Journal of Computational Mathematics
Vol.42, No.1, 2024, 1–23.
http://www.global-sci.org/jcm
doi:10.4208/jcm.2202-m2021-0290

ANALYSIS OF THE IMPLICIT-EXPLICIT ULTRA-WEAK DISCONTINUOUS GALERKIN METHOD FOR CONVECTION-DIFFUSION PROBLEMS*

Haijin Wang and Anping Xu
School of Science, Nanjing University of Posts and Telecommunications, Nanjing 210023, China
Email: hjwang@njupt.edu.cn, 1020082215@njupt.edu.cn
Qi Tao[1)]
Beijing Computational Science Research Center, Beijing 100193, China
Email: taoqi@csrc.ac.cn

Abstract

In this paper, we first present the optimal error estimates of the semi-discrete ultra-weak discontinuous Galerkin method for solving one-dimensional linear convection-diffusion equations. Then, coupling with a kind of Runge-Kutta type implicit-explicit time discretization which treats the convection term explicitly and the diffusion term implicitly, we analyze the stability and error estimates of the corresponding fully discrete schemes. The fully discrete schemes are proved to be stable if the time-step $\tau \leq \tau_0$, where τ_0 is a constant independent of the mesh-size h. Furthermore, by the aid of a special projection and a careful estimate for the convection term, the optimal error estimate is also obtained for the third order fully discrete scheme. Numerical experiments are displayed to verify the theoretical results.

Mathematics subject classification: 65M12, 65M15, 65M60.

Key words: The ultra-weak discontinuous Galerkin method, Convection-diffusion, Implicit-explicit time discretization, Stability, Error estimate.

1. Introduction

Among the time discretization methods for solving convection-diffusion problems, explicit time discretization results in severe time step restriction, while pure implicit time discretization always requires solving large non-linear systems of equations. In [19], a kind of Runge-Kutta (RK) type implicit-explicit (IMEX) time discretization [1] coupled with the local discontinuous Galerkin (LDG) spatial discretization [7] was studied for one dimensional linear convection-diffusion equations. The corresponding fully discrete IMEX-LDG schemes were proved to be unconditionally stable under the time step restriction $\tau \leq \tau_0$, where τ_0 depends only on the coefficients of convection and diffusion and not on the mesh size. The similar results were also extended to non-linear problems in [20] and to multi-dimensional cases in [21]. Later, the stability of IMEX time discretization combined with the embedded DG method [10], the

* Received October 14, 2021 / Revised version received January 4, 2022 / Accepted February 17, 2022 / Published online December 7, 2022 /
1) Corresponding author

(σ, μ)-family DG method [12] and the directed DG method [14] was investigated in [8, 16, 22], respectively.

The stability mechanism of the aforementioned fully discrete methods lies in that, the anti-dissipation of the explicit discretization for the convection term can be controlled by the stability provided by the implicit discretization for the diffusion term. In this paper, we concern about whether the same mechanism is inherent to the ultra-weak discontinuous Galerkin (UWDG) method coupled with IMEX time discretization.

The UWDG method was developed to solve time dependent partial differential equations (PDEs) containing high order spatial derivatives by Cheng and Shu [5]. Unlike the LDG method, the UWDG method does not introduce any auxiliary variables. The main idea of the UWDG method is to apply integration by parts repeatedly and to move all the spatial derivatives from the trial function to the test function in the weak formulations. The UWDG method has been successfully applied to kinds of high order PDEs. In [9], Fu and Shu designed an energy-conserving UWDG method for the generalized KdV equation. The UWDG method for generalized stochastic KdV equations was studied in [13], for Schrödinger equations was studied in [3,4]. Recently, the UWDG method combined with the LDG method to solve the PDEs with high order spatial derivatives was developed by Tao *et al.* [15,18]. It is worth pointing out that, most of the above works focus on the theoretical analysis for the semi-discrete UWDG method. Although IMEX time discretization is used in numerical experiments, such as in [3, 13], there is no theoretical analysis for the fully discrete IMEX-UWDG scheme. Meanwhile, the error estimates of the semi-discrete UWDG method for convection-diffusion problems [5] are not optimal, but numerical experiments show optimal accuracy. As far as the authors know, there is no theoretical analysis to fill this gap so far.

In this work, we will first present the optimal error estimates of the semi-discrete UWDG scheme for solving one-dimensional linear convection-diffusion equations with periodic boundary conditions. The main technique is a special projection to be defined following from [3]. The projection can eliminate the projection errors involved in the diffusion part, but the projection errors involved in the convection part can not be eliminated, so traditional treatment will lose accuracy. By the aid of the stability provided by diffusion discretization, we obtain optimal error estimates for the semi-discrete UWDG scheme.

We will also perform the analysis of stability and error estimates for some fully discrete IMEX-UWDG schemes. Typically, three specific RK type IMEX schemes coupled with the UWDG spatial discretization will be considered. By energy analysis, we prove similar stability results to that for the IMEX-LDG method in [19]. Different from the LDG method, where the discretization of the diffusion part can be converted into some inner products of auxiliary variables, there are no auxiliary variables which can be used in the UWDG method. With respect to the UWDG discretization, we make full use of the symmetric and dissipative properties, which will help us to build up negative definite quadratic forms about the implicit discretization of diffusion part, so as to obtain the desired stability results. Along the similar line of stability analysis and by the aid of the special projection mentioned above, we also carry out the optimal error estimates for the third order IMEX-UWDG scheme.

The paper is organized as follows. We first present the semi-discrete UWDG scheme for the model problem and give its optimal error estimates in Section 2. Then we give the stability analysis of three specific fully discrete IMEX-UWDG schemes in Section 3. In Section 4, we give optimal error estimates for the third order fully discrete IMEX-UWDG scheme. Numerical results are given in Section 5 to verify the main theoretical results. In Section 6, we give some concluding remarks.

2. The Semi-Discrete UWDG Method and Error Estimates

2.1. The semi-discrete UWDG scheme

In this subsection, we would like to present the definition of the semi-discrete UWDG scheme for a linear convection-diffusion problem in one dimension

$$U_t + cU_x - dU_{xx} = 0, \tag{2.1a}$$

$$U(x,0) = U_0(x) \tag{2.1b}$$

for $x \in \Omega = (a,b)$ and $t \in (0,T]$, where c and $d > 0$ are coefficients of convection and diffusion, respectively. Without loss of generality, we assume $c > 0$. We only consider periodic boundary condition in this paper. Moreover, we assume that the initial solution $U_0(x)$ is in $L^2(\Omega)$ and is smooth enough.

Let $\mathcal{T}_h = \{I_j = (x_{j-\frac{1}{2}}, x_{j+\frac{1}{2}})\}_{j=1}^N$ be a partition of Ω, where $x_{\frac{1}{2}} = a$ and $x_{N+\frac{1}{2}} = b$ are the two boundary points. We denote the cell length as $h_j = x_{j+\frac{1}{2}} - x_{j-\frac{1}{2}}$ for $j = 1,\ldots,N$, and define $h = \max_j h_j$. In addition, we assume $\mathcal{T}_h$ is quasi-uniform, that is, there exists a positive constant ν such that $h_j/h \geq \nu$ for all j, during mesh refinements.

The discontinuous finite element space is defined as

$$V_h = \left\{ v \in L^2(\Omega) : v|_{I_j} \in \mathcal{P}_k(I_j),\ \forall j = 1,\ldots,N \right\}, \tag{2.2}$$

here $\mathcal{P}_k(I_j)$ is the space of polynomials in I_j of degree no more than $k \geq 1$. For any piecewise function p, there are two traces along the right-hand side and left-hand side of each element boundary points, denoted by p^+ and p^-, respectively, and the "jump" is denoted by $[p] = p^+ - p^-$.

Following [5], the semi-discrete UWDG scheme is defined as follows: Find $u \in V_h$, such that in each cell I_j, the variation formulation

$$\begin{aligned}&(u_t, v)_j - c(u, v_x)_j + c(\check{u}v^-)_{j+\frac{1}{2}} - c(\check{u}v^+)_{j-\frac{1}{2}} - d(u, v_{xx})_j\\ &- d(\widetilde{u_x}v^-)_{j+\frac{1}{2}} + d(\widetilde{u_x}v^+)_{j-\frac{1}{2}} + d(\hat{u}v_x^-)_{j+\frac{1}{2}} - d(\hat{u}v_x^+)_{j-\frac{1}{2}} = 0\end{aligned} \tag{2.3}$$

holds for any $v \in V_h$. Here $\check{u}$, $\hat{u}$ and $\widetilde{u_x}$ are numerical fluxes that are chosen to be

$$\check{u} = u^-, \quad \hat{u} = u^+, \quad \widetilde{u_x} = u_x^- + \lambda[u], \tag{2.4}$$

where $(\cdot,\cdot)_j$ denotes the inner product in $L^2(I_j)$, and $\lambda = \frac{C_0}{h}$ is a positive penalty coefficient with large enough positive constant C_0.

Remark 2.1. We can also take the following numerical fluxes:

$$\check{u} = u^-, \quad \hat{u} = u^-, \quad \widetilde{u_x} = u_x^+ + \lambda[u].$$

Remark 2.2. When $k = 0$, we should take $\lambda = \frac{1}{h}$, otherwise, the scheme will not be consistent. A simple explanation is that, the above scheme reduces to a finite difference scheme

$$(u_j)_t = -c\frac{u_j - u_{j-1}}{h} + d\lambda\frac{u_{j+1} - 2u_j + u_{j-1}}{h},$$

if $k = 0$ and uniform mesh is adopted, we can see that only $\lambda = \frac{1}{h}$ gives consistent scheme in this case. We will consider the case $k \geq 1$ in this paper.

We rewrite (2.3) in the following variation form:

$$(u_t, v)_j = \mathcal{H}_j(u, v) + \mathcal{L}_j(u, v), \tag{2.5}$$

where

$$\mathcal{H}_j(u,v) = c\Big[(u, v_x)_j - u^-_{j+\frac{1}{2}} v^-_{j+\frac{1}{2}} + u^-_{j-\frac{1}{2}} v^+_{j-\frac{1}{2}}\Big] = -c\Big[(u_x, v)_j + [u]_{j-\frac{1}{2}} v^+_{j-\frac{1}{2}}\Big], \tag{2.6a}$$

$$\begin{aligned}\mathcal{L}_j(u,v) = d\Big[&(u, v_{xx})_j + (u_x)^-_{j+\frac{1}{2}} v^-_{j+\frac{1}{2}} - (u_x)^-_{j-\frac{1}{2}} v^+_{j-\frac{1}{2}} - u^+_{j+\frac{1}{2}} (v_x)^-_{j+\frac{1}{2}} \\ &+ u^+_{j-\frac{1}{2}} (v_x)^+_{j-\frac{1}{2}} + \lambda\left([u]_{j+\frac{1}{2}} v^-_{j+\frac{1}{2}} - [u]_{j-\frac{1}{2}} v^+_{j-\frac{1}{2}}\right)\Big].\end{aligned} \tag{2.6b}$$

We would like to take the initial condition $u(x, 0)$ as the projection $\mathbb{P}_h$ (which is to be defined in (2.22)) of the initial solution $U_0(x)$.

