\le&\int_{\mathbb{R}^{nm}}K(y_1,\cdots,y_m)\left\|\prod_{j=1}^m f_j(|y_j|\cdot)\right\|_{\dot{M}^{q,\lambda}(\mathbb{R}^n,|x|^\alpha,|x|^\gamma)}dy_1\cdots dy_m\\
\le&\int_{\mathbb{R}^{nm}}K(y_1,\cdots,y_m)\prod_{j=1}^m\|f_j(|y_j|\cdot)\|_{\dot{M}^{q_j,\lambda_j}(\mathbb{R}^n,|x|^\alpha,|x|^{\frac{q_j\gamma_j}{q}})}dy_1\cdots dy_m.
\end{aligned}$$

Using Lemma 2.1, we can deduce that

$$\begin{aligned}
&\|T(f_1,\cdots,f_m)\|_{\dot{M}^{q,\lambda}(\mathbb{R}^n,|x|^\alpha,|x|^\gamma)}\\
\leq&\int_{\mathbb{R}^{nm}}K(y_1,\cdots,y_m)\prod_{j=1}^m|y_j|^{-d(\lambda_j,q_j,\alpha,\frac{q_j\gamma_j}{q})}dy_1\cdots dy_m\prod_{j=1}^m\|f_j\|_{\dot{M}^{q_j,\lambda_j}(\mathbb{R}^n,|x|^\alpha,|x|^{\frac{q_j\gamma_j}{q}})}\\
\leq&C_m\prod_{j=1}^m\|f_j\|_{\dot{M}^{q_j,\lambda_j}(\mathbb{R}^n,|x|^\alpha,|x|^{\frac{q_j\gamma_j}{q}})}.
\end{aligned}$$

Now, we will show that the operator norm of T is equal to C_m. Taking

$$f_j(x)=|x|^{-d(\lambda_j,q_j,\alpha,\frac{q_j\gamma_j}{q})},\quad j=1,\cdots,m,$$

we calculate that

$$T(f_1,\cdots,f_m)(x)=C_m|x|^{-d(\lambda,q,\alpha,\gamma)},$$

which yields

$$\|T(f_1,\cdots,f_m)\|_{\dot{M}^{q,\lambda}(\mathbb{R}^n,|x|^\alpha,|x|^\gamma)}=C_m\prod_{j=1}^m\|f_j\|_{\dot{M}^{q_j,\lambda_j}(\mathbb{R}^n,|x|^\alpha,|x|^{\frac{q_j\gamma_j}{q}})}.$$

This finishes the proof of Theorem 2.2. □

Proof of Corollary 2.1. Let q' be the conjugate number of q and $g\in L^{q'}(\mathbb{R}^n,|x|^\gamma)$. Using duality identity and Hölder's inequality, and making a change of variables, we obtain the following sequence of inequalities

$$\begin{aligned}
&|\langle T(f_1,\cdots,f_m),g\rangle|\\
\leq&\int_{\mathbb{R}^{nm}}|K(y_1,\cdots,y_m)|\int_{\mathbb{R}^n}|g(x)||f_1(|x|y_1)|\cdots|f_m(|x|y_m)||x|^\gamma dxdy_1\cdots dy_m\\
\leq&\int_{\mathbb{R}^{nm}}|K(y_1,\cdots,y_m)|\|g\|_{L^{q'}(\mathbb{R}^n,|x|^\gamma)}\prod_{i=1}^m\left(\int_{\mathbb{R}^n}|f_i(|x|y_i)|^{q_i}|x|^{\frac{q_i\gamma_i}{q}}dx\right)^{\frac{1}{q_i}}dy_1\cdots dy_m\\
=&D_m\|g\|_{L^{q'}(\mathbb{R}^n,|x|^\gamma)}\|f_1\|_{L^{q_1}(\mathbb{R}^n,|x|^{\frac{q_1\gamma_1}{q}})}\cdots\|f_m\|_{L^{q_m}(\mathbb{R}^n,|x|^{\frac{q_m\gamma_m}{q}})}.
\end{aligned}\tag{2.10}$$

This proves the first part of our theorem.

