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CONVERGENCE ANALYSIS OF SOME FINITE ELEMENT PARALLEL ALGORITHMS FOR THE STATIONARY INCOMPRESSIBLE MHD EQUATIONS*

Xiaojing Dong[1)]
Hunan Key Laboratory for Computation and Simulation in Science and Engineering, Key Laboratory of Intelligent Computing & Information Processing of Ministry of Education, School of Mathematics and Computational Science, Xiangtan University, Xiangtan 411105, China;
Institute of Applied Physics and Computational Mathematics, Beijing 100088, China
Email: dongxiaojing99@xtu.edu.cn
Yinnian He
School of Mathematics and Statistics, Xi'an Jiaotong University, Xi'an 710049, China
Email: heyn@mail.xjtu.edu.cn

Abstract

By combination of iteration methods with the partition of unity method (PUM), some finite element parallel algorithms for the stationary incompressible magnetohydrodynamics (MHD) with different physical parameters are presented and analyzed. These algorithms are highly efficient. At first, a global solution is obtained on a coarse grid for all approaches by one of the iteration methods. By parallelized residual schemes, local corrected solutions are calculated on finer meshes with overlapping sub-domains. The subdomains can be achieved flexibly by a class of PUM. The proposed algorithm is proved to be uniformly stable and convergent. Finally, one numerical example is presented to confirm the theoretical findings.

Mathematics subject classification: 35Q30, 65M60, 65N30, 76D05.
Key words: Partition of unity method, Local and parallel algorithm, Finite element method, Iteration methods, Magnetohydrodynamics.

1. Introduction

The stationary incompressible MHD equations [1] in a Lipschitz polygon/polyhedron $\Omega \subset R^d$ $(d = 2, 3)$ with homogeneous Dirichlet boundary conditions are described as

$$-R_e^{-1}\Delta\mathbf{u} + \mathbf{u}\cdot\nabla\mathbf{u} + \nabla p - S_c\mathrm{curl}\,\mathbf{B}\times\mathbf{B} = \mathbf{f}, \tag{1.1}$$

$$\mathrm{div}\;\mathbf{u} = 0, \tag{1.2}$$

$$S_cR_{e_m}^{-1}\mathrm{curl}\;\mathrm{curl}\;\mathbf{B} - S_c\mathrm{curl}\;(\mathbf{u}\times\mathbf{B}) - \nabla r = \mathbf{g}, \tag{1.3}$$

$$\mathrm{div}\;\mathbf{B} = 0, \tag{1.4}$$

$$\mathbf{u}|_{\partial\Omega} = 0, \quad \mathbf{B}\times\mathbf{n}|_{\partial\Omega} = 0, \quad r|_{\partial\Omega} = 0, \tag{1.5}$$

where R_e and R_{e_m} are the Reynolds numbers of hydrodynamic and magnetic, respectively, $\mathbf{n}$ is the unit outward normal vector on $\partial\Omega$, S_c is the coupling number of the two fields. $\mathbf{u}$ represents

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1) Corresponding author

fluid velocity field, $\mathbf{B}$ magnetic field strength, p hydrodynamic pressure and r magnetic pseudo-pressure. Let $\mathbf{g}$ be solenoidal.

The governing MHD model is strongly nonlinear because the classical equations of Maxwell and Navier-Stokes are coupled. This physical system describes the relationship between incompressible flows with electrically conducting property and the existing magnetic field. It has important applications in numerous areas of science, e.g., process metallurgy and MHD ion propulsion, see [2, 3].

Recently, finite element methods (FEM) for numerically solving MHD equations have become an attractive topic for the community of scientific computing. Based on the exact penalty constraint idea on magnetic, a stabilized FE formulation was studied in [4]. Stabilized FEM motivated by residual-based stabilizations was investigated in [5]. Divergence-cleaning algorithm in continuous FEM was given in [6]. A divergence-free discontinuous FEM was analyzed in [7]. To treat the nonlinear terms efficiently, three classical FE iterative methods were proposed and the stability and convergence related to physical parameters and iterations were proved by Dong *et al.* [8]. By using the Lagrange multiplier associated with the magnetic divergence constraint, a double-saddle-point FE formulation was given and analyzed in [9], and a mixed discontinuous Galerkin scheme of this version was proposed by Houston *et al.* [10]. The mixed FEMs with exactly preserving mass conservation of hydrodynamics and Gauss law of magnetic were studied in [11] and [12–15], respectively. Some robust solvers for finite element discrete system was designed in [16–18]. As for the time-dependent MHD equations, the stabilized nodal-based FEMs were proposed in [19], Euler semi-implicit fully discrete FE schemes were analyzed by Prohl [20] and He [21], the Crank-Nicolson extrapolation fully discrete FE scheme was analyzed by Dong and He [22].

It has been proven practically that two-level FEM [23, 24] is a high-efficiency technique to solve partial differential equations numerically, since it can reduce the cost of computing. This method has been applied to treat the nonlinear terms and coupled terms in the MHD problem in [25–27]. According to the observation of the behavior of a FE solution, [28] proposed parallel FEMs based on local algorithms. [28] obtained low frequencies component governing the global properties of the solution by using coarse mesh, and then approximates high frequencies one by solving the resulted local residual subproblems on several subdomains with the fine grids. This numerical algorithm is of high performance for few communications between blocks. Thus, it has been developed and extended to various problems, such as, Navier-Stokes equations [29–31], MHD equations [32, 33], etc.

Inspired by the algorithm in [28] and two-level FEM with respect to different physical parameters for the stationary MHD [27], in this article, we mainly extend the recent work [34, 35] to some local and parallel FE iterative algorithms (LPFEIAs) related to different physical parameters (Explicit in Theorem 3.2) for problem (1.1)-(1.5). The extensions to the previous studies [34, 35] are explained clearly before Theorem 3.12. According to different stable conditions of three classical m-iteration methods, we combine FEM with different iterations on a globally coarse mesh to obtain FE iterative solutions $(\mathbf{u}_H^m, \mathbf{B}_H^m, p_H^m, r_H^m)$ first, then we correct them by different linearized residual schemes in parallel on some local overlapping subdomains Ω_j to seek the correction solutions $(\mathbf{u}_{mh}^j, \mathbf{B}_{mh}^j, p_{mh}^j, r_{mh}^j), j = 1, \dots, J$, where J is the number of the subdomains, m is the iterative step and mesh sizes satisfy h $(h \ll H)$. Moreover, the uniform stability and convergence of each algorithm is analyzed.

The paper is divided into 4 sections. The next section is devoted to giving some notation preparation and providing some results of FEM for the problem (1.1)-(1.5). In Section 3, some

LPFEIAs with respect to different physical parameters are proposed, and the stability and error bounds of each algorithm are proved. In Section 4, a series of numerical results are shown to validate our theoretical analysis.