For convenience of analysis, we denote

$$(v, r) = \sum_{j=1}^{N} (v, r)_j,$$

which is the inner product in $L^2(\Omega)$. Let

$$\mathcal{H}(\cdot,\cdot) = \sum_{j=1}^{N} \mathcal{H}_j(\cdot,\cdot), \quad \mathcal{L} = \sum_{j=1}^{N} \mathcal{L}_j(\cdot,\cdot).$$

After summing over $j = 1, \ldots, N$ in the variation formulations (2.5), we get the semi-discrete UWDG scheme in the global form

$$(u_t, v) = \mathcal{H}(u, v) + \mathcal{L}(u, v). \tag{2.7}$$

2.2. Preliminaries

In this subsection, we will present some notations and some properties of the UWDG spatial discretization method.

2.2.1. Notations and the inverse inequality

We will use the standard notations in Sobolev spaces. We use $\|\cdot\|_D$ to denote the standard L^2 norm in D. For any integer $s \geq 1$, let $H^s(D)$ represent the space in which the function itself and the derivatives up to the s-th order are all in $L^2(D)$, the corresponding norm is denoted by $\|\cdot\|_{H^s(D)}$. In addition, we omit the subscript D if $D = \Omega$. We would also like to use C to denote a generic positive constant that is independent of h and may have a different value in each occurrence.

We consider the following broken Sobolev space:

$$H^s(\mathcal{T}_h) = \left\{ w \in L^2(\Omega) : w|_{I_j} \in H^s(I_j), \forall j = 1, \ldots, N \right\} \tag{2.8}$$

equipped with norm

$$\|w\|_{H^s(\mathcal{T}_h)} = \sqrt{\sum_{j=1}^{N} \|w\|^2_{H^s(I_j)}},$$

for any given integer $s \geq 1$. In addition, we define the following norms (or semi-norms):

$$\llbracket w \rrbracket = \sqrt{\sum_{j=1}^{N} [w]^2_{j-\frac{1}{2}}}\ , \tag{2.9}$$

$$\|w\|_{\Gamma_h} = \sqrt{\sum_{j=1}^{N} \|w\|^2_{\partial I_j}}\ , \tag{2.10}$$

$$\|w\|_h = \sqrt{\|w_x\|^2 + h^{-1}\llbracket w \rrbracket^2}\ , \tag{2.11}$$

for arbitrary $w \in H^1(\mathcal{T}_h)$, where

$$\|w\|_{\partial I_j} = \sqrt{\left(w^+_{j-\frac{1}{2}}\right)^2 + \left(w^-_{j+\frac{1}{2}}\right)^2}$$

is the L^2 norm on the boundary of I_j and $\|w_x\|^2 = \sum_{j=1}^{N} \|w_x\|^2_{I_j}$.

From [17], we have the following properties for polynomials defined in $[-1, 1]$.

Lemma 2.1. *There exist positive constants $C_1 \leq \sqrt{3}$ and $C_2 \leq \frac{\sqrt{2}}{2}$, such that*

$$\|p_x\|_{L^2([-1,1])} \leq C_1 k^2 \|p\|_{L^2([-1,1])}, \tag{2.12}$$

$$|p(x)| \leq C_2(k+1)\|p\|_{L^2([-1,1])}, \qquad \forall x \in [-1, 1] \tag{2.13}$$

for any $p \in \mathcal{P}_k([-1, 1])$.

Owing to Lemma 2.1 and the standard scaling argument, we have the following inverse inequalities for any $v \in V_h$:

$$\|v_x\|_{I_j} \leq 2\sqrt{3}k^2 h_j^{-1}\|v\|_{I_j} \leq 2\sqrt{3}k^2(\nu h)^{-1}\|v\|_{I_j}, \tag{2.14a}$$

$$\left|v^{\pm}_{j\mp\frac{1}{2}}\right| \leq (k+1)h_j^{-\frac{1}{2}}\|v\|_{I_j} \leq (k+1)(\nu h)^{-\frac{1}{2}}\|v\|_{I_j}, \tag{2.14b}$$

$$\left|(v_x)^{\pm}_{j\mp\frac{1}{2}}\right| \leq k h_j^{-\frac{1}{2}}\|v_x\|_{I_j} \leq k(\nu h)^{-\frac{1}{2}}\|v_x\|_{I_j}, \tag{2.14c}$$

where ν is the mesh parameter defined before, which equals 1 for uniform mesh.

2.2.2. Properties of the UWDG spatial discretization

The semi-definiteness and boundedness properties of the operator $\mathcal{H}$ are given in the following two lemmas. We omit the proofs and refer readers to [23] for more details.

Lemma 2.2. *For any $v \in V_h$, there holds the equality*

$$\mathcal{H}(v, v) = -\frac{c}{2}\llbracket v \rrbracket^2. \tag{2.15}$$

Lemma 2.3. *For any $u, v \in V_h$, there hold the following inequalities:*

$$|\mathcal{H}(u, v)| \leq c\left(\|u_x\| + \sqrt{\mu h^{-1}}\llbracket u \rrbracket\right)\|v\|, \tag{2.16a}$$

$$|\mathcal{H}(u, v)| \leq c\left(\|v_x\| + \sqrt{\mu h^{-1}}\llbracket v \rrbracket\right)\|u\|. \tag{2.16b}$$

Here $\mu = (k+1)^2\nu$.

The next lemmas introduce the properties of the operator $\mathcal{L}$. Lemmas 2.4 and 2.5 state the symmetric and dissipative properties of $\mathcal{L}$, respectively. They will play an important role in the stability analysis.

Lemma 2.4. *For any $u, v \in V_h$, there holds*

$$\mathcal{L}(u,v) = \mathcal{L}(v,u). \tag{2.17}$$

Proof. From (2.6b) and integration by parts, we obtain

$$\begin{aligned}\mathcal{L}(u,v) =& d\sum_{j=1}^{N}\Big[-(u_x,v_x)_j + (u_x)^-_{j+\frac12}v^-_{j+\frac12} - (u_x)^-_{j-\frac12}v^+_{j-\frac12} - u^+_{j+\frac12}(v_x)^-_{j+\frac12}\\ &+ u^-_{j+\frac12}(v_x)^-_{j+\frac12} + \lambda\big([u]_{j+\frac12}v^-_{j+\frac12} - [u]_{j-\frac12}v^+_{j-\frac12}\big)\Big]\\ =& d\sum_{j=1}^{N}\Big[-(u_x,v_x)_j - (u_x)^-_{j+\frac12}[v]_{j+\frac12} - [u]_{j+\frac12}(v_x)^-_{j+\frac12} - \lambda[u]_{j+\frac12}[v]_{j+\frac12}\Big], \end{aligned}\tag{2.18}$$

owing to the periodic boundary condition. Thus, we get $\mathcal{L}(u,v) = \mathcal{L}(v,u)$ from (2.18). □

Lemma 2.5. *For any $v \in V_h$, there hold*

$$\mathcal{L}(v,v) \le -\frac{d}{2}|||v|||_h^2, \tag{2.19}$$

$$|||v|||_h^2 \le -\frac{2}{d}\mathcal{L}(v,v), \tag{2.20}$$

if the penalty coefficient $\lambda = \frac{C_0}{h}$ with

$$C_0 \ge \frac12 + \frac{2k^2}{\nu}. \tag{2.21}$$

Proof. From (2.18) we get

$$\begin{aligned}\mathcal{L}(v,v) =& -d\left[\|v_x\|^2 + \lambda[\![v]\!]^2 + 2\sum_{j=1}^{N}(v_x)^-_{j+\frac12}[v]_{j+\frac12}\right]\\ \le& -d\|v_x\|^2 - d\lambda[\![v]\!]^2 + d\left[\varepsilon h\sum_{j=1}^{N}\Big((v_x)^-_{j+\frac12}\Big)^2 + \frac{1}{\varepsilon h}\sum_{j=1}^{N}[v]^2_{j+\frac12}\right]\\ \le& -d\|v_x\|^2 - d\lambda[\![v]\!]^2 + d\varepsilon k^2\nu^{-1}\|v_x\|^2 + \frac{d}{\varepsilon h}[\![v]\!]^2,\end{aligned}$$

for arbitrary positive constant ε, where the Young's inequality and the inverse inequality (2.14c) were used. Noting that $\lambda = \frac{C_0}{h}$ and taking $\varepsilon = \frac{\nu}{2k^2}$, we have

$$\begin{aligned}\mathcal{L}(v,v) \le& -d\|v_x\|^2 - d\frac{C_0}{h}[\![v]\!]^2 + \frac{d}{2}\|v_x\|^2 + \frac{2k^2 d}{\nu h}[\![v]\!]^2\\ =& -\frac{d}{2}\|v_x\|^2 - d\left(C_0 - \frac{2k^2}{\nu}\right)h^{-1}[\![v]\!]^2.\end{aligned}$$

Thus, if $C_0 \ge \frac12 + \frac{2k^2}{\nu}$, then

$$\mathcal{L}(v,v) \le -d\min\left\{\frac12, C_0 - \frac{2k^2}{\nu}\right\}\left(\|v_x\|^2 + h^{-1}[\![v]\!]^2\right) \le -\frac{d}{2}|||v|||_h^2.$$

Therefore, we can obtain (2.19). In addition, from (2.19) we can easily get (2.20). □

Remark 2.3. From the proof of Lemma 2.5, we can also get $\mathcal{L}(v,v) \le 0$ if $\lambda \ge \frac{k^2}{\nu h}$. This implies that the semi-discrete UWDG scheme (2.7) is stable under the condition $\lambda \ge \frac{k^2}{\nu h}$.

Lemma 2.6. *Let $\boldsymbol{u} = (u_1, \cdots, u_n)^\top$, $\boldsymbol{v} = (v_1, \cdots, v_n)^\top$. Define*

$$\underline{\mathcal{L}}(\boldsymbol{u}, \boldsymbol{v}) = \sum_{i=1}^{n} \mathcal{L}(u_i, v_i),$$

if $\lambda \ge \frac{k^2}{\nu h}$, then we have

(a) $\underline{\mathcal{L}}(\boldsymbol{u}, \boldsymbol{u}) \le 0$.

(b) For any symmetric matrix $A \in \mathbb{R}^{n\times n}$, $\underline{\mathcal{L}}(\boldsymbol{u}, A\boldsymbol{v}) = \underline{\mathcal{L}}(A\boldsymbol{u}, \boldsymbol{v})$.