For the second part, we will show that if the kernel K is nonnegative, then the operator norm $\|T\|$ of T is equal to D_m. For a positive integer N and $i=1,\cdots,m$, we define the sequences of functions g_N and $f_{i,N}$ by

$$g_N(x)=|x|^{-\frac{n+\gamma}{q'}+\frac{1}{q'N}}\chi_{B(0,1)}(x)\quad\text{and}\quad f_{i,N}=|x|^{-\frac{n}{q_i}-\frac{\gamma_i}{q}+\frac{1}{q_iN}}\chi_{B(0,1)}(x).\tag{2.11}$$

By a simple computation, we have

$$\|g_N\|^{q'}_{L^{q'}(\mathbb{R}^n,|x|^\gamma)}=\|f_{i,N}\|^{q_i}_{L^{q_i}(\mathbb{R}^n,|x|^{\frac{q_i\gamma_i}{q}})}$$

$$
\begin{aligned}
&=\|g_N\|_{L^{q'}(\mathbb{R}^n,|x|^\gamma)}\|f_{1,N}\|_{L^{q_1}(\mathbb{R}^n,|x|^{\frac{q_1\gamma_1}{q}})}\cdots\|f_{m,N}\|_{L^{q_m}(\mathbb{R}^n,|x|^{\frac{q_m\gamma_m}{q}})}\\
&=N|\mathbb{S}^{n-1}|.
\end{aligned}
$$

Therefore, we have

$$
\begin{aligned}
&|\langle T(f_{1,N},\cdots,f_{m,N}),g_N\rangle|\\
=&\int_{B(0,1)}|x|^{-\frac{n+\gamma}{q'}+\frac{1}{q'N}}|x|^\gamma\int_{\mathbb{R}^{nm}}K(y_1,\cdots,y_m)\prod_{i=1}^m f_{i,N}(|x|y_i)dy_1\cdots dy_m dx\\
=&\int_{B(0,1)}|x|^{-\frac{n+\gamma}{q'}+\frac{1}{q'N}}|x|^\gamma\int_{(B(0,\frac{1}{|x|}))^m}K(y_1,\cdots,y_m)\prod_{i=1}^m(|x|y_i)^{-\frac{n}{q_i}-\frac{\gamma_i}{q}+\frac{1}{q_iN}}dy_1\cdots dy_m dx\\
=&\int_{B(0,1)}|x|^{-n+\frac{1}{N}}\int_{(B(0,\frac{1}{|x|}))^m}K(y_1,\cdots,y_m)\prod_{i=1}^m y_i^{-\frac{n}{q_i}-\frac{\gamma_i}{q}+\frac{1}{q_iN}}dy_1\cdots dy_m dx\\
=&-N|\mathbb{S}^{n-1}|\int_1^\infty(r^{-\frac{1}{N}})'\left(\int_{(B(0,r))^m}K(y_1,\cdots,y_m)\prod_{i=1}^m y_i^{-\frac{n}{q_i}-\frac{\gamma_i}{q}+\frac{1}{q_iN}}dy_1\cdots dy_m\right)dr\\
=&N|\mathbb{S}^{n-1}|^{m+1}\int_0^1\cdots\int_0^1 K(r_1,\cdots,r_m)\prod_{j=1}^m r_j^{-\frac{n}{q_j}-\frac{\gamma_j}{q}+\frac{1}{q_jN}+n-1}dr_1\cdots dr_m+\sum_{i=1}^m L_i, \qquad (2.12)
\end{aligned}
$$

where L_i is defined as

$$
\begin{aligned}
L_i=&N|\mathbb{S}^{n-1}|^{m+1}\int_1^\infty r_i^{-\frac{1}{N}}\int_0^{r_i}\cdots\int_0^{r_i}K(r_1,\cdots,r_m)\\
&\times\prod_{j=1}^m r_j^{-\frac{n}{q_j}-\frac{\gamma_j}{q}+\frac{1}{q_jN}+n-1}dr_1\cdots\widehat{dr_i}\cdots dr_m dr_i. \qquad (2.13)
\end{aligned}
$$