2. A FE Scheme for Stationary MHD Equations

This section is devoted to giving the notation and some results of the FE solution to the MHD equations (1.1)-(1.5). Let

$$\mathbf{H}_0^1(\Omega) = \{\mathbf{v}|\mathbf{v} \in \mathbf{H}^1(\Omega), \mathbf{v}|_{\partial\Omega} = \mathbf{0}\}, \quad \mathbf{H}_0(\mathrm{curl}, \Omega) = \{\mathbf{C}|\mathbf{C} \in \mathbf{H}(\mathrm{curl}, \Omega), \mathbf{C} \times \mathbf{n}|_{\partial\Omega} = 0\}$$

and

$$L_0^2(\Omega) = \left\{ q \in L^2(\Omega) : \int_\Omega q d\mathbf{x} = 0 \right\}$$

be the Sobolev spaces, $\|\cdot\|_{0,\Omega}$ be the L^2 norm, and $\|\cdot\|_{-1,\Omega}$ be the norm of $\mathbf{H}^{-1}(\Omega)$. Denote the graph norms on $\mathbf{H}_0^1(\Omega) \times \mathbf{H}_0(\mathrm{curl}, \Omega)$ and $L_0^2(\Omega) \times H_0^1(\Omega)$ by $\|(\mathbf{u}, \mathbf{B})\|_{E,\Omega} = (\|\nabla \mathbf{u}\|_{0,\Omega}^2 + \|\mathbf{B}\|_{\mathrm{curl}}^2)^{1/2}$, and $\|(p, r)\|_{B,\Omega} = (\|p\|_{0,\Omega}^2 + \|\nabla r\|_{0,\Omega}^2)^{1/2}$, respectively, where $\|\mathbf{B}\|_{\mathrm{curl}} = (\|\mathbf{B}\|_{0,\Omega}^2 + \|\mathrm{curl}\ \mathbf{B}\|_{0,\Omega}^2)^{1/2}$.

A weak form of the MHD system (1.1)-(1.5) is to seek $(\mathbf{u}, \mathbf{B}, p, r) \in \mathbf{H}_0^1(\Omega) \times \mathbf{H}_0(\mathrm{curl}, \Omega) \times L_0^2(\Omega) \times H_0^1(\Omega)$ such that

$$\begin{aligned} & a_0((\mathbf{u}, \mathbf{B}), (\mathbf{v}, \mathbf{C})) + a_1((\mathbf{u}, \mathbf{B}), (\mathbf{u}, \mathbf{B}), (\mathbf{v}, \mathbf{C})) + b(p, r; \mathbf{v}, \mathbf{C}) - b(q, s; \mathbf{u}, \mathbf{B}) \\ = & \mathbf{F}((\mathbf{v}, \mathbf{C})), \quad \forall (\mathbf{v}, \mathbf{C}, q, s) \in \mathbf{H}_0^1(\Omega) \times \mathbf{H}_0(\mathrm{curl}, \Omega) \times L_0^2(\Omega) \times H_0^1(\Omega), \end{aligned} \tag{2.1}$$

where

$$\begin{aligned} & a_0((\mathbf{u}, \mathbf{B}), (\mathbf{v}, \mathbf{C})) = R_e^{-1}(\nabla \mathbf{u}, \nabla \mathbf{v}) + S_c R_{e_m}^{-1}(\mathrm{curl}\ \mathbf{B}, \mathrm{curl}\ \mathbf{C}), \\ & b_p(\mathbf{v}, q) = -(\mathrm{div}\ \mathbf{v}, q), \ \ b_r(s, \mathbf{C}) = -(\nabla s, \mathbf{C}), \\ & a_1((\mathbf{u}, \mathbf{B}), (\mathbf{w}, \mathbf{\Phi}), (\mathbf{v}, \mathbf{C})) \\ = & \frac{1}{2}(\mathbf{u} \cdot \nabla \mathbf{w}, \mathbf{v}) - \frac{1}{2}(\mathbf{u} \cdot \nabla \mathbf{v}, \mathbf{w}) - S_c(\mathrm{curl}\ \mathbf{\Phi} \times \mathbf{B}, \mathbf{v}) + S_c(\mathrm{curl}\ \mathbf{C} \times \mathbf{B}, \mathbf{w}), \\ & b(q, s; \mathbf{v}, \mathbf{C}) = b_p(\mathbf{v}, q) + b_r(s, \mathbf{C}), \quad \mathbf{F}((\mathbf{v}, \mathbf{C})) = < \mathbf{f}, \mathbf{v} > + (\mathbf{g}, \mathbf{C}) \end{aligned}$$

for $\mathbf{u}, \mathbf{w}, \mathbf{v} \in \mathbf{H}_0^1(\Omega), \mathbf{B}, \mathbf{\Phi}, \mathbf{C} \in \mathbf{H}_0(\mathrm{curl}, \Omega), q \in L_0^2(\Omega), s \in H_0^1(\Omega)$. Taking $(\mathbf{v}, \mathbf{C}, q, r) = (\mathbf{0}, \nabla r, 0, 0)$ in (2.1) and noticing that $\nabla \cdot \mathbf{g} = 0$, $\mathrm{curl} \nabla r = \mathbf{0}$, we have $\|\nabla r\|_{0,\Omega} = 0$, which implies that $r \equiv 0$ in Ω by $r|_{\partial\Omega} = 0$.

We recall some properties of $a_0(\cdot, \cdot)$ and $a_1(\cdot, \cdot, \cdot)$ from [9,36]: for $\mathbf{u}, \mathbf{w}, \mathbf{v} \in \mathbf{H}_0^1(\Omega), \mathbf{B}, \mathbf{\Phi}, \mathbf{C} \in \mathbf{H}_0(\mathrm{curl}, \Omega)$, there have

$$a_0((\mathbf{w}, \mathbf{\Phi}), (\mathbf{v}, \mathbf{C})) \le C_{\max} \|(\mathbf{w}, \mathbf{\Phi})\|_{E,\Omega} \|(\mathbf{v}, \mathbf{C})\|_{E,\Omega}, \tag{2.2}$$

$$a_0((\mathbf{w}, \mathbf{\Phi}), (\mathbf{w}, \mathbf{\Phi})) \ge C_{\min} \|(\mathbf{w}, \mathbf{\Phi})\|_{E,\Omega}^2, \tag{2.3}$$

$$a_1((\mathbf{u}, \mathbf{B}), (\mathbf{w}, \mathbf{\Phi}), (\mathbf{w}, \mathbf{\Phi})) = 0, \tag{2.4}$$