(c) For any symmetric positive definite matrix $B \in \mathbb{R}^{n\times n}$, $\underline{\mathcal{L}}(\boldsymbol{u}, B\boldsymbol{u}) \le 0$.

Proof. Please refer to [16]. □

The stability and error analysis of the semi-discrete UWDG scheme for (2.1) had been given in [5]. However, the error estimates therein was suboptimal. In the next subsection, we present the optimal error estimates for the semi-discrete UWDG scheme. The main technique is the special projection proposed in [3].

2.3. Optimal error estimates for the semi-discrete UWDG scheme

We first present the definition of the special projection $\mathbb{P}_h$. For any periodic function $w \in \mathcal{H}^s(\Omega)$ with $s \ge 2$, the projection $\mathbb{P}_h w \in V_h$ is defined as follows: if $k \ge 2$, then in each element $I_j = (x_{j-\frac{1}{2}}, x_{j+\frac{1}{2}})$

$$(\mathbb{P}_h w - w, v)_j = 0, \quad \forall v \in \mathcal{P}_{k-2}(I_j), \tag{2.22a}$$

$$\big(\widehat{\mathbb{P}_h w}\big)_{j-\frac{1}{2}} = w_{j-\frac{1}{2}}, \tag{2.22b}$$

$$\big(\widetilde{(\mathbb{P}_h w)_x}\big)_{j+\frac{1}{2}} = (w_x)_{j+\frac{1}{2}}, \tag{2.22c}$$

where $\widehat{\mathbb{P}_h w}$ and $\widetilde{(\mathbb{P}_h w)_x}$ are defined in the same manner as the definition of numerical flux in (2.4)

$$\big(\widehat{\mathbb{P}_h w}\big)_{j-\frac{1}{2}} = (\mathbb{P}_h w)^+_{j-\frac{1}{2}}, \quad \big(\widetilde{(\mathbb{P}_h w)_x}\big)_{j+\frac{1}{2}} = \big((\mathbb{P}_h w)_x\big)^-_{j+\frac{1}{2}} + \lambda[\mathbb{P}_h w]_{j+\frac{1}{2}}.$$

If $k=1$, then only (2.22b) and (2.22c) are needed. This projection can eliminate the projection errors of the diffusion part, both in the interior of element and at the boundary of element. So it plays a key role in getting the optimal error estimates. From Lemma 3.1 in [3], we know that the above projection is a local projection under the setting of numerical flux (2.4). In [3], the unique existence and optimal approximation properties of the projection $\mathbb{P}_h w$ for $w \in W^{k+1,\infty}(\Omega)$ were discussed, with the setting of general numerical flux. In fact, the regularity assumption $w \in W^{k+1,\infty}(\Omega)$ can be improved to $w \in \mathcal{H}^{k+1}(\Omega)$. For the purpose of optimal error estimates and for the completeness of this paper, we present the following lemma and its proof.

Lemma 2.7. *The projection $\mathbb{P}_h$ exists uniquely when*

$$\Gamma_j = \lambda - \frac{k^2}{h_j} \ne 0, \quad \forall j. \tag{2.23}$$

In addition, assume $w \in \mathcal{H}^s(\Omega)$ $(s \geq 2)$ and denote $\eta = w - \mathbb{P}_h w$, then

$$\|\eta\| + h^{\frac{1}{2}}\|\eta\|_{\Gamma_h} \leq Ch^{\min(k+1,s)}\|w\|_{H^s(\Omega)}\left(1 + \frac{\lambda}{\min_j |\Gamma_j|}\right), \tag{2.24}$$

where C depends on k but is independent of h and w.

Proof. The proof of this lemma follows from Appendix A.1 in [3], but with a slight difference. In what follows we would like to use the projection P_h proposed in [5] to prove the existence, uniqueness and optimal approximation of the projection $\mathbb{P}_h$. The projection P_h is defined as follows:

$$(P_h w - w, v)_j = 0, \quad \forall\ v \in \mathcal{P}_{k-2}(I_j), \tag{2.25a}$$

$$(P_h w)^+_{j-\frac{1}{2}} = w_{j-\frac{1}{2}}, \tag{2.25b}$$

$$\big((P_h w)_x\big)^-_{j+\frac{1}{2}} = (w_x)_{j+\frac{1}{2}}. \tag{2.25c}$$

When $k = 1$, only conditions (2.25b) and (2.25c) are needed. The existence and uniqueness of P_h can be verified straightforwardly since it is defined element-wise. By the standard scaling argument [6], we can obtain the following approximation property:

$$\|P_h w - w\|_{I_j} + h^{\frac{1}{2}}\|P_h w - w\|_{\partial I_j} \leq Ch^{\min(k+1,s)}\|w\|_{H^s(I_j)}, \quad \forall j, \tag{2.26}$$

where $C > 0$ is a bounded constant which is independent of h, j and w.

Denote $Ew = \mathbb{P}_h w - P_h w$, then it satisfies

$$(Ew, v)_j = 0, \qquad \forall v \in \mathcal{P}_{k-2}(I_j), \tag{2.27a}$$

$$Ew^+ = 0 \qquad \text{at} \quad x_{j-\frac{1}{2}}, \tag{2.27b}$$

$$Ew^- - \frac{(Ew)^-_x}{\lambda} = (w - P_h w)^- \quad \text{at} \quad x_{j+\frac{1}{2}}. \tag{2.27c}$$

To prove that $\mathbb{P}_h w$ exists uniquely, we only need to prove that Ew exists uniquely. For this purpose, we express Ew as

$$Ew(x) = \sum_{\ell=0}^{k} \alpha_{j,\ell} P_{j,\ell}(x) = \sum_{\ell=0}^{k} \alpha_{j,\ell} P_\ell(\hat{x}), \quad x \in I_j,$$

where $\{P_\ell(\hat{x})\}_{\ell=0}^k$ are standard Legendre polynomials in $[-1, 1]$, $P_{j,\ell}(x) = P_\ell(\hat{x})$ by an affine mapping $\hat{x} = \frac{2(x-x_j)}{h_j}$ for $x \in I_j$. From (2.27a) and the orthogonality property of Legendre polynomials, we get

$$\alpha_{j,\ell} = 0, \quad \text{for} \quad \ell = 0, \ldots, k-2, \quad j = 1, \ldots, N.$$

Then, from (2.27b), (2.27c) and the properties $P_\ell(\pm 1) = (\pm 1)^\ell$, $P'_\ell(\pm 1) = \frac{1}{2}(\pm 1)^{\ell-1}\ell(\ell+1)$, we can obtain

$$\mathcal{M}_j \begin{bmatrix} \alpha_{j,k-1} \\ \alpha_{j,k} \end{bmatrix} = \begin{bmatrix} \phi_j \\ \psi_j \end{bmatrix},$$

where

$$\mathcal{M}_j = \begin{bmatrix} (-1)^{k-1} & (-1)^k \\ 1 - \frac{k(k-1)}{\lambda h_j} & 1 - \frac{k(k+1)}{\lambda h_j} \end{bmatrix},$$

and $\phi_j = 0$, $\psi_j = (w - P_h w)^-|_{x_{j+\frac{1}{2}}}$. We can calculate that

$$\det\mathcal{M}_j = 2(-1)^{k-1}\left(1 - \frac{k^2}{\lambda h_j}\right) = 2(-1)^{k-1}\frac{\Gamma_j}{\lambda}.$$

Thus, Ew exists uniquely if $\det\mathcal{M}_j \neq 0$, $\forall j$, i.e, $\Gamma_j \neq 0$. Moreover, we can solve that

$$\alpha_{j,k-1} = \alpha_{j,k} = \frac{(-1)^{k-1}\psi_j}{\det\mathcal{M}_j}.$$

With the help of (2.26), we get

$$|\alpha_{j,\ell}| \leq Ch^{\min(k+1,s)-\frac{1}{2}}\|w\|_{H^s(I_j)}|\det\mathcal{M}_j|^{-1}, \quad \ell = k-1, k.$$

Hence,

$$\|Ew\|_{I_j} = \left(\alpha_{j,k-1}^2\|P_{j,k-1}\|_{I_j}^2 + \alpha_{j,k}^2\|P_{j,k}\|_{I_j}^2\right)^{\frac{1}{2}} \leq Ch^{\min(k+1,s)}\|w\|_{H^s(I_j)}|\det\mathcal{M}_j|^{-1},$$

where the last inequality used $\|P_{j,\ell}\|_{I_j} = \mathcal{O}(h^{1/2})$, the constant C depends on k. Furthermore,

$$\|Ew\|_{\partial I_j} = \sqrt{\left(Ew_{j-\frac{1}{2}}^+\right)^2 + \left(Ew_{j+\frac{1}{2}}^-\right)^2} = |\alpha_{j,k-1} + \alpha_{j,k}| \leq |\alpha_{j,k-1}| + |\alpha_{j,k}|.$$

As a consequence, combining the above estimates, we obtain

$$\|Ew\| + h^{\frac{1}{2}}\|Ew\|_{\Gamma_h} \leq Ch^{\min(k+1,s)}\|w\|_{H^s(\Omega)}\frac{\lambda}{\min_j|\Gamma_j|}. \tag{2.28}$$

It together with (2.26) leads to (2.24). □

Remark 2.4. Noting that $\lambda = \frac{C_0}{h}$, so $\frac{\lambda}{\min_j|\Gamma_j|} = \mathcal{O}(1)$, thus we get the optimal approximation property of projection

$$\|\eta\| + h^{\frac{1}{2}}\|\eta\|_{\Gamma_h} \leq Ch^{\min(k+1,s)}\|w\|_{H^s(\Omega)}. \tag{2.29}$$

Lemma 2.8. *For any $w \in \mathcal{H}^s(\Omega)$ with $s \geq 2$ and $v \in V_h$, we have*

$$\mathcal{L}(\mathbb{P}_h w - w, v) = 0. \tag{2.30}$$

Proof. From the definition of $\mathbb{P}_h$ and (2.6b), we obtain the conclusion directly. □

To derive the optimal error estimates, we assume that the exact solution of (2.1) satisfies

$$U, U_t \in L^\infty(0,T;H^{k+1}). \tag{2.31}$$

As the traditional error analysis in finite element methods, we divide the error between the exact solution and the numerical solution into two parts, namely

$$e = U - u = \xi - \eta, \tag{2.32}$$

where

$$\xi = \mathbb{P}_h U - u, \quad \eta = \mathbb{P}_h U - U. \tag{2.33}$$

Owing to the smoothness of U, we get

$$(e_t, v) = \mathcal{H}(e, v) + \mathcal{L}(e, v) = \mathcal{H}(\xi, v) + \mathcal{L}(\xi, v) - \mathcal{H}(\eta, v), \quad \forall v \in V_h, \tag{2.34}$$

where Lemma 2.8 was used. By using the Cauchy-Schwarz inequality, (2.29) and the Young's inequality, we get

$$|\mathcal{H}(\eta, v)| \le Cch^{k+1}\left(\|v_x\| + h^{-\frac{1}{2}}[\![v]\!]\right) \le c^2\varepsilon |\!|\!| v |\!|\!|_h^2 + Ch^{2k+2}, \tag{2.35}$$

for arbitrary $\varepsilon > 0$.