Here, $\widehat{dr_i}$ means that we do not integrate with respect to the variable r_i. The last equality follows from integration by parts and the observation that, if we let

$$
W(z_1,\cdots,z_m)=\int_0^{z_1}\cdots\int_0^{z_m}K(r_1,\cdots,r_m)\prod_{j=1}^m r_j^{-\frac{n}{q_j}-\frac{\gamma_j}{q}+\frac{1}{q_jN}+n-1}dr_1\cdots dr_m,
$$

then

$$
\begin{aligned}
&\frac{d}{dx}W(x,\cdots,x)\\
=&\sum_{i=1}^m\frac{\partial W}{\partial z_i}(x,\cdots,x)\int_0^x\cdots\int_0^x K(r_1,\cdots,\overset{(i)}{x},\cdots,r_m)x^{-\frac{n}{q_i}-\frac{\gamma_i}{q}+\frac{1}{q_iN}+n-1}\\
&\times\prod_{j\neq i}r_j^{-\frac{n}{q_j}-\frac{\gamma_j}{q}+\frac{1}{q_jN}+n-1}dr_1\cdots\widehat{dr_i}\cdots dr_m,
\end{aligned}
$$

where the upper index (i) means that x replaces the variable r_i in the i-th position. By means of (2.12), we have

$$\begin{aligned}&\frac{|\langle T(f_{1,N},\cdots,f_{m,N}),g_n\rangle|}{\|g_N\|_{L^{q'}(\mathbb{R}^n,|x|^{\gamma})}\|f_{1,N}\|_{L^{q_1}(\mathbb{R}^n,|x|^{\frac{q_1\gamma_1}{q}})}\cdots\|f_{m,N}\|_{L^{q_m}(\mathbb{R}^n,|x|^{\frac{q_m\gamma_m}{q}})}}\\ =&|\mathbb{S}^{n-1}|^m\int_0^1\cdots\int_0^1 K(r_1,\cdots,r_m)\prod_{j=1}^m r_j^{-\frac{n}{q_j}-\frac{\gamma_j}{q}+\frac{1}{q_jN}+n-1}dr_1\cdots dr_m+\sum_{i=1}^m\frac{L_i}{N|\mathbb{S}^{n-1}|}.\end{aligned}\tag{2.14}$$

Let now E_i denote the domain of integration in the integral L_i defined by (2.13), that is,

$$E_i=\{(r_1,\cdots,r_m)\in(0,\infty)^m:1\le r_i<\infty,\ 0\le r_j\le r_i,\ j\neq i\}.$$

Taking into account that $1/q_1+\cdots+1/q_m=1/q$, we can bound the integrand of $L_i/(N|\mathbb{S}^{n-1}|)$ on E_i as follows:

$$\begin{aligned}&r_i^{-\frac{1}{N}}K(r_1,\cdots,r_m)\prod_{j=1}^m r_j^{-\frac{n}{q_j}-\frac{\gamma_j}{q}+\frac{1}{q_jN}+n-1}\\ \le& r_i^{-\frac{1}{N}+\frac{1}{p_1N}+\cdots+\frac{1}{p_mN}}K(r_1,\cdots,r_m)\prod_{j=1}^m r_j^{-\frac{n}{q_j}-\frac{\gamma_j}{q}+n-1}\\ =&r_i^{-\frac{1}{p'N}}K(r_1,\cdots,r_m)\prod_{j=1}^m r_j^{-\frac{n}{q_j}-\frac{\gamma_j}{q}+n-1}\\ \le& K(r_1,\cdots,r_m)\prod_{j=1}^m r_j^{-\frac{n}{q_j}-\frac{\gamma_j}{q}+n-1}.\end{aligned}$$

For the integrand of the first term in (2.14) on $[0,1]^m$, we also have that

$$K(r_1,\cdots,r_m)\prod_{j=1}^m r_j^{-\frac{n}{q_j}-\frac{\gamma_j}{q}+\frac{1}{q_jN}+n-1}\le K(r_1,\cdots,r_m)\prod_{j=1}^m r_j^{-\frac{n}{q_j}-\frac{\gamma_j}{q}+n-1}.$$

The condition (2.1) of the kernel K is equivalent to

$$D_m=|\mathbb{S}^{n-1}|^m\int_0^\infty\cdots\int_0^\infty K(r_1,\cdots,r_m)\prod_{i=1}^m r_i^{-\frac{n}{q_i}-\frac{\gamma_i}{q}+n-1}dr_1\cdots dr_m<\infty.\tag{2.15}$$