$$a_1((\mathbf{u}, \mathbf{B}), (\mathbf{w}, \mathbf{\Phi}), (\mathbf{v}, \mathbf{C})) \le \widehat{N} \|(\mathbf{u}, \mathbf{B})\|_{E,\Omega} \|(\mathbf{w}, \mathbf{\Phi})\|_{E,\Omega} \|(\mathbf{v}, \mathbf{C})\|_{E,\Omega}, \tag{2.5}$$

where $C_{\max} = \max\{R_e^{-1}, S_c R_{e_m}^{-1}\}, C_{\min} = \min\{R_e^{-1}, S_c \lambda_0 R_{e_m}^{-1}\}$, λ_0 and $\widehat{N}$ are positive constants related only to Ω [36, 37]

$$\|\mathbf{B}\|_{\mathrm{curl}}^2 \le \lambda_0^{-1} \|\mathrm{curl} \mathbf{B}\|_{0,\Omega}^2, \quad \forall \mathbf{B} \in \mathbf{H}_0(\mathrm{curl}, \Omega), \tag{2.6}$$

$$\widehat{N} = \sup_{\mathbf{u}, \mathbf{w}, \mathbf{v} \in \mathbf{H}_0^1(\Omega), \mathbf{B}, \mathbf{\Phi}, \mathbf{C} \in \mathbf{H}_0(\mathrm{curl}, \Omega)} \frac{a_1((\mathbf{u}, \mathbf{B}), (\mathbf{w}, \mathbf{\Phi}), (\mathbf{v}, \mathbf{C}))}{\|(\mathbf{u}, \mathbf{B})\|_{E,\Omega} \|(\mathbf{w}, \mathbf{\Phi})\|_{E,\Omega} \|(\mathbf{v}, \mathbf{C})\|_{E,\Omega}}. \tag{2.7}$$

From [9], there has the following inf-sup condition:

$$\sup_{\mathbf{v}\in\mathbf{H}_0^1(\Omega),\mathbf{C}\in\mathbf{H}_0(\mathrm{curl},\Omega)} \frac{b(q,s;\mathbf{v},\mathbf{C})}{\|(\mathbf{v},\mathbf{C})\|_{E,\Omega}} \geq \beta\|(q,s)\|_{B,\Omega}, \quad \forall q\in L_0^2(\Omega), \quad s\in H_0^1(\Omega), \tag{2.8}$$

where $\beta>0$ is a constant related to Ω.

Based on (2.2)-(2.8), some properties of the solution to (2.1) can be derived in [9, 36].

Theorem 2.1. *Assume that* $\mathbf{f}\in\mathbf{H}^{-1}(\Omega),\mathbf{g}\in\mathbf{L}^2(\Omega),$ *and* $\nabla\cdot\mathbf{g}=0$. *Suppose*

$$0<\vartheta=\frac{\widehat{N}\|\mathbf{F}\|_{*,\Omega}}{(C_{\min})^2}<1, \tag{2.9}$$

then the problem (2.1) *has a unique solution* $(\mathbf{u},\mathbf{B},p,r)\in\mathbf{H}_0^1(\Omega)\times\mathbf{H}_0(\mathrm{curl},\Omega)\times L_0^2(\Omega)\times H_0^1(\Omega)$ *satisfying*

$$C_{\min}\|(\mathbf{u},\mathbf{B})\|_{E,\Omega}\leq\|\mathbf{F}\|_{*,\Omega}, \tag{2.10}$$

where $\|\mathbf{F}\|_{*,\Omega}=(\|\mathbf{f}\|_{-1,\Omega}+\|\mathbf{g}\|_{0,\Omega})^{1/2}$.

Let $T_\mu(\Omega)=\{K\}$ $(\mu=H,h)$ be a family of regular and uniform triangulations or tetrahedrons partition of Ω, μ the mesh size. t-th order inf-sup stable Galerkin finite element spaces (for example, Taylor-Hood element $(P_t,P_{t-1}), t\geq 2$, Mini element $(P_1^b,P_1), t=1$) are employed to approximate $\mathbf{u}$ and p, which are denoted by $(\mathbf{X}_0^\mu(\Omega),S_0^\mu(\Omega))$. Denote by $\mathbf{Y}_n^\mu(\Omega)\subset\mathbf{H}_0(\mathrm{curl},\Omega)$ the discrete space for $\mathbf{B}$, which is the l-th order Nédélec edge element space [38]. The FE space for r is defined by $Q_0^\mu(\Omega)=\{s_\mu\in H_0^1(\Omega): s_\mu|_K\in P_{l+1}(K),\forall K\subset T_\mu(\Omega)\}$.

Then the FE scheme of (2.1) is to find $(\mathbf{u}_\mu,\mathbf{B}_\mu,p_\mu,r_\mu)\in\mathbf{X}_0^\mu(\Omega)\times\mathbf{Y}_n^\mu(\Omega)\times S_0^\mu(\Omega)\times Q_0^\mu(\Omega)$ such that

$$\begin{aligned}
&a_0((\mathbf{u}_\mu,\mathbf{B}_\mu),(\mathbf{v},\mathbf{C}))+a_1((\mathbf{u}_\mu,\mathbf{B}_\mu),(\mathbf{u}_\mu,\mathbf{B}_\mu),(\mathbf{v},\mathbf{C}))+b(p_\mu,r_\mu;\mathbf{v},\mathbf{C})=\mathbf{F}((\mathbf{v},\mathbf{C})),\\
&b(q,s;\mathbf{u}_\mu,\mathbf{B}_\mu)=0, \quad \forall(\mathbf{v},\mathbf{C},q,s)\in\mathbf{X}_0^\mu(\Omega)\times\mathbf{Y}_n^\mu(\Omega)\times S_0^\mu(\Omega)\times Q_0^\mu(\Omega).
\end{aligned} \tag{2.11}$$

The existence, uniqueness of the solution to this discrete system, and error estimate results are as follows [9, 36]:

Theorem 2.2. *Suppose* (2.9) *holds. Then the FE scheme* (2.11) *has a unique solution* $(\mathbf{u}_\mu,\mathbf{B}_\mu,p_\mu,r_\mu)\in\mathbf{X}_0^\mu(\Omega)\times\mathbf{Y}_n^\mu(\Omega)\times S_0^\mu(\Omega)\times Q_0^\mu(\Omega)$ *satisfying the following energy estimate:*

$$C_{\min}\|(\mathbf{u}_\mu,\mathbf{B}_\mu)\|_{E,\Omega}\leq\|\mathbf{F}\|_{*,\Omega}. \tag{2.12}$$

Moreover, we assume that the solution of the problem (2.1) $\mathbf{u}\in\mathbf{H}^{1+\gamma}(\Omega),\mathrm{curl}\ \mathbf{B}\in\mathbf{H}^\delta(\Omega)$, $p\in H^\gamma(\Omega),r\in H^{1+\delta}(\Omega)$ $(\gamma,\delta>\frac{1}{2})$ *hold, then*

$$\begin{aligned}
&C_{\min}\|(\mathbf{u}-\mathbf{u}_\mu,\mathbf{B}-\mathbf{B}_\mu)\|_{E,\Omega}+\|(p-p_\mu,r-r_\mu)\|_{B,\Omega}\\
&\leq\kappa\mu^\tau(\|\mathbf{u}\|_{1+\gamma,\Omega}+\|\mathbf{B}\|_{\delta,\Omega}+\|\nabla\times\mathbf{B}\|_{\delta,\Omega}+\|p\|_{\gamma,\Omega}+\|r\|_{1+\delta,\Omega}),
\end{aligned} \tag{2.13}$$

where $\tau=\min\{\gamma,t,\delta,l\}$, $\kappa>0$ *is a constant related to* (Ω,R_e,R_{e_m},S_c).