Taking $v = \xi$ in (2.34), we get

$$(\xi_t, \xi) = (\eta_t, \xi) + \mathcal{H}(\xi, \xi) + \mathcal{L}(\xi, \xi) - \mathcal{H}(\eta, \xi). \tag{2.36}$$

Exploiting the Cauchy-Schwarz and the Young's inequality, Lemmas 2.2, 2.5 and (2.35), we can obtain

$$\frac{1}{2}\frac{d}{dt}\|\xi\|^2 \le \|\eta_t\|\|\xi\| - \frac{c}{2}[\![\xi]\!]^2 - \frac{d}{2}|\!|\!|\xi|\!|\!|_h^2 + c^2\varepsilon|\!|\!|\xi|\!|\!|_h^2 + Ch^{2k+2} \le \|\xi\|^2 + Ch^{2k+2}, \tag{2.37}$$

if $\varepsilon \le \frac{d}{2c^2}$. Hence by the Gronwall inequality we get

$$\|\xi\| \le Ch^{k+1}. \tag{2.38}$$

As a result, by the triangle inequality and (2.29), we arrive at the following theorem.

Theorem 2.1. *Assume U is the exact solution of* (2.1) *satisfying* (2.31)*, let $u \in V_h$ with $k \ge 1$ be the solution of* (2.7)*. Under the conditions of* (2.21) *and* (2.23)*, we have*

$$\|U(t) - u(t)\| \le Ch^{k+1}, \tag{2.39}$$

where the constant C depends on c, d, λ, k and the regularity of U, but not on h.

3. The Fully Discrete Schemes and Their Stability Analysis

In this section, we study the stability of several fully discrete UWDG schemes. With respect to the time discretization, we would like to adopt those IMEX schemes considered in [19]. We omit the detailed introduction of IMEX time discretization methods to save space, one can refer to [1, 2, 11, 19].

3.1. Fully discrete schemes

Let $\{t^n = n\tau\}_{n=0}^M$ be the uniform partition of the time interval $[0, T]$, here τ is time step. Denote u^n as the numerical solution at time level t^n, let $u^{n,\ell}$ be the numerical solution at intermediate stages $t^{n,\ell}$, the numerical solution of u^{n+1} is obtained by the following fully discrete IMEX-UWDG schemes.

The first order IMEX-UWDG scheme

$$(u^{n+1}, v) = (u^n, v) + \tau\mathcal{H}(u^n, v) + \tau\mathcal{L}(u^{n+1}, v) \tag{3.1}$$

for arbitrary $v \in V_h$.

The second order IMEX-UWDG scheme

$$(u^{n,1},v) = (u^n,v) + \gamma\tau\mathcal{H}(u^n,v) + \gamma\tau\mathcal{L}(u^{n,1},v), \tag{3.2a}$$
$$\begin{aligned}(u^{n+1},v) = {} & (u^n,v) + \delta\tau\mathcal{H}(u^n,v) + (1-\delta)\tau\mathcal{H}(u^{n,1},v) \\ & + (1-\gamma)\tau\mathcal{L}(u^{n,1},v) + \gamma\tau\mathcal{L}(u^{n+1},v)\end{aligned} \tag{3.2b}$$

for arbitrary $v \in V_h$, here $\gamma = 1 - \frac{\sqrt{2}}{2}$, $\delta = 1 - \frac{1}{2\gamma}$.

The third order IMEX-UWDG scheme

$$(u^{n,\ell},v) = (u^n,v) + \tau\sum_{i=0}^{3}\Big(a_{\ell i}\mathcal{H}(u^{n,i},v) + \hat{a}_{\ell i}\mathcal{L}(u^{n,i},v)\Big), \quad \ell = 1,2,3, \tag{3.3a}$$
$$(u^{n+1},v) = (u^n,v) + \tau\sum_{i=0}^{3}\Big(b_i\mathcal{H}(u^{n,i},v) + \hat{b}_i\mathcal{L}(u^{n,i},v)\Big) \tag{3.3b}$$

for arbitrary $v \in V_h$. The coefficients are listed in Table 3.1.

Table 3.1: The coefficients $a_{\ell i}, \hat{a}_{\ell i}, b_i$ and $\hat{b}_i$.

	$a_{\ell i}$				$\hat{a}_{\ell i}$			
ℓ \ i	0	1	2	3	0	1	2	3
1	γ	0	0	0	0	γ	0	0
2	$\frac{1+\gamma}{2} - \alpha_1$	α_1	0	0	0	$\frac{1-\gamma}{2}$	γ	0
3	0	$1-\alpha_2$	α_2	0	0	β_1	β_2	γ
	b_i				$\hat{b}_i$			
	0	β_1	β_2	γ	0	β_1	β_2	γ

In Table 3.1, γ is the middle root of $6x^3 - 18x^2 + 9x - 1 = 0$, $\gamma \approx 0.435866521508459$. In addition, $\beta_1 = -\frac{3}{2}\gamma^2 + 4\gamma - \frac{1}{4}$, $\beta_2 = \frac{3}{2}\gamma^2 - 5\gamma + \frac{5}{4}$ and $\alpha_2 = \frac{\frac{1}{3} - 2\gamma^2 - 2\beta_2\alpha_1\gamma}{\gamma(1-\gamma)}$, where α_1 is a free parameter, we take $\alpha_1 = -0.35$ which is the same as the choice in [2].

3.2. Stability analysis

Theorem 3.1. *There exists a positive constant τ_0 such that if $\tau \le \tau_0$, then under the condition* (2.21)*, the solution of scheme* (3.1)-(3.3) *satisfies*

$$\|u^{n+1}\| \le \|u^n\|, \quad \forall n. \tag{3.4}$$

Here τ_0 is proportional to d/c^2 but is independent of the mesh size h, it may have different values for different schemes.

3.2.1. Proof for the first-order scheme

Following [19], we take $v = u^{n+1}$ in (3.1) to get

$$(u^{n+1} - u^n, u^{n+1}) = \tau\mathcal{H}(u^n,u^{n+1}) + \tau\mathcal{L}(u^{n+1},u^{n+1}). \tag{3.5}$$

According to Lemma 2.5, we have

$$\mathcal{L}(u^{n+1}, u^{n+1}) \le -\frac{d}{2}\|\!|u^{n+1}|\!\|_h^2.$$

Then (3.5) is equivalent to

$$\frac{1}{2}\|u^{n+1}\|^2 - \frac{1}{2}\|u^n\|^2 + \frac{1}{2}\|u^{n+1} - u^n\|^2 + \frac{d}{2}\tau\|\!|u^{n+1}|\!\|_h^2 \le \tau\mathcal{H}(u^n, u^{n+1}) \doteq R_0. \tag{3.6}$$

Adding and subtracting a term $\tau\mathcal{H}(u^{n+1}, u^{n+1})$, we get

$$\begin{aligned} R_0 =& \tau\mathcal{H}(u^{n+1}, u^{n+1}) - \tau\mathcal{H}(u^{n+1} - u^n, u^{n+1}) \\ =& -\frac{c}{2}\tau[\![u^{n+1}]\!]^2 - \tau\mathcal{H}(u^{n+1} - u^n, u^{n+1}), \end{aligned} \tag{3.7}$$

where the property (2.15) was used in the last equality. Hence, by (2.16b) and the Young's inequality, we get

$$\begin{aligned} R_0 \le& c\tau\left(\|u_x^{n+1}\| + \sqrt{\mu h^{-1}}[\![u^{n+1}]\!]\right)\|u^{n+1} - u^n\| \\ \le& \frac{1}{2}\|u^{n+1} - u^n\|^2 + c^2\tau^2\max\{1, \mu\}\|\!|u^{n+1}|\!\|_h^2. \end{aligned} \tag{3.8}$$

Consequently, if $c^2\tau^2\max\{1,\mu\} \le \frac{d}{2}\tau$, i.e, $\tau \le \tau_0 = \frac{d}{2c^2\max\{1,\mu\}}$, then we get (3.4). □

3.2.2. Proof for the second-order scheme

From (3.2a) and (3.2b), we have

$$(u^{n,1} - u^n, v) = \gamma\tau\mathcal{H}(u^n, v) + \gamma\tau\mathcal{L}(u^{n,1}, v), \tag{3.9a}$$

$$\begin{aligned} (u^{n+1} - u^{n,1}, v) =& (\delta - \gamma)\tau\mathcal{H}(u^n, v) + (1-\delta)\tau\mathcal{H}(u^{n,1}, v) \\ &+ (1-2\gamma)\tau\mathcal{L}(u^{n,1}, v) + \gamma\tau\mathcal{L}(u^{n+1}, v). \end{aligned} \tag{3.9b}$$

Taking $v = u^{n,1}$ in (3.9a) and $v = u^{n+1}$ in (3.9b), and adding up the two equations, we get

$$\frac{1}{2}\|u^{n+1}\|^2 - \frac{1}{2}\|u^n\|^2 + \frac{1}{2}\|u^{n+1} - u^{n,1}\|^2 + \frac{1}{2}\|u^{n,1} - u^n\|^2 =: R_1 + R_2, \tag{3.10}$$

where

$$\begin{aligned} R_1 =& \gamma\tau\mathcal{H}(u^n, u^{n,1}) + (\delta - \gamma)\,\tau\mathcal{H}(u^n, u^{n+1}) + (1-\delta)\tau\mathcal{H}(u^{n,1}, u^{n+1}), \\ R_2 =& \gamma\tau\mathcal{L}(u^{n,1}, u^{n,1}) + (1-2\gamma)\tau\mathcal{L}(u^{n,1}, u^{n+1}) + \gamma\tau\mathcal{L}(u^{n+1}, u^{n+1}). \end{aligned}$$

According to Lemma 2.4, we get

$$(1-2\gamma)\tau\mathcal{L}(u^{n,1}, u^{n+1}) = \frac{1-2\gamma}{2}\tau\left[\mathcal{L}(u^{n,1}, u^{n+1}) + \mathcal{L}(u^{n+1}, u^{n,1})\right].$$

Therefore, if we denote $\mathbf{u}^n = (u^{n,1}, u^{n+1})^\top$, then R_2 can be represented as

$$R_2 = \tau\underline{\mathcal{L}}(\mathbf{u}^n, \mathbb{A}\mathbf{u}^n), \tag{3.11}$$

where

$$\mathbb{A} = \begin{pmatrix} \gamma & \frac{1-2\gamma}{2} \\ \frac{1-2\gamma}{2} & \gamma \end{pmatrix}. \tag{3.12}$$

Since $\mathbb{A}$ is symmetric positive definite, we can get $R_2 \leq 0$ according to Lemma 2.6.