From the assumption (2.15), we can use the Lebesgue Dominated Convergence theorem. It implies that

$$\lim_{N\to\infty}\frac{L_i}{N|\mathbb{S}^{n-1}|}=|\mathbb{S}^{n-1}|^m\int_1^\infty\int_0^{r_i}\cdots\int_0^{r_i}K(r_1,\cdots,r_m)\prod_{j=1}^m r_j^{-\frac{n}{q_j}-\frac{\gamma_j}{q}+n-1}dr_1\cdots dr_m.\tag{2.16}$$

and

$$\lim_{N\to\infty} |\mathbb{S}^{n-1}|^m \int_0^1 \cdots \int_0^1 K(r_1,\cdots,r_m) \prod_{j=1}^m r_j^{-\frac{n}{q_j}-\frac{\gamma_j}{q}+n-1} dr_1\cdots dr_m$$
$$=|\mathbb{S}^{n-1}|^m \int_0^1 \cdots \int_0^1 K(r_1,\cdots,r_m) \prod_{j=1}^m r_j^{-\frac{n}{q_j}-\frac{\gamma_j}{q}+n-1} dr_1\cdots dr_m. \tag{2.17}$$

Furthermore, we have

$$[0,1]^m \bigcup \left(\bigcup_{i=1}^m E_i\right) = (0,\infty)^m,$$

and for $i,j=1,\cdots,m$, any of the intersection sets $[0,1]^m \bigcap E_i$, $E_i \bigcap E_j$, $i\neq j$, has Lebesgue measure zero in $\mathbb{R}^m$. Consequently, (2.14), (2.16) and (2.17) imply that

$$\|T\|_{L^{q_1}(\mathbb{R}^n,|x|^{\frac{q_1\gamma_1}{q}})\times\cdots\times L^{q_m}(\mathbb{R}^n,|x|^{\frac{q_m\gamma_m}{q}})\to L^q(\mathbb{R}^n,|x|^\gamma)}$$
$$=\lim_{N\to\infty} \frac{|\langle T(f_{1,N},\cdots,f_{m,N}),g_n\rangle|}{\|g_N\|_{L^{q'}(\mathbb{R}^n,|x|^\gamma)}\|f_{1,N}\|_{L^{q_1}(\mathbb{R}^n,|x|^{\frac{q_1\gamma_1}{q}})}\cdots\|f_{m,N}\|_{L^{q_m}(\mathbb{R}^n,|x|^{\frac{q_m\gamma_m}{q}})}}$$
$$=D_m.$$

We are done. □

3 An application: sharp constant for the Hardy-Littlewood-Pólya operator

By taking particular kernel K in operator T defined by (2.2) or (2.3), we can obtain sharp bound for the Hardy-Littlewood-Pólya operator on two power weighted central and non-central homogeneous Morrey spaces. Our main results in this section are as followed.

Theorem 3.1. *Let $m\in\mathbb{N}$, f_i be radial functions in radial functions in $L^{q_j,\lambda}(\mathbb{R}^n,|x|^\alpha,|x|^{\frac{q_j\gamma_j}{q}})$, $1\le q<\infty$. Assume that $m,\alpha,q,q_j,\lambda,\lambda_j$ with $j=1,\cdots,m$ as same as in Theorem* 2.1. *Assume also that*

$$n\lambda-\frac{\gamma}{q}+\alpha\left(\lambda+\frac{1}{q}\right)<0 \quad \text{and} \quad n\lambda_j-\frac{\gamma_j}{q}+\alpha\left(\lambda_j+\frac{1}{q_j}\right)>-n \ \text{ with } j=1,\cdots,m. \tag{3.1}$$

Then $\mathcal{P}$ is bounded from $\prod_{j=1}^m L^{q_j,\lambda_j}(\mathbb{R}^n,|x|^\alpha,|x|^{\gamma_j})$ to $L^{q,\lambda}(\mathbb{R}^n,|x|^\alpha,|x|^\gamma)$.

Furthermore, if $\alpha\neq -n$, $-1/q_j<\lambda_j$ and $q\lambda=q_j\lambda_j$ with $j=1,\cdots,m$, then

$$\|\mathcal{P}\|_{\prod_{j=1}^m L^{q_j,\lambda_j}(\mathbb{R}^n,|x|^\alpha,|x|^{\gamma_j})\to L^{q,\lambda}(\mathbb{R}^n,|x|^\alpha,|x|^\gamma)}$$
$$=|\mathbb{S}^{n-1}|^m \frac{mnq}{-nq\lambda+\gamma-\alpha(q\lambda+1)} \prod_{j=1}^m \frac{qq_j}{nqq_j(\lambda_j+1)-q_j\lambda_j+\alpha q(q_j\lambda_j+1)}.$$