3. LPFEIAs Based on PUM for Stationary MHD Equations

This section aims to combine three iterations with PUM, we extend the work [34, 35] to the LPFEA related to different physical parameters for the problem (1.1)-(1.5).

Let $T_{H_p}(\Omega)$ be a regular triangulation or tetrahedron whose mesh size is H_p. $T_H(\Omega)$ and $T_h(\Omega)$ are nested regular meshes of Ω with $h < H \le H_p$. H_p is fixed and could be independent of mesh sizes h and H. Denote $D^j = \mathrm{supp}\phi_j \cap \Omega$ by the union of triangules/tetrahedra sharing the common vertex $x_j \in T_{H_p}(\Omega)$, whose Lagrangian basis function is ϕ_j such that $\phi_j(x_i) = \delta_{j,i}$. Let $D^{j,0} = D^j$, extend one layer of its neighbors to obtain

$$D^{j,1} = \bigcup_{x_i \in D^{j,0}} D^i. \tag{3.1}$$

Similarly, we can get the two layers of oversampling

$$D^{j,2} = \bigcup_{x_i \in D^{j,1}} D^i. \tag{3.2}$$

Gradually, we arrive at the k layers of oversampling $D^{j,k}$, who can be viewed as the local subdomain corresponding to x_i for partition of unity. In the extreme, we could extend to the full domain Ω. See Fig. 3.1 for the cases of $k = 0$, $k = 1$ and $k = 2$.

Obviously, $\{D^{j,k}\}$ form an open coverage of Ω and $\{\phi_j\}$ are the corresponding partition of unity function satisfying

$$\mathrm{supp}\phi_j \subset D^{j,k}, \qquad \forall\, 1 \le j \le J, \tag{3.3}$$

$$\sum_j \phi_j \equiv 1, \qquad \forall\, x \in \Omega, \tag{3.4}$$

$$\|\phi_j\|_{L^\infty(\Omega)} \le C_p, \qquad \forall\, 1 \le j \le J, \tag{3.5}$$

$$\|\nabla\phi_j\|_{L^\infty(\Omega)} \le \frac{C_p}{\mathrm{diam}(D^{j,k})} \le C_p H_p^{-1}, \qquad \forall\, 1 \le j \le J. \tag{3.6}$$

Denote by $\mathbf{X}_0^h(D^{j,k}), \mathbf{Y}_n^h(D^{j,k}), S_0^h(D^{j,k}), Q_0^h(D^{j,k})$ the FE spaces on $D^{j,k}$. They can be treated as the restriction of $\mathbf{X}_0^h(\Omega), \mathbf{Y}_n^h(\Omega), S_0^h(\Omega), Q_0^h(\Omega)$ on $D^{i,k}$, which have the same boundary conditions on $\partial D^{j,k} \cap \partial\Omega$ and zero on $\partial D^{j,k}$ located in the interior of Ω.

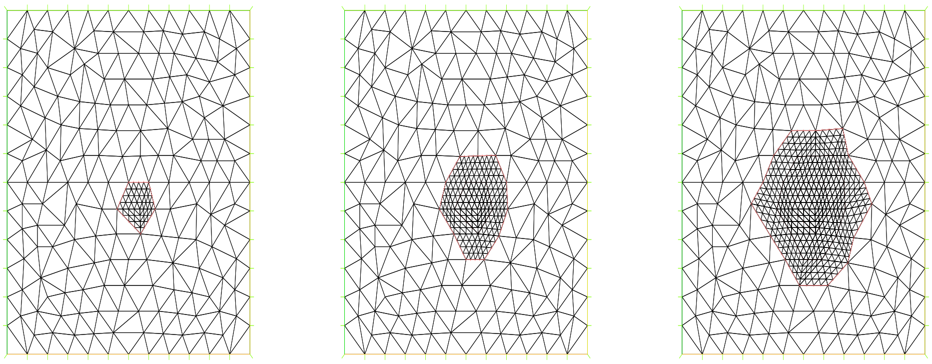

Fig. 3.1. A subdomain: (left) no oversampling $D^{j,0}$, (middle) one layer of oversampling $D^{j,1}$, (right) two layers of oversampling $D^{j,2}$.

Motivated by [28], the LPFEA for MHD equations (1.1)-(1.5) can be designed as follows:

Algorithm 3.1. LPFEA for MHD

Step I. Given a initial solution $(\mathbf{u}_H, \mathbf{B}_H, p_H, r_H) \in \mathbf{X}_0^H(\Omega) \times \mathbf{Y}_n^H(\Omega) \times S_0^H(\Omega) \times Q_0^H(\Omega)$ by

$$\begin{aligned} & a_0((\mathbf{u}_H, \mathbf{B}_H), (\mathbf{v}, \mathbf{C})) + a_1((\mathbf{u}_H, \mathbf{B}_H), (\mathbf{u}_H, \mathbf{B}_H), (\mathbf{v}, \mathbf{C})) + b(p_H, r_H; \mathbf{v}, \mathbf{C}) \\ & \quad -b(q, s; \mathbf{u}_H, \mathbf{B}_H) = \mathbf{F}((\mathbf{v}, \mathbf{C})), \\ & \quad \forall (\mathbf{v}, \mathbf{C}, q, s) \in \mathbf{X}_0^H(\Omega) \times \mathbf{Y}_n^H(\Omega) \times S_0^H(\Omega) \times Q_0^H(\Omega). \end{aligned} \tag{3.7}$$