To estimate R_1, we follow [19] to rewrite R_1 as

$$\begin{aligned} R_1 =& \gamma\tau\mathcal{H}(u^{n,1}, u^{n,1}) + (1-\gamma)\tau\mathcal{H}(u^{n+1}, u^{n+1}) - \gamma\tau\mathcal{H}(u^{n,1} - u^n, u^{n,1}) \\ &- (1-\gamma)\tau\mathcal{H}(u^{n+1} - u^{n,1}, u^{n+1}) + \tau\mathcal{H}(u^{n,1} - u^n, u^{n+1}), \end{aligned}$$

here we used the fact that $\delta - \gamma = -1$. Then by (2.15) we have

$$\begin{aligned} R_1 =& -\frac{c}{2}\gamma\tau[\![u^{n,1}]\!]^2 - \frac{c}{2}(1-\gamma)\tau[\![u^{n+1}]\!]^2 - \gamma\tau\mathcal{H}(u^{n,1} - u^n, u^{n,1}) \\ &- (1-\gamma)\tau\mathcal{H}(u^{n+1} - u^{n,1}, u^{n+1}) + \tau\mathcal{H}(u^{n,1} - u^n, u^{n+1}). \end{aligned}$$

Similar to (3.8), we have

$$R_1 \leq C_{\mu,\gamma}c^2\tau^2\left(|\!|\!|u^{n,1}|\!|\!|_h^2 + |\!|\!|u^{n+1}|\!|\!|_h^2\right) + \frac{1}{2}\left(\|u^{n,1} - u^n\|^2 + \|u^{n+1} - u^{n,1}\|^2\right), \tag{3.13}$$

where $C_{\mu,\gamma} > 0$ depends on μ and γ.

Consequently, by using (3.10), (3.13) and Lemma 2.5, we get

$$\frac{1}{2}\|u^{n+1}\|^2 - \frac{1}{2}\|u^n\|^2 \leq R_2 - \frac{2C_{\mu,\gamma}c^2\tau}{d}\tau\left[\mathcal{L}(u^{n,1}, u^{n,1}) + \mathcal{L}(u^{n+1}, u^{n+1})\right].$$

Denoting $\frac{2C_{\mu,\gamma}c^2\tau}{d} = \phi_0$, we have

$$\frac{1}{2}\|u^{n+1}\|^2 - \frac{1}{2}\|u^n\|^2 \leq \tau\underline{\mathcal{L}}(\mathbf{u}^n, \mathbb{B}\mathbf{u}^n), \tag{3.14}$$

where $\mathbb{B} = \mathbb{A} - \phi_0\mathbb{I}$, with $\mathbb{I}$ being the identity matrix. We can verify that $\mathbb{B}$ is positive definite when $\phi_0 \leq 2\gamma - \frac{1}{2}$, i.e $\frac{2C_{\mu,\gamma}c^2\tau}{d} \leq 2\gamma - \frac{1}{2}$, that is $\tau \leq \frac{(2\gamma - \frac{1}{2})d}{2C_{\mu,\gamma}c^2}$. Hence, according to Lemma 2.6, we get (3.4). $\square$

3.2.3. Proof for the third-order scheme

Following [19, 20], we introduce a series of notations

$$\begin{aligned} &\mathbb{E}_1 w^n = w^{n,1} - w^n, && \mathbb{E}_2 w^n = w^{n,2} - 2w^{n,1} + w^n, \\ &\mathbb{E}_3 w^n = 2w^{n,3} + w^{n,2} - 3w^{n,1}, && \mathbb{E}_4 w^n = w^{n+1} - w^{n,3}, \\ &\mathbb{E}_{31} w^n = w^{n,3} + w^{n,2} - 2w^{n,1}, && \mathbb{E}_{32} w^n = w^{n,3} - w^{n,1} \end{aligned} \tag{3.15}$$

for any function w. With these notations, we can rewrite the scheme (3.3) as

$$(\mathbb{E}_\ell u^n, v) = \Phi_\ell(\boldsymbol{u}^n, v) + \Psi_\ell(\boldsymbol{u}^n, v), \quad \ell = 1, 2, 3, 4, \tag{3.16}$$

where $\boldsymbol{u}^n = (u^n, u^{n,1}, u^{n,2}, u^{n,3})$ and

$$\Phi_\ell(\boldsymbol{u}^n, v) = \sum_{i=0}^{3}\sigma_{\ell i}\tau\mathcal{H}(u^{n,i}, v), \tag{3.17a}$$

$$\Psi_\ell(\boldsymbol{u}^n, v) = \theta_{\ell 1}\tau\mathcal{L}(u^{n,1}, v) + \theta_{\ell 2}\tau\mathcal{L}(u^{n,2} - 2u^{n,1}, v) + \theta_{\ell 3}\tau\mathcal{L}(u^{n,3}, v) \tag{3.17b}$$

for $\ell = 1, 2, 3, 4$. Here and below, $w^{n,0} = w^n$ for any function w. The coefficients $\sigma_{\ell i}$ and $\theta_{\ell i}$ are given in Table 3.2. Please refer to [19] for more details.

Table 3.2: The coefficients $\sigma_{\ell i}$ and $\theta_{\ell i}$ in (3.17).

	$\sigma_{\ell i}$				$\theta_{\ell i}$		
$\ell \backslash i$	0	1	2	3	1	2	3
1	γ	0	0	0	γ	0	0
2	$\frac{1-3\gamma}{2} - \alpha_1$	α_1	0	0	$\frac{1-\gamma}{2}$	γ	0
3	$\frac{1-5\gamma}{2} - \alpha_1$	$2(1-\alpha_2)+\alpha_1$	$2\alpha_2$	0	$2(\frac{9}{4} - \frac{11}{4}\gamma - \beta_1)$	$2(1-\beta_1-\frac{\gamma}{2})$	2γ
4	0	$\alpha_2 - \beta_2 - \gamma$	$\beta_2 - \alpha_2$	γ	0	0	0

Taking the test functions $v = u^{n,1}, u^{n,2} - 2u^{n,1}, u^{n,3}$ and $2u^{n+1}$ in (3.16), for $\ell = 1, 2, 3, 4$, respectively. Summing up these equations, we get the energy equation

$$\|u^{n+1}\|^2 - \|u^n\|^2 + \mathcal{S} = \mathcal{T}_c + \mathcal{T}_d, \tag{3.18}$$

where

$$\mathcal{S} = \frac{1}{2}\Big(\|\mathbb{E}_1 u^n\|^2 + \|\mathbb{E}_2 u^n\|^2 + \|\mathbb{E}_{31} u^n\|^2 + \|\mathbb{E}_{32} u^n\|^2 + 2\|\mathbb{E}_4 u^n\|^2\Big), \tag{3.19a}$$

$$\mathcal{T}_c = \Phi_1(\boldsymbol{u}^n, u^{n,1}) + \Phi_2(\boldsymbol{u}^n, u^{n,2} - 2u^{n,1}) + \Phi_3(\boldsymbol{u}^n, u^{n,3}) + 2\Phi_4(\boldsymbol{u}^n, u^{n+1}), \tag{3.19b}$$

$$\mathcal{T}_d = \Psi_1(\boldsymbol{u}^n, u^{n,1}) + \Psi_2(\boldsymbol{u}^n, u^{n,2} - 2u^{n,1}) + \Psi_3(\boldsymbol{u}^n, u^{n,3}) + 2\Psi_4(\boldsymbol{u}^n, u^{n+1}) \tag{3.19c}$$

are given in the same form as that in [19].

We would like to estimate $\mathcal{T}_d$ firstly. Denote $\mathbf{w}^n = (u^{n,1}, u^{n,2} - 2u^{n,1}, u^{n,3})^\top$, then similarly to (3.11), we can get

$$\mathcal{T}_d = \tau \underline{\mathcal{L}}(\mathbf{w}^n, \mathbb{C}\mathbf{w}^n), \tag{3.20}$$

where

$$\mathbb{C} = \frac{1}{2}\begin{pmatrix} 2\theta_{11} & \theta_{21} & \theta_{31} \\ \theta_{21} & 2\theta_{22} & \theta_{32} \\ \theta_{31} & \theta_{32} & 2\theta_{33} \end{pmatrix}. \tag{3.21}$$

We can verify that the eigenvalues of $\mathbb{C}$ are all positive, so $\mathbb{C}$ is positive definite. Hence $\mathcal{T}_d \leq 0$ according to Lemma 2.6.

To estimate $\mathcal{T}_c$, we rewrite it as

$$\begin{aligned} \mathcal{T}_c =& \sigma_{10}\tau\mathcal{H}(u^{n,1}, u^{n,1}) - (\sigma_{20} + \sigma_{21})\tau\mathcal{H}(u^{n,2} - 2u^{n,1}, u^{n,2} - 2u^{n,1}) \\ &+ (\sigma_{30} + \sigma_{31} + \sigma_{32})\tau\mathcal{H}(u^{n,3}, u^{n,3}) + \sum_{i=1}^{3} T_i, \end{aligned}$$

where

$$T_1 = 2(\sigma_{42} - \sigma_{43})\tau\mathcal{H}(u^{n,1}, u^{n+1} - u^{n,3}) - \sigma_{10}\tau\mathcal{H}(u^{n,1} - u^n, u^{n,1}), \tag{3.22a}$$

$$\begin{aligned} T_2 =& 2\sigma_{42}\tau\mathcal{H}(u^{n,2} - 2u^{n,1}, u^{n+1} - u^{n,3}) + \sigma_{21}\tau\mathcal{H}(u^{n,1} - u^n, u^{n,2} - 2u^{n,1}) \\ &+ (\sigma_{20} + \sigma_{21})\tau\mathcal{H}(u^{n,2} - 2u^{n,1} + u^n, u^{n,2} - 2u^{n,1}), \end{aligned} \tag{3.22b}$$

$$\begin{aligned} T_3 =& 2\sigma_{43}\tau\mathcal{H}(u^{n,3}, u^{n+1} - u^{n,3}) + (2\sigma_{42} + \sigma_{32})\mathcal{H}(u^{n,2} - 2u^{n,1} + u^n, u^{n,3}) \\ &+ (-\sigma_{30} + 2\sigma_{42} + \sigma_{32})\tau\mathcal{H}(u^{n,1} - u^n, u^{n,3}) \\ &- (\sigma_{30} + \sigma_{31} + \sigma_{32} - 2\sigma_{43})\tau\mathcal{H}(u^{n,3} - u^{n,1}, u^{n,3}). \end{aligned} \tag{3.22c}$$