Theorem 3.2. *Assume that m, α, q, q_j, λ, λ_j with $j=1,\cdots,m$ are as same as in Theorem 3.1. Then $\mathcal{P}$ is bounded from $\prod_{j=1}^m \dot{M}^{q_j,\lambda_j}(\mathbb{R}^n,|x|^\alpha,|x|^{\gamma_j})$ to $\dot{M}^{q,\lambda}(\mathbb{R}^n,|x|^\alpha,|x|^\gamma)$.*

Furthermore, if $\alpha\neq -n$, $-1/q_j<\lambda_j$ and $q\lambda=q_j\lambda_j$ with $j=1,\cdots,m$, then

$$\begin{aligned}&\|\mathcal{P}\|_{\prod_{j=1}^m \dot{M}^{q_j,\lambda_j}(\mathbb{R}^n,|x|^\alpha,|x|^{\gamma_j})\to \dot{M}^{q,\lambda}(\mathbb{R}^n,|x|^\alpha,|x|^\gamma)}\\ =&|\mathbb{S}^{n-1}|^m\frac{mnq}{-nq\lambda+\gamma-\alpha(q\lambda+1)}\prod_{j=1}^m\frac{qq_j}{nqq_j(\lambda_j+1)-q_j\lambda_j+\alpha q(q_j\lambda_j+1)}.\end{aligned}$$

Proof of Theorems 3.1 *and* 3.2. If we take the kernel

$$K(y_1,\cdots,y_m)=(\max(1,|y_1|,\cdots,|y_m|))^{-mn}$$

in Theorems 2.1 and 2.2, respectively, then all things reduce to calculating

$$C_m=\int_{\mathbb{R}^{nm}}\frac{1}{(\max(1,|y_1|,\cdots,|y_m|))^{mn}}\prod_{j=1}^m|y_j|^{-d(\lambda_j,q_j,\alpha,\frac{q_j\gamma_j}{q})}dy_1\cdots dy_m.$$

Using the polar coordinates transformation, we obtain

$$C_m=|\mathbb{S}^{n-1}|^m\int_0^\infty\cdots\int_0^\infty\frac{1}{(\max(1,r_1,\cdots,r_m))^{mn}}\prod_{j=1}^m r_j^{-d(\lambda_j,q_j,\alpha,\frac{q_j\gamma_j}{q})+n-1}dr_1\cdots dr_m.$$

Since C_m was precisely calculated in [2, Lemma 3], we omit the details. Thus, we give the sharp bounds in Theorems 3.1 and 3.2. □

4 A further result

In this section, we will give sharp bound for the n-dimensional Hausdorff operator on weighted Morrey spaces.

Theorem 4.1. *Let $1\leq q<\infty$, $-1/q<\lambda<0$ and $\alpha,\gamma\in\mathbb{R}$ with $\alpha\neq -n$. Assume that a nonnegative function Φ on $\mathbb{R}^n$ satisfies*

$$F=\int_{\mathbb{R}^n}\frac{\Phi(y)}{|y|^{n+\lambda(\alpha+n)+(\alpha-\gamma)/q}}dy<\infty.$$

Then we have

$$\|\mathcal{H}_\Phi\|_{L^{q,\lambda}(\mathbb{R}^n,|x|^\alpha,|x|^\gamma)\to L^{q,\lambda}(\mathbb{R}^n,|x|^\alpha,|x|^\gamma)}=F,$$

where the n-dimensional Hausdorff operator is defined by

$$\mathcal{H}_\Phi(f)(x)=\int_{\mathbb{R}^n}\frac{\Phi(y)}{|y|^n}f(x/|y|)dy.$$

Proof. The upper bound was obtained in Theorem 1.2 of [10]. To show the constant F is best possible, we just need to take the test function

$$f(x)=|x|^{\lambda(\alpha+n)+(\alpha-\gamma)/q}.$$

By Lemma 2.2, Theorem 4.1 holds. □

Remark 4.1. Theorem 4.1 can be straightforwardly extended to the multilinear and the product setting.

Acknowledgements

The author would like to thank the anonymous referee for many valuable comments and suggestions. This work was supported by National Natural Science Foundation of China (Grant No. 11871452), Beijing Information Science and Technology University Foundation (Grant No. 2025031), Natural Science Foundation of Henan Province (Grant No. 202300410338) and the Nanhu Scholar Program for Young Scholars of Xinyang Normal University.

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