Step II. Find local correction solutions $(\mathbf{u}_h^j, \mathbf{B}_h^j, p_h^j, r_h^j) \in \mathbf{X}_0^h(D^{j,k}) \times \mathbf{Y}_n^h(D^{j,k}) \times S_0^h(D^{j,k}) \times Q_0^h(D^{j,k}), j = 1, \dots, J$ in parallel by one of the following three correction schemes:

$$a_0((\mathbf{u}_h^j, \mathbf{B}_h^j), (\mathbf{v}, \mathbf{C})) + b(p_h^j, r_h^j; \mathbf{v}, \mathbf{C}) - b(q, s; \mathbf{u}_h^j, \mathbf{B}_h^j) = \mathbf{R}_H((\mathbf{v}, \mathbf{C})), \tag{3.8}$$

$$\begin{aligned} & a_0((\mathbf{u}_h^j, \mathbf{B}_h^j), (\mathbf{v}, \mathbf{C})) + a_1((\mathbf{u}_h^j, \mathbf{B}_h^j), (\mathbf{u}_H, \mathbf{B}_H), (\mathbf{v}, \mathbf{C})) \\ & \quad +a_1((\mathbf{u}_H, \mathbf{B}_H), (\mathbf{u}_h^j, \mathbf{B}_h^j), (\mathbf{v}, \mathbf{C})) + b(p_h^j, r_h^j; \mathbf{v}, \mathbf{C}) \\ & \quad -b(q, s; \mathbf{u}_h^j, \mathbf{B}_h^j) = \mathbf{R}_H((\mathbf{v}, \mathbf{C})), \end{aligned} \tag{3.9}$$

$$\begin{aligned} & a_0((\mathbf{u}_h^j, \mathbf{B}_h^j), (\mathbf{v}, \mathbf{C})) + a_1((\mathbf{u}_H, \mathbf{B}_H), (\mathbf{u}_h^j, \mathbf{B}_h^j), (\mathbf{v}, \mathbf{C})) \\ & \quad +b(p_h^j, r_h^j; \mathbf{v}, \mathbf{C}) - b(q, s; \mathbf{u}_h^j, \mathbf{B}_h^j) = \mathbf{R}_H((\mathbf{v}, \mathbf{C})) \end{aligned} \tag{3.10}$$

for all $(\mathbf{v}, \mathbf{C}, q, s) \in \mathbf{X}_0^h(D^{j,k}) \times \mathbf{Y}_n^h(D^{j,k}) \times S_0^h(D^{j,k}) \times Q_0^h(D^{j,k})$, where

$$\begin{aligned} & \mathbf{R}_H((\mathbf{v}, \mathbf{C})) = \mathbf{F}((\mathbf{v}, \mathbf{C})) - a_0((\mathbf{u}_H, \mathbf{B}_H), (\mathbf{v}, \mathbf{C})) \\ & - a_1((\mathbf{u}_H, \mathbf{B}_H), (\mathbf{u}_H, \mathbf{B}_H), (\mathbf{v}, \mathbf{C})) - b(p_H, r_H; \mathbf{v}, \mathbf{C}) \ \ +b(q, s; \mathbf{u}_H, \mathbf{B}_H). \end{aligned}$$

Step III. Update $(\mathbf{u}^j, \mathbf{B}^j, p^j, r^j) = (\mathbf{u}_H + \mathbf{u}_h^j, \mathbf{B}_H + \mathbf{B}_h^j, p_H + p_h^j, r_H + r_h^j)$ in $D^{j,k}$.

Step IV. Assemble the solution

$$\mathbf{u}^h = \sum_{j=1}^J \phi_j \mathbf{u}^j, \quad \mathbf{B}^h = \sum_{j=1}^J \phi_j \mathbf{B}^j, \quad p^h = \sum_{j=1}^J \phi_j p^j, \quad r^h = \sum_{j=1}^J \phi_j r^j.$$

Algorithm 3.1 is designed based on the double-saddle point scheme presented in [9], which is different from the version based on exact penalty scheme in our previous work [34]. For the $(\mathbf{u}^h, \mathbf{B}^h, p^h, r^h)$ from Algorithm 3.1 by both the three corrections, we can derive their errors between the solution $(\mathbf{u}_h, \mathbf{B}_h, p_h, r_h)$ of (2.11).

Theorem 3.1. *Under the assumptions of Theorem 2.2, then there holds*

$$\|(\mathbf{u}_h - \mathbf{u}^h, \mathbf{B}_h - \mathbf{B}^h)\|_{E,\Omega} + \|(p_h - p^h, r_h - r^h)\|_{B,\Omega} \le C_1 H^{2\tau}, \tag{3.11}$$

where

$$\begin{aligned} & C_1 = \big(C_{R_e, R_{em}, S_c} + \vartheta\big) C_0(\|\mathbf{u}\|_{1+\gamma,\Omega} + \|\mathbf{B}\|_{\delta,\Omega} + \|\nabla \times \mathbf{B}\|_{\delta,\Omega} + \|p\|_{\gamma,\Omega} + \|r\|_{1+\delta,\Omega}), \\ & C_{R_e, R_{em}, S_c} = C_{\max}/C_{\min}. \end{aligned}$$

Proof. For the solutions $(\mathbf{u}^h, \mathbf{B}^h, p^h, r^h)$ obtained from (3.7) with correction schemes (3.9) and (3.10), the proof of (3.11) can be deduced similar to the proof of Theorem 3.2 in [34]. For the solution $(\mathbf{u}^h, \mathbf{B}^h, p^h, r^h)$ obtained from (3.7) with correction scheme (3.8), since the trilinear term $a_1(\cdot,\cdot,\cdot)$ disappears in the left hand of the correction scheme (3.8) compared to the other two correction schemes, (3.11) can be derived naturally. $\square$

Remark 3.1. The constant C_0 appeared in (3.11) depends on the number of the subdomain $D^{j,k}$ $(j = 1,\dots,J)$ and the constant C_p in (3.5) and (3.6). The relationship of dependance can be found in the proofs of Theorem 3.3 in [31] and Theorem 3.2 in [34]. From now on, we assume that J is a fixed constant, which allows the algorithm to be suitable for those parallel architectures with a moderate number of processors.