Using property (2.15) and noting that those coefficients σ_{10}, $-(\sigma_{20}+\sigma_{21})$ and $(\sigma_{30}+\sigma_{31}+\sigma_{32})$ are all positive, we get

$$\mathcal{T}_c \leq \sum_{i=1}^{3} T_i.$$

By the aid of Lemma 2.3, we get

$$\mathcal{T}_c \leq C_{\star,\mu} c\tau \Big(\|u^{n,1}\|_h + \|u^{n,2} - 2u^{n,1}\|_h + \|u^{n,3}\|_h \Big) \mathcal{T}_0,$$

where $C_{\star,\mu} > 0$ depends on the coefficients in (3.22) and μ, and

$$\mathcal{T}_0 = \|u^{n+1} - u^{n,3}\| + \|u^{n,1} - u^n\| + \|u^{n,2} - 2u^{n,1} + u^n\| + \|u^{n,3} - u^{n,1}\|.$$

Then by the Young's inequality and Lemma 2.5, we have

$$\begin{aligned} \mathcal{T}_c &\leq \varepsilon C_{\star,\mu}^2 c^2 \tau^2 \left(\|u^{n,1}\|_h^2 + \|u^{n,2} - 2u^{n,1}\|_h^2 + \|u^{n,3}\|_h^2 \right) + \frac{3}{4\varepsilon} \mathcal{T}_0^2 \\ &\leq -\frac{2\varepsilon C_{\star,\mu}^2 c^2 \tau^2}{d} \left[\mathcal{L}(u^{n,1}, u^{n,1}) + \mathcal{L}(u^{n,2} - 2u^{n,1}, u^{n,2} - 2u^{n,1}) + \mathcal{L}(u^{n,3}, u^{n,3}) \right] + \frac{3}{4\varepsilon} \mathcal{T}_0^2 \end{aligned} \tag{3.23}$$

for arbitrary $\varepsilon > 0$. Taking $\varepsilon = 6$ and denoting $\phi_1 = \frac{12 C_{\star,\mu}^2 c^2 \tau}{d}$, then

$$\mathcal{T}_c \leq -\phi_1 \tau \left[\mathcal{L}(u^{n,1}, u^{n,1}) + \mathcal{L}(u^{n,2} - 2u^{n,1}, u^{n,2} - 2u^{n,1}) + \mathcal{L}(u^{n,3}, u^{n,3}) \right] + \mathcal{S}, \tag{3.24}$$

where $\mathcal{S}$ has been defined in (3.19a). Owing to (3.18), (3.20) and (3.24) we have

$$\|u^{n+1}\|^2 - \|u^n\|^2 \leq \tau \underline{\mathcal{L}}(\mathbf{w}^n, \mathbb{C}' \mathbf{w}^n), \tag{3.25}$$

where $\mathbb{C}' = \mathbb{C} - \phi_1 \mathbb{I}$. We can verify that $\mathbb{C}'$ is positive definite if $\phi_1 < \frac{\gamma}{4}$, i.e. $\tau \leq \tau_0 = \frac{d\gamma}{48 C_{\star,\mu}^2 c^2}$. Thus, $\|u^{n+1}\| \leq \|u^n\|$. □

Remark 3.1. In [22], Wang and Zhang also discussed the stability analysis of the first and second order fully discrete IMEX-DDG schemes. Even though both the DDG and the UWDG methods are based on the primal formulation without auxiliary variables and have the dissipative property, the DDG discretization considered in [22] is not symmetric. With the help of the symmetric and dissipative properties of the UWDG discretization, we obtain the stability of the above three fully discrete IMEX-UWDG schemes.

4. Error Estimates for Fully Discrete Schemes

With the stability results in the previous section and the optimal error estimates for the semi-discrete UWDG scheme discussed in Subsection 2.3, it is conceptually straightforward, although technical, to obtain the optimal error estimates for the fully discrete IMEX-UWDG schemes considered in previous section. We will only give the optimal error estimates for the third order IMEX-UWDG scheme (3.3) as an example. Following [19,20], we introduce reference functions $U^{(\ell)}$ for $\ell = 0, 1, 2, 3$ as follows. Let $U^{(0)} = U$ be the exact solution of the problem (2.1) and define

$$U^{(\ell)} = U^{(0)} + \tau \sum_{i=0}^{3} \left(-c a_{\ell i} U_x^{(i)} + d \hat{a}_{\ell i} U_{xx}^{(i)} \right), \quad \ell = 1, 2, 3. \tag{4.1}$$

At each stage time level $t^{n,\ell}$, the reference function is denoted by $U^{n,\ell} = U^{(\ell)}(x,t^n)$, and the corresponding error is denoted by

$$e^{n,\ell} = U^{n,\ell} - u^{n,\ell} = \xi^{n,\ell} - \eta^{n,\ell},$$

where

$$\xi^{n,\ell} = \mathbb{P}_h U^{n,\ell} - u^{n,\ell}, \quad \eta^{n,\ell} = \mathbb{P}_h U^{n,\ell} - U^{n,\ell}.$$

To derive the optimal error estimates, we assume the exact solution U satisfies

$$U, D_t U \in L^\infty(0,T;H^{k+1}), \quad U_t^{(4)} \in L^\infty(0,T;L^2). \tag{4.2}$$

By the above smoothness assumption, and owing to the linear structure of the projection $\mathbb{P}_h$, it follows from (2.29) that

$$\|\eta^{n,\ell}\| + h^{\frac{1}{2}}\|\eta^{n,\ell}\|_{\Gamma_h} \le Ch^{k+1}, \tag{4.3a}$$

$$\|\mathbb{E}_{\ell+1}\eta^n\| + h^{\frac{1}{2}}\|\mathbb{E}_{\ell+1}\eta^n\|_{\Gamma_h} \le Ch^{k+1}\tau \tag{4.3b}$$

for any n and $\ell = 0,1,2,3$, where the bounded constant C depends only on the regularity of U and not on n, h and τ.

Next, we present the estimates for ξ^n. To this end, we set up the error equations firstly. From [19] we have

$$(\mathbb{E}_\ell U^n, v) = \Phi_\ell(\boldsymbol{U}^n, v) + \Psi_\ell(\boldsymbol{U}^n, v) + \delta_{4\ell}(\zeta^n, v), \quad \ell = 1,2,3,4, \tag{4.4}$$

where $\boldsymbol{U}^n = (U^n, U^{n,1}, U^{n,2}, U^{n,3})$, $\delta_{4\ell} = 1$ for $\ell = 4$ and $\delta_{4\ell} = 0$ for $\ell = 1,2,3$, ζ^n is the local truncation error which satisfies

$$\|\zeta^n\| \le C\tau^4, \tag{4.5}$$

where C only depends on the regularity of U.

Subtracting (3.16) from (4.4), we get the error equation

$$(\mathbb{E}_\ell \xi_u^n, v) = \Phi_\ell(\boldsymbol{\xi}^n, v) + \Psi_\ell(\boldsymbol{\xi}^n, v) - \Phi_\ell(\boldsymbol{\eta}^n, v) + (\mathbb{E}_\ell \eta^n + \delta_{4\ell}\zeta^n, v), \quad \ell = 1,2,3,4, \tag{4.6}$$

here we used $\Psi_\ell(\boldsymbol{\eta}^n, v) = 0$ by Lemma 2.8. Here $\boldsymbol{\xi}^n = (\xi^n, \xi^{n,1}, \xi^{n,2}, \xi^{n,3})$ and $\boldsymbol{\eta}^n = (\eta^n, \eta^{n,1}, \eta^{n,2}, \eta^{n,3})$.

Taking $v = \xi^{n,1}$, $\xi^{n,2} - 2\xi^{n,1}$, $\xi^{n,3}$, $2\xi^{n+1}$ in (4.6), for $\ell = 1,2,3,4$ respectively, summing up these equations leads to the energy equation

$$\|\xi^{n+1}\|^2 - \|\xi^n\|^2 + \mathcal{S}' = I + II + III + IV, \tag{4.7}$$

where

$$\mathcal{S}' = \frac{1}{2}\Big(\|\mathbb{E}_1\xi^n\|^2 + \|\mathbb{E}_2\xi^n\|^2 + \|\mathbb{E}_{31}\xi^n\|^2 + \|\mathbb{E}_{32}\xi^n\|^2 + 2\|\mathbb{E}_4\xi^n\|^2\Big), \tag{4.8}$$

and

$$I = \sum_{\ell=1}^{4} \Phi_\ell(\boldsymbol{\xi}^n, v_\ell), \qquad II = \sum_{\ell=1}^{4} \Psi_\ell(\boldsymbol{\xi}^n, v_\ell),$$

$$III = -\sum_{\ell=1}^{4} \Phi_\ell(\boldsymbol{\eta}^n, v_\ell), \quad IV = \sum_{\ell=1}^{4} (\mathbb{E}_\ell \eta^n + \delta_{4\ell}\zeta^n, v_\ell).$$

Here and below we adopt the following simplified notations:

$$v_1 = \xi^{n,1}, \quad v_2 = \xi^{n,2} - 2\xi^{n,1}, \quad v_3 = \xi^{n,3}, \quad v_4 = 2\xi^{n+1}.$$

In what follows we estimate the right hand side of (4.7) term by term. Along the same procedure as the estimates for $\mathcal{T}_c$ with taking $\varepsilon = 7$ in (3.23), we get

$$I \le -\phi_2 \tau \sum_{\ell=1}^{3} \mathcal{L}(v_\ell, v_\ell) + \frac{6}{7}\mathcal{S}', \tag{4.9}$$

where $\phi_2 = \frac{14C_{\star,\mu}^2 c^2 \tau}{d}$. Similar to the estimate for $\mathcal{T}_d$ in (3.20), we get

$$II = \tau \underline{\mathcal{L}}(\mathbf{v}^n, \mathbb{C}\mathbf{v}^n), \tag{4.10}$$

where $\mathbf{v}^n = (v_1, v_2, v_3)^\top$, and $\mathbb{C}$ is defined in (3.21).

The estimate for III is more complicated. If we simply make use of (2.35), then we will get

$$III \le c^2 \varepsilon \tau \sum_{\ell=1}^{4} \|v_\ell\|_h^2 + Ch^{2k+2}\tau$$

for any positive ε. However, there is no stability term to bound the term $\|v_4\|_h^2 = \|\xi^{n+1}\|_h^2$, so we need to find another way. We would like to adopt the trick used in [20]. According to the definition of Φ_ℓ in (3.17a), we split III into two terms Z_1 and Z_2, with

$$\begin{aligned} Z_1 =& -\tau \sum_{\ell=1}^{3}\sum_{i=0}^{3} \sigma_{\ell i}\mathcal{H}(\eta^{n,i}, v_\ell) - 2\tau \sum_{i=0}^{3} \sigma_{4i}\mathcal{H}(\eta^{n,i}, v_3), \\ Z_2 =& -2\tau \sum_{i=0}^{3} \sigma_{4i}\mathcal{H}(\eta^{n,i}, \mathbb{E}_4\xi^n). \end{aligned}$$

By a simple use of (2.35), we get

$$Z_1 \le c^2 \varepsilon_0 \tau \sum_{\ell=1}^{3} \|v_\ell\|_h^2 + Ch^{2k+2}\tau \le -\frac{2c^2\varepsilon_0}{d}\tau \sum_{\ell=1}^{3} \mathcal{L}(v_\ell, v_\ell) + Ch^{2k+2}\tau \tag{4.11}$$

for arbitrary $\varepsilon_0 > 0$, where Lemma 2.5 was used in the second step.