3.1. Three iterative methods

The three iterative methods based on FEM studied in [8,36] are described by $(\mathbf{u}_H^m, \mathbf{B}_H^m, p_H^m, r_H^m) \in \mathbf{X}_0^H(\Omega) \times \mathbf{Y}_n^H(\Omega) \times S_0^H(\Omega) \times Q_0^H(\Omega)$ by

Iteration I (Stokes-type iteration)

$$\begin{aligned}
&a_0((\mathbf{u}_H^n, \mathbf{B}_H^n), (\mathbf{v}, \mathbf{C})) + a_1((\mathbf{u}_H^{n-1}, \mathbf{B}_H^{n-1}), (\mathbf{u}_H^{n-1}, \mathbf{B}_H^{n-1}), (\mathbf{v}, \mathbf{C})) \\
&\quad + b(p_H^n, r_H^n; \mathbf{v}, \mathbf{C}) = \mathbf{F}((\mathbf{v}, \mathbf{C})), \qquad (3.12)\\
&b(q, s; \mathbf{u}_H^n, \mathbf{B}_H^n) = 0. \qquad (3.13)
\end{aligned}$$

Iteration II (Newton iteration)

$$\begin{aligned}
&a_0((\mathbf{u}_H^n, \mathbf{B}_H^n), (\mathbf{v}, \mathbf{C})) + a_1((\mathbf{u}_H^{n-1}, \mathbf{B}_H^{n-1}), (\mathbf{u}_H^n, \mathbf{B}_H^n), (\mathbf{v}, \mathbf{C})) \\
&\quad + a_1((\mathbf{u}_H^n, \mathbf{B}_H^n), (\mathbf{u}_H^{n-1}, \mathbf{B}_H^{n-1}), (\mathbf{v}, \mathbf{C})) + b(p_H^n, r_H^n; \mathbf{v}, \mathbf{C}) \\
&= \mathbf{F}((\mathbf{v}, \mathbf{C})) + a_1((\mathbf{u}_H^{n-1}, \mathbf{B}_H^{n-1}), (\mathbf{u}_H^{n-1}, \mathbf{B}_H^{n-1}), (\mathbf{v}, \mathbf{C})), \qquad (3.14)\\
&b(q, s; \mathbf{u}_H^n, \mathbf{B}_H^n) = 0. \qquad (3.15)
\end{aligned}$$

Iteration III (Oseen-type iteration)

$$\begin{aligned}
&a_0((\mathbf{u}_H^n, \mathbf{B}_H^n), (\mathbf{v}, \mathbf{C})) + a_1((\mathbf{u}_H^{n-1}, \mathbf{B}_H^{n-1}), (\mathbf{u}_H^n, \mathbf{B}_H^n), (\mathbf{v}, \mathbf{C})) \\
&\quad + b(p_H^n, r_H^n; \mathbf{v}, \mathbf{C}) = \mathbf{F}((\mathbf{v}, \mathbf{C})), \qquad (3.16)\\
&b(q, s; \mathbf{u}_H^n, \mathbf{B}_H^n) = 0 \qquad (3.17)
\end{aligned}$$

for $n = 1,\dots,m$, where $(\mathbf{u}_H^0, \mathbf{B}_H^0, p_H^0, r_H^0)$ is given by

$$a_0((\mathbf{u}_H^0, \mathbf{B}_H^0), (\mathbf{v}, \mathbf{C})) + b(p_H^0, r_H^0; \mathbf{v}, \mathbf{C}) - b(q, s; \mathbf{u}_H^0, \mathbf{B}_H^0) = \mathbf{F}((\mathbf{v}, \mathbf{C})) \qquad (3.18)$$

for all $(\mathbf{v}, \mathbf{C}, q, s) \in \mathbf{X}_0^H(\Omega) \times \mathbf{Y}_n^H(\Omega) \times S_0^H(\Omega) \times Q_0^H(\Omega)$.

$(\mathbf{u}_H^m, \mathbf{B}_H^m, p_H^m, r_H^m)$ got from Iterations I-III are of uniform stability and convergence, which have been proven in [8,36].

Theorem 3.2. *Suppose that* $(\mathbf{u}_H, \mathbf{B}_H, p_H, r_H)$ *is the solution of* (2.11). ϑ *is defined by* (2.9). *Set* $(\mathbf{u}_{ite}^m, \mathbf{B}_{ite}^m) = (\mathbf{u}_H - \mathbf{u}_H^m, \mathbf{B}_H - \mathbf{B}_H^m), (p_{ite}^m, r_{ite}^m) = (p_H - p_H^m, r_H - r_H^m)$. *If* $0 < \vartheta \le \frac{2}{5}$, *then* $(\mathbf{u}_H^m, \mathbf{B}_H^m, p_H^m, r_H^m)$ *derived by Iteration I satisfy*

$$C_{\min}\|(\mathbf{u}_H^m, \mathbf{B}_H^m)\|_{E,\Omega} \le \frac{6}{5}\|\mathbf{F}\|_{*,\Omega}, \ \ \|(p_H^m, r_H^m)\|_{B,\Omega} \le \frac{1}{\beta} C_{R_e,R_{e_m},S_c}\|\mathbf{F}\|_{*,\Omega}, \qquad (3.19)$$

$$C_{\min}\|(\mathbf{u}_{ite}^m, \mathbf{B}_{ite}^m)\|_{E,\Omega} + \|(p_{ite}^m, r_{ite}^m)\|_{B,\Omega} \le C C_{R_e,R_{e_m},S_c}\Big(\frac{11}{5}\vartheta\Big)^m \|\mathbf{F}\|_{*,\Omega}. \qquad (3.20)$$

If $0<\vartheta\le\frac{5}{11}$, *then* $(\mathbf{u}_H^m,\mathbf{B}_H^m,p_H^m,r_H^m)$ *derived by Iteration II satisfy*

$$C_{\min}\|(\mathbf{u}_H^m,\mathbf{b}_H^m)\|_{E,\Omega}\le\frac{4}{3}\|\mathbf{F}\|_{*,\Omega},\ \ \|(p_H^m,r_H^m)\|_{B,\Omega}\le\frac{1}{\beta}C_{R_e,R_{e_m},S_c}\|\mathbf{F}\|_{*,\Omega},\tag{3.21}$$

$$C_{\min}\|(\mathbf{u}_{ite}^m,\mathbf{B}_{ite}^m)\|_{E,\Omega}+\|(p_{ite}^m,r_{ite}^m)\|_{B,\Omega}\le CC_{R_e,R_{e_m},S_c}\Big(\frac{15}{13}\vartheta\Big)^{2^m-1}\|\mathbf{F}\|_{*,\Omega}.\tag{3.22}$$

If $0<\vartheta<1$, *then* $(\mathbf{u}_H^m,\mathbf{B}_H^m,p_H^m,r_H^m)$ *derived by Iteration III satisfy*

$$C_{\min}\|(\mathbf{u}_H^m,\mathbf{b}_H^m)\|_{E,\Omega}\le\|\mathbf{F}\|_{*,\Omega},\ \ \|(p_H^m,r_H^m)\|_{B,\Omega}\le\frac{1}{\beta}C_{R_e,R_{e_m},S_c}\|\mathbf{F}\|_{*,\Omega},\tag{3.23}$$

$$C_{\min}\|(\mathbf{u}_{ite}^m,\mathbf{B}_{ite}^m)\|_{E,\Omega}+\|(p_{ite}^m,r_{ite}^m)\|_{B,\Omega}\le CC_{R_e,R_{e_m},S_c}\vartheta^m\|\mathbf{F}\|_{*,\Omega},\tag{3.24}$$

where $C>0$ *is a general constant which is independent of* H,h *and* m.

Motivated by LPFEA [28] and FE iterative methods with different physical parameters for stationary MHD problem [27], applying the above three iterations to seek an initial solution of the nonlinear problem on a coarse mesh, we present some LPFEIAs related to different physical parameters.

3.2. LPFEIAs based on PUM with $0<\vartheta\le 2/5$

According to Theorem 3.2, when $0<\vartheta\le\frac{2}{5}$, we can choose nine kinds of LPFEIAs by combining the iterative solution $(\mathbf{u}_H^m,\mathbf{B}_H^m,p_H^m,r_H^m)$ from Iterations I, II and III on $T_H(\Omega)$ with three local corrections in parallel on $T_h(D^{j,k}),j=1,\ldots,J$. The LPFEIAs are proposed as follows:

Step I. Find a coarse solution $(\mathbf{u}_H^m,\mathbf{B}_H^m,p_H^m,r_H^m)\in\mathbf{X}_0^H(\Omega)\times\mathbf{Y}_n^H(\Omega)\times S_0^H(\Omega)\times Q_0^H(\Omega)$ by Iterations I, II and III.

Step II. Solve local corrections $(\mathbf{u}_{mh}^j,\mathbf{B}_{mh}^j,p_{mh}^j,r_{mh}^j)\in\mathbf{X}_0^h(D^{j,k})\times\mathbf{Y}_n^h(D^{j,k})\times S_0^h(D^{j,k})\times Q_0^h(D^{j,k})$ $(j=1,\ldots,J)$ in parallel by three corrections

Correction I (Stokes-type correction)

$$a_0((\mathbf{u}_{mh}^j,\mathbf{B}_{mh}^j),(\mathbf{v},\mathbf{C}))+b(p_{mh}^j,r_{mh}^j;\mathbf{v},\mathbf{C})-b(q,s;\mathbf{u}_{mh}^j,\mathbf{B}_{mh}^j)=\mathbf{R}_H^m((\mathbf{v},\mathbf{C})).\tag{3.25}$$

Correction II (Newton correction)

$$\begin{aligned}&a_0((\mathbf{u}_{mh}^j,\mathbf{B}_{mh}^j),(\mathbf{v},\mathbf{C}))+a_1((\mathbf{u}_H^m,\mathbf{B}_H^m),(\mathbf{u}_{mh}^j,\mathbf{B}_{mh}^j),(\mathbf{v},\mathbf{C}))\\&\quad+a_1((\mathbf{u}_{mh}^j,\mathbf{B}_{mh}^j),(\mathbf{u}_H^m,\mathbf{B}_H^m),(\mathbf{v},\mathbf{C}))+b(p_{mh}^j,r_{mh}^j;\mathbf{v},\mathbf{C})\\&\quad-b(q,s;\mathbf{u}_{mh}^j,\mathbf{B}_{mh}^j)=\mathbf{R}_H^m((\mathbf{v},\mathbf{C})).\end{aligned}\tag{3.26}$$

Correction III (Oseen-type correction)

$$\begin{aligned}&a_0((\mathbf{u}_{mh}^j,\mathbf{B}_{mh}^j),(\mathbf{v},\mathbf{C}))+a_1((\mathbf{u}_H^m,\mathbf{B}_H^m),(\mathbf{u}_{mh}^j,\mathbf{B}_{mh}^j),(\mathbf{v},\mathbf{C}))\\&\quad+b(p_{mh}^j,r_{mh}^j;\mathbf{v},\mathbf{C})-b(q,s;\mathbf{u}_{mh}^j,\mathbf{B}_{mh}^j)=\mathbf{R}_H^m((\mathbf{v},\mathbf{C}))\end{aligned}\tag{3.27}$$

for all $(\mathbf{v},\mathbf{C},q,s)\in\mathbf{X}_0^h(D^{j,k})\times\mathbf{Y}_n^h(D^{j,k})\times S_0^h(D^{j,k})\times Q_0^h(D^{j,k})$, where

$$\begin{aligned}\mathbf{R}_H^m((\mathbf{v},\mathbf{C}))=&\ \mathbf{F}((\mathbf{v},\mathbf{C}))-a_0((\mathbf{u}_H^m,\mathbf{B}_H^m),(\mathbf{v},\mathbf{C}))\\&-a_1((\mathbf{u}_H^m,\mathbf{B}_H^m),(\mathbf{u}_H^m,\mathbf{B}_H^m),(\mathbf{v},\mathbf{C}))\\&-b(p_H^m,r_H^m;\mathbf{v},\mathbf{C})+b(q,s;\mathbf{u}_H^m,\mathbf{B}_H^m).\end{aligned}$$

Step III. Set $(\mathbf{u}_m^j, \mathbf{B}_m^j, p_m^j, r_m^j) = (\mathbf{u}_H^m + \mathbf{u}_{mh}^j, \mathbf{B}_H^m + \mathbf{B}_{mh}^j, p_H^m + p_{mh}^j, r_H^m + r_{mh}^j)$ in $D^{j,k}$.

Step IV. Assemble the solution:

$$\mathbf{u}_m^h = \sum_{j=1}^{J} \phi_j \mathbf{u}_m^j, \quad \mathbf{B}_m^h = \sum_{j=1}^{J} \phi_j \mathbf{B}_m^j, \quad p_m^h = \sum_{j=1}^{J} \phi_j p_m^j, \quad r_m^h = \sum_{j=1}^{J} \phi_j r_m^j.$$

The correction schemes (3.25)-(3.27) are linearized by $(\mathbf{u}_H^m, \mathbf{B}_H^m, p_H^m, r_H^m)$ from Step I, while the correction scheme studied in our previous work [35] is employed by Picard's iteration. Compared to the numerical methods for MHD studied in [9, 27, 33], the advantages of the proposed method are that it not only improves efficiency, but also provides a flexible and manageable way to decompose the whole computing domain.