Noting that $\sigma_{41} = -\sigma_{42} - \sigma_{43}$, so

$$Z_2 = -2\sigma_{42}\tau\mathcal{H}(\eta^{n,2} - \eta^{n,1}, \mathbb{E}_4\xi^n) - 2\sigma_{43}\tau\mathcal{H}(\eta^{n,3} - \eta^{n,1}, \mathbb{E}_4\xi^n).$$

Hence by the Cauchy-Schwarz inequality, (4.3b) and the inverse inequalities (2.14a) and (2.14b), we get

$$Z_2 \le Ch^k \tau^2 \|\mathbb{E}_4\xi^n\| \le \varepsilon_1 \|\mathbb{E}_4\xi^n\|^2 + Ch^{2k}\tau^4 \tag{4.12}$$

for arbitrary $\varepsilon_1 > 0$.

Furthermore, we use the Cauchy-Schwarz inequality, the Young's inequality and (4.3b), (4.5) to deal with the term IV, namely,

$$IV \le \varepsilon_2 \tau \sum_{\ell=1}^{4} \|v_\ell\|^2 + C(h^{2k+2}\tau + \tau^7) \tag{4.13}$$

for arbitrary $\varepsilon_2 > 0$. Noting that $v_1 = \xi^n + \mathbb{E}_1\xi^n$, $v_2 = \mathbb{E}_2\xi^n - \xi^n$, $v_3 = \mathbb{E}_{31}\xi^n - v_2$, $v_4 = 2(v_3 + \mathbb{E}_4\xi^n)$. So (4.13) will become

$$IV \leq \tilde{C}\tau\|\xi^n\|^2 + \tau\mathcal{S}' + C(h^{2k+2}\tau + \tau^7), \tag{4.14}$$

if ε_2 is chosen properly, where $\tilde{C}$ is a positive constant which is independent of h and τ.

As a result, from(4.7), (4.9)–(4.12) and (4.14), we have

$$\begin{aligned}\|\xi_u^{n+1}\|^2 - \|\xi_u^n\|^2 + \mathcal{S}' \leq \tilde{C}\tau\|\xi^n\|^2 + \tau\underline{\mathcal{L}}(\mathbf{v}^n, \mathbb{C}\mathbf{v}^n) - \tau\left(\phi_2 + \frac{2c^2\varepsilon_0}{d}\right)\sum_{\ell=1}^{3}\mathcal{L}(v_\ell, v_\ell)\\ + \left(\frac{6}{7} + 2\varepsilon_1 + \tau\right)\mathcal{S}' + C(h^{2k+2}\tau + h^{2k}\tau^4 + \tau^7).\end{aligned} \tag{4.15}$$

Taking ε_0 and ε_1 small enough, for example, taking $\varepsilon_0 = \frac{d\gamma}{40c^2}$ and $\varepsilon_1 = \frac{1}{112}$, then we can get that, if

$$\phi_2 \leq \frac{\gamma}{5}, \quad \tau \leq \frac{1}{8},$$

i.e,

$$\tau \leq \tau_0 = \min\left\{\frac{d\gamma}{70C_{\star,\mu}^2 c^2}, \frac{1}{8}\right\}, \tag{4.16}$$

then

$$\|\xi^{n+1}\|^2 - \|\xi^n\|^2 \leq \tilde{C}\tau\|\xi^n\|^2 + C(h^{2k+2}\tau + h^{2k}\tau^4 + \tau^7). \tag{4.17}$$

By the aid of the discrete Gronwall's inequality, we obtain

$$\begin{aligned}\|\xi^n\| &\leq e^{\tilde{C}n\tau}\|\xi^0\| + C(h^{k+1} + h^k\tau^{\frac{3}{2}} + \tau^3)\\ &\leq C(h^{k+1} + h^{2k} + \tau^3) \leq C(h^{k+1} + \tau^3),\end{aligned} \tag{4.18}$$

where we have used $\xi_u^0 = 0$, the Young's inequality and the condition $k \geq 1$.

Finally, by the triangle inequality and (4.3a), we can get the main error estimate, which is stated in the following theorem.

Theorem 4.1. *Let U be the exact solution of problem* (2.1)*, which satisfies* (4.2)*, let u be the numerical solution of scheme* (3.3)*. Under the conditions* (2.21) *and* (2.23)*, there exists a positive constant τ_0 that is independent of h, such that if $\tau \leq \tau_0$, then*

$$\max_{n\tau\leq T}\|U(t^n) - u^n\| \leq C(h^{k+1} + \tau^3), \tag{4.19}$$

where T is the final computing time, and the bounded constant $C > 0$ depends on k and the regularity of U but does not depend on h and τ.

Remark 4.1. Following [20], we can also get the optimal error estimate of the fully discrete IMEX-UWDG schemes for convection-diffusion problems with a nonlinear convection term, if we adopt monotone numerical flux in the discretization of the convection part.

5. Numerical Experiments

In this section, we give some numerical examples to confirm the theoretical results for the fully discrete IMEX-UWDG schemes considered in Section 3. Since the first order scheme (3.1) with penalty parameter $\lambda = 1/h$ and polynomial space $k = 0$ is the same as the first order IMEX-LDG scheme with $k = 0$, we only present the results for the second order scheme (3.2) with $k = 1$ and the third order scheme (3.3) with $k = 2$.

Example 5.1. We consider problem (2.1) in the interval $(-\pi, \pi)$, with the exact solution

$$U(x,t) = e^{-dt}\sin(x - ct). \tag{5.1}$$

Obviously, the periodic boundary condition is satisfied by the exact solution.

Table 5.1 lists the maximum time step τ_0 to ensure that the L^2 norm of numerical solution decreases with time, for different choice of penalty parameters. In this test, we take uniform mesh with mesh size $h = \frac{2\pi}{640}$. The final computing time is $T = 5000$. In our test, τ_0 is obtained numerically by a bisection search, i.e, we first set two initial values $\tau_1 = 0$ and $\tau_2 = 10$, and set $\tau = \frac{\tau_1+\tau_2}{2}$ in each loop, if the L^2-norm of numerical solution is decreasing in the sense that $\|u^{n+1}\| - \|u^n\| \le 1.E-24$ (considering the machine rounding error), then we set $\tau_1 = \tau$; otherwise, we set $\tau_2 = \tau$; continue the loop until $|\tau_1 - \tau_2| \le 0.001$. The results show that τ_0 is approximately proportional to $\frac{d}{c^2}$, which validates our theoretical stability properties.

Tables 5.2 and 5.3 show the L^2 errors and orders of accuracy for the schemes (3.2) and (3.3), for solving (2.1) with $d = 0.1$ on both uniform and nonuniform meshes. In this section, the nonuniform meshes are generated by randomly perturbing each node in the uniform mesh by up 20%. In these tests, time step is taken as $\tau = h$ and the final computing time is $T = 10$. Optimal error accuracy can be observed from both tables.

Example 5.2. We consider the following viscous Burgers' equation with a source term

$$U_t + UU_x = dU_{xx} + g(x,t), \tag{5.2a}$$

$$U(x,0) = \sin(x), \tag{5.2b}$$

where $g(x,t) = \frac{1}{2}e^{-2dt}\sin(2x)$, $x \in (-\pi,\pi)$ and $t \in (0,T]$. The exact solution of (5.2) is $U(x,t) = e^{-dt}\sin(x)$, and the periodic boundary condition is satisfied by the exact solution.

We display the L^2 errors and orders of accuracy for the schemes (3.2) and (3.3), for solving (5.2) on uniform and nonuniform meshes in Tables 5.4 and 5.5, respectively. We only show the results of $\lambda = \frac{5}{h}$ for the second order scheme and $\lambda = \frac{10}{h}$ for the third order scheme as examples to illustrate the optimal order of accuracy.

Table 5.1: The maximum time step τ_0 to ensure that the L^2-norm of numerical solution decreases with time for the schemes (3.2) and (3.3), $T = 5000$, $h = 2\pi/640$.

scheme	$d = 0.01$			$c = 0.1$		
	$c = 0.05$	$c = 0.1$	$c = 0.2$	$d = 0.01$	$d = 0.02$	$d = 0.04$
2nd order, $\lambda = 3/h$	5.535	1.380	0.341	1.380	2.767	5.538
2nd order, $\lambda = 5/h$	5.543	1.387	0.347	1.387	2.770	5.540
3rd order, $\lambda = 9/h$	4.699	1.083	0.242	1.083	2.349	4.974
3rd order, $\lambda = 12/h$	5.405	1.295	0.302	1.295	2.702	5.537

Table 5.2: The second order scheme (3.2) with $k = 1$, $T = 10$, $\tau = h$.

mesh type	λ	N	$c = 1$		$c = 0.1$		$c = 0.01$	
			L^2 error	order	L^2 error	order	L^2 error	order
uniform	$3/h$	40	2.39E+08	-	1.41E-03	-	1.47E-03	-
		80	6.75E-03	35.04	3.59E-04	1.98	3.68E-04	2.00
		160	1.69E-03	2.00	9.04E-05	1.99	9.20E-05	2.00
		320	4.22E-04	2.00	2.27E-05	1.99	2.30E-05	2.00
		640	1.05E-04	2.00	5.68E-06	2.00	5.75E-06	2.00
	$5/h$	40	2.70E-02	-	9.31E-04	-	9.08E-04	-
		80	6.76E-03	2.00	2.31E-04	2.01	2.27E-04	2.00
		160	1.69E-03	2.00	5.77E-05	2.00	5.67E-05	2.00
		320	4.23E-04	2.00	1.44E-05	2.00	1.42E-05	2.00
		640	1.06E-04	2.00	3.60E-06	2.00	3.54E-06	2.00
nonuniform	$5/h$	40	1.21E+02	-	1.22E-03	-	1.21E-03	-
		80	6.76E-03	14.13	2.88E-04	2.08	2.78E-04	2.12
		160	1.69E-03	2.00	7.49E-05	1.95	7.54E-05	1.89
		320	4.23E-04	2.00	1.85E-05	2.02	1.83E-05	2.04
		640	1.06E-04	2.00	4.62E-06	2.00	4.61E-06	1.99
	$8/h$	40	2.70E-02	-	1.11E-03	-	1.12E-03	-
		80	6.77E-03	2.00	2.84E-04	1.97	2.69E-04	2.06
		160	1.69E-03	2.00	6.94E-05	2.03	6.82E-05	1.98
		320	4.24E-04	2.00	1.74E-05	2.00	1.69E-05	2.01
		640	1.06E-04	2.00	4.40E-06	1.98	4.31E-06	1.97

Table 5.3: The third order scheme (3.3) with $k = 2$, $T = 10$, $\tau = h$.

mesh type	λ	N	$c = 1$		$c = 0.1$		$c = 0.01$	
			L^2 error	order	L^2 error	order	L^2 error	order
uniform	$9/h$	40	1.68E+10	-	1.26E-05	-	1.26E-05	-
		80	5.61E-05	48.09	1.57E-06	3.00	1.57E-06	3.00
		160	7.02E-06	3.00	1.97E-07	3.00	1.97E-07	3.00
		320	8.78E-07	3.00	2.46E-08	3.00	2.46E-08	3.00
		640	1.10E-07	3.00	3.08E-09	3.00	3.08E-09	3.00
	$12/h$	40	8.33E+03	-	1.11E-05	-	1.11E-05	-
		80	5.61E-05	27.15	1.39E-06	3.00	1.38E-06	3.00
		160	7.02E-06	3.00	1.73E-07	3.00	1.73E-07	3.00
		320	8.78E-07	3.00	2.16E-08	3.00	2.16E-08	3.00
		640	1.10E-07	3.00	2.70E-09	3.00	2.70E-09	3.00
nonuniform	$12/h$	40	1.20E+12	-	1.53E-05	-	1.48E-05	-
		80	5.61E-05	54.25	2.03E-06	2.91	1.84E-06	3.01
		160	7.02E-06	3.00	2.43E-07	3.06	2.41E-07	2.93
		320	8.78E-07	3.00	3.05E-08	3.00	2.97E-08	3.02
		640	1.10E-07	3.00	3.80E-09	3.00	3.84E-09	2.95
	$15/h$	40	6.01E+05	-	1.45E-05	-	1.31E-05	-
		80	5.61E-05	33.32	1.72E-06	3.08	1.75E-06	2.90
		160	7.02E-06	3.00	2.24E-07	2.94	2.27E-07	2.95
		320	8.78E-07	3.00	2.80E-08	3.00	2.84E-08	3.00
		640	1.10E-07	3.00	3.47E-09	3.01	3.50E-09	3.02

Table 5.4: The second order scheme with $k = 1, \lambda = 5/h$ and the third order scheme with $k = 2, \lambda = 10/h$ for Burgers' equation (5.2) on uniform meshes. $T = 10$, $\tau = h$ for $d = 1$ and $d = 0.1$, $\tau = 0.5h$ for $d = 0.01$.

scheme	N	$d=1$		$d=0.1$		$d=0.01$	
		L^2 error	order	L^2 error	order	L^2 error	order
2nd order, $\lambda = 5/h$	40	1.22E-06	-	8.69E-04	-	3.58E-03	-
	80	3.07E-07	2.00	2.17E-04	2.00	6.34E-04	2.50
	160	7.67E-08	2.00	5.42E-05	2.00	1.37E-04	2.21
	320	1.92E-08	2.00	1.35E-05	2.00	3.28E-05	2.06
	640	4.79E-09	2.00	3.39E-06	2.00	8.11E-06	2.02
3rd order, $\lambda = 10/h$	40	7.45E-08	-	1.16E-05	-	2.78E-05	-
	80	9.75E-09	2.93	1.45E-06	3.00	3.56E-06	2.96
	160	1.25E-09	2.97	1.81E-07	3.00	4.46E-07	3.00
	320	1.58E-10	2.98	2.26E-08	3.00	5.57E-08	3.00
	640	1.98E-11	2.99	2.83E-09	3.00	6.96E-09	3.00

Table 5.5: The second order scheme with $k = 1, \lambda = 5/h$ and the third order scheme with $k = 2, \lambda = 10/h$ for Burgers' equation (5.2) on nonuniform meshes. $T = 10$, $\tau = h$ for $d = 1$ and $d = 0.1$, $\tau = 0.3h$ for $d = 0.01$.

scheme	N	$d=1$		$d=0.1$		$d=0.01$	
		L^2 error	order	L^2 error	order	L^2 error	order
2nd order, $\lambda = 5/h$	40	7.83E-07	-	1.22E-03	-	5.26E-03	-
	80	1.84E-07	2.09	2.94E-04	2.05	9.07E-04	2.54
	160	4.80E-08	1.94	7.40E-05	1.99	1.96E-04	2.21
	320	1.08E-08	2.16	1.85E-05	2.00	4.60E-05	2.09
	640	2.67E-09	2.01	4.65E-06	2.00	1.14E-05	2.01
3rd order, $\lambda = 10/h$	40	7.41E-08	-	1.93E-05	-	4.23E-05	-
	80	9.73E-09	2.93	2.49E-06	2.96	5.23E-06	3.01
	160	1.25E-09	2.97	3.04E-07	3.04	6.93E-07	2.92
	320	1.58E-10	2.98	3.86E-08	2.98	9.67E-08	2.84
	640	1.98E-11	2.99	4.78E-09	3.01	1.22E-08	2.98

6. Concluding Remarks

We present the optimal error estimates of the semi-discrete UWDG method for solving linear convection-diffusion equations with periodic boundary conditions in one dimension. The UWDG method coupled with three specific IMEX time discretizations are shown to be stable if the time step is bounded from above by a positive constant which is independent of the mesh size. The symmetric and dissipative properties of the UWDG method play an important role in the stability analysis. We also present optimal error estimates for the third order fully discrete IMEX-UWDG scheme, under the same temporal condition as in stability analysis. Ongoing work includes the study of IMEX-UWDG schemes with general numerical flux setting. We also would like to generalize the results of this paper to multi-dimensional problems.

Acknowledgments. Research sponsored by NSFC grants 11871428 and 12071214, and Nature Science Research Program for Colleges and Universities of Jiangsu Province grant 20KJB110011. Research is supported in part by NSFC grants U1930402 and the fellowship of China Postdoctoral Science Foundation (No. 2020TQ0030).

References

[1] U.M. Ascher, S.J. Ruuth and R.J. Spiteri, Implicit-explicit Runge-Kutta methods for time-dependent partial differential equations, *Appl. Numer. Math.*, **25** (1997), 151–167.

[2] M.P. Calvo, J. de Frutos and J. Novo, Linearly implicit Runge-Kutta methods for advection-reaction-diffusion equations, *Appl. Numer. Math.*, **37** (2001), 535–549.

[3] A. Chen, F. Li and Y. Cheng, An ultra-weak discontinuous Galerkin method for Schrödinger equation in one dimension, *J. Sci. Comput.*, **78** (2019), 772–815.

[4] A. Chen, Y. Cheng, Y. Liu and M. Zhang, Superconvergence of ultra-weak discontinuous Galerkin methods for the linear Schrödinger equation in one dimension, *J. Sci. Comput.*, **82** (2020), 22.

[5] Y. Cheng and C.W. Shu, A discontinuous Galerkin finite element method for time dependent partial differential equations with higher order derivatives, *Math. Comput.*, **77** (2008), 699–730.

[6] P.G. Ciarlet, The Finite Element Method for Elliptic Problems, North–Holland, Amsterdam, New York, 1978.

[7] B. Cockburn and C.W. Shu, The local discontinuous Galerkin method for time-dependent convection-diffusion systems, *SIAM. J. Numer. Anal.*, **35** (1998), 2440–2463.

[8] G. Fu and C.W. Shu, Analysis of an embedded discontinuous Galerkin method with implicit-explicit time-marching for convection-diffusion problems, *Int. J. Numer. Anal. Model.*, **347** (2017), 305–327.

[9] G. Fu and C.W. Shu, An energy-conserving ultra-weak discontinuous Galerkin method for the generalized Korteweg-De Vries equation, *J. Comput. Appl. Math.*, **349** (2019), 41–51.

[10] S. Güzey, B. Cockburn, and H.K. Stolarski, The embedded discontinuous Galerkin method: application to linear shell problems, *Internat. J. Numer. Methods Engrg.*, **70** (2007), 757–790.

[11] C.A. Kennedy and M.H. Carpenter, Additive Runge-Kutta schemes for convection-diffusion-reactive equations, *Appl. Numer. Math.*, **44** (2003), 139–181.

[12] B.V. Leer, M. Lo, R. Gitik and S. Nomura, A venerable family of discontinuous Galerkin schemes for diffusion revisited, in: Z.J. Wang (Ed.), Adaptive High-Order Methods in Computational Fluid Dynamics, World Scientific, (2011), 185–201.

[13] Y. Li, C.W. Shu and S. Tang, An ultra-weak discontinuous Galerkin method with IMEX time-marching for generalized stochastic KdV equations, *J. Sci. Comput.*, **82** (2020), 61.

[14] H. Liu and J. Yan, The direct discontinuous Galerkin (DDG) methods for diffusion problems, *SIAM J. Numer. Anal.*, **47** (2009), 675–698.

[15] Y. Liu, Q. Tao and C.W. Shu, Analysis of optimal superconvergence of an ultraweak-local discontinuous Galerkin method for a time dependent fourth-order equation, *ESAIM: M2AN.*, **54** (2020), 1797–1820.

[16] S. Ortleb, L^2-stability analysis of IMEX-(σ, μ)DG schemes for linear advection-diffusion equations, *Appl. Numer. Math.*, **147** (2020), 43–65.

[17] M.A. Reyna and F. Li, Operator bounds and time step conditions for DG and central DG methods, *J. Sci. Comput.*, **62** (2015), 532–554.

[18] Q. Tao, Y. Xu and C.W. Shu, An ultraweak-local discontinuous Galerkin method for PDEs with high order spatial derivatives, *Math. Comput.*, **89** (2020), 2753–2783.

[19] H.J. Wang, C.W. Shu and Q. Zhang, Stability and error estimates of local discontinuous Galerkin methods with implicit-explicit time-marching for advection-diffusion problems, *SIAM J. Numer. Anal.*, **53** (2015), 206–227.

[20] H.J. Wang, C.W. Shu and Q. Zhang, Stability analysis and error estimates of local discontinuous Galerkin methods with implicit-explicit time-marching for nonlinear convection-diffusion problems, *Appl. Math. Comput.*, **272** (2016), 237–258.

[21] H.J. Wang, S.P. Wang, Q. Zhang and C.W. Shu, Local discontinuous Galerkin methods with implicit-explicit time-marching for multi-dimensional convection-diffusion problems, *ESAIM: M2AN.*, **50** (2016), 1083–1105.

[22] H.J. Wang and Q. Zhang, The direct discontinuous Galerkin methods with implicit-explicit time marching for convection-diffusion problems, *Commun. Appl. Math. Comput.*, (2021), 10.1007/s42967-020-00114-1.

[23] Q. Zhang and F.Z. Gao, A fully-discrete local discontinuous Galerkin method for convection-dominated Sobolev equation, *J. Sci. Comput.*, **51** (2012), 107–134.