Lemma 3.1. *Suppose that the conditions of Theorem* 2.2 *hold and* $0 < \vartheta \le \frac{2}{5}$. *Then* $(\mathbf{u}_{mh}^j, \mathbf{B}_{mh}^j, p_{mh}^j, r_{mh}^j)$ *obtained by Iterations I, II and III with Corrections I, II and III satisfy*

$$\begin{aligned}
&C_{\min} C_{R_e,R_{e_m},S_c} \|(\mathbf{u}_{mh}^j, \mathbf{B}_{mh}^j)\|_{E,D^{j,k}} + \|(p_{mh}^j, r_{mh}^j)\|_{B,D^{j,k}} \\
\le\ & C(C_{R_e,R_{e_m},S_c})^2 \|\mathbf{F}\|_{*,\Omega}, \qquad (3.28)\\
&C_{\min} \|(\mathbf{u}^j - \mathbf{u}_m^j, \mathbf{B}^j - \mathbf{B}_m^j)\|_{E,D^{j,k}} + \|(p^j - p_m^j, r^j - r_m^j)\|_{B,D^{j,k}} \\
\le\ & C\big(C_{\min} \|(\mathbf{u}_H - \mathbf{u}_H^m, \mathbf{B}_H - \mathbf{B}_H^m)\|_{E,D^{j,k}} + \|(p_H - p_H^m, r_H - r_H^m)\|_{B,D^{j,k}}\big), \qquad (3.29)
\end{aligned}$$

where $(\mathbf{u}^j, \mathbf{B}^j, p^j, r^j)$ *and* $(\mathbf{u}_H, \mathbf{B}_H, p_H, r_H)$ *are from Algorithm* 3.1 *and* $(\mathbf{u}_H^m, \mathbf{B}_H^m, p_H^m, r_H^m)$ *is from Iterations I-III.*

Proof. The proof is divided into two parts.

Part 1. This part is devoted to analyze $(\mathbf{u}_{mh}^j, \mathbf{B}_{mh}^j, p_{mh}^j, r_{mh}^j)$ obtained by Iterations I, II and III with Correction II. Taking $(q, s) = (0, 0)$ in (3.26), and using (2.8), (2.2) and (2.5) give

$$\begin{aligned}
\|(p_{mh}^j, r_{mh}^j)\|_{B,D^{j,k}} \le \frac{1}{\beta} \Big(& C_{\max} \big(\|(\mathbf{u}_{mh}^j, \mathbf{B}_{mh}^j)\|_{E,D^{j,k}} + \|(\mathbf{u}_H^m, \mathbf{B}_H^m)\|_{E,D^{j,k}} \big) \\
& + \widehat{N} \big(2\|(\mathbf{u}_H^m, \mathbf{B}_H^m)\|_{E,D^{j,k}} \|(\mathbf{u}_{mh}^j, \mathbf{B}_{mh}^j)\|_{E,D^{j,k}} + \|(\mathbf{u}_H^m, \mathbf{B}_H^m)\|_{E,D^{j,k}}^2 \big) \\
& + \|\mathbf{F}\|_{*,D^{j,k}} + \sqrt{d} \|(p_H^m, r_H^m)\|_{B,D^{j,k}} \Big). \qquad (3.30)
\end{aligned}$$

Note that $\mathbf{X}_0^h(D^{j,k}) \subset \mathbf{X}_0^h(\Omega), S_0^h(D^{j,k}) \subset S_0^h(\Omega), \mathbf{Y}_n^h(D^{j,k}) \subset \mathbf{Y}_n^h(\Omega)$ and $Q_0^h(D^{j,k}) \subset Q_0^h(\Omega)$. Setting $(\mathbf{v}, \mathbf{C}, q, s) = (\mathbf{u}_{mh}^j, \mathbf{B}_{mh}^j, p_{mh}^j, r_{mh}^j)$ in (3.26), and applying (2.4)-(2.5), we conclude from Theorem 3.2 and $0 < \vartheta \le \frac{2}{5}$ that

$$\begin{aligned}
&\frac{7}{15} C_{\min} \|(\mathbf{u}_{mh}^j, \mathbf{B}_{mh}^j)\|_{E,D^{j,k}}^2 \\
\le\ & \left(1 - \max\left\{\frac{6}{5}, \frac{4}{3}, 1\right\} \vartheta\right) C_{\min} \|(\mathbf{u}_{mh}^j, \mathbf{B}_{mh}^j)\|_{E,D^{j,k}}^2 \\
\le\ & C_{\min} \left(1 - \frac{\widehat{N}}{(C_{\min})^2} \cdot C_{\min} \|(\mathbf{u}_H^m, \mathbf{B}_H^m)\|_{E,D^{j,k}}\right) \|(\mathbf{u}_{mh}^j, \mathbf{B}_{mh}^j)\|_{E,D^{j,k}}^2 \\
\le\ & \big(C_{\min} - \widehat{N} \|(\mathbf{u}_H^m, \mathbf{B}_H^m)\|_{E,D^{j,k}}\big) \|(\mathbf{u}_{mh}^j, \mathbf{B}_{mh}^j)\|_{E,D^{j,k}}^2 \\
\le\ & a_0((\mathbf{u}_{mh}^j, \mathbf{B}_{mh}^j), (\mathbf{u}_{mh}^j, \mathbf{B}_{mh}^j)) + a_1((\mathbf{u}_{mh}^j, \mathbf{B}_{mh}^j), (\mathbf{u}_H^m, \mathbf{B}_H^m), (\mathbf{u}_{mh}^j, \mathbf{B}_{mh}^j)) \\
=\ & \mathbf{R}_H^m((\mathbf{u}_{mh}^j, \mathbf{B}_{mh}^j)). \qquad (3.31)
\end{aligned}$$

Applying (3.30) to (3.31), and using Cauchy-Schwarz inequalities yields

$$
\begin{aligned}
&\frac{7}{15}C_{\min}\|(\mathbf{u}^j_{mh},\mathbf{B}^j_{mh})\|^2_{E,D^{j,k}}\\
\le\ &\|\mathbf{F}\|_{*,D^{j,k}}\|(\mathbf{u}^j_{mh},\mathbf{B}^j_{mh})\|_{E,D^{j,k}}+C_{\max}\|(\mathbf{u}^m_H,\mathbf{B}^m_H)\|_{E,D^{j,k}}\|(\mathbf{u}^j_{mh},\mathbf{B}^j_{mh})\|_{E,D^{j,k}}\\
&\quad+\widehat{N}\|(\mathbf{u}^m_H,\mathbf{B}^m_H)\|^2_{E,D^{j,k}}\|(\mathbf{u}^j_{mh},\mathbf{B}^j_{mh})\|_{E,D^{j,k}}\\
&\quad+\sqrt{d}\|(p^m_H,r^m_H)\|_{B,D^{j,k}}\|(\mathbf{u}^j_{mh},\mathbf{B}^j_{mh})\|_{E,D^{j,k}}+\sqrt{d}\|(\mathbf{u}^m_H,\mathbf{B}^m_H)\|_{E,D^{j,k}}\|(p^j_{mh},r^j_{mh})\|_{B,D^{j,k}}\\
\le\ &\frac{15}{4C_{\min}}\|\mathbf{F}\|^2_{*,D^{j,k}}+\frac{1}{15}C_{\min}\|(\mathbf{u}^j_{mh},\mathbf{B}^j_{mh})\|^2_{E,D^{j,k}}\\
&\quad+\frac{15(C_{\max})^2}{4C_{\min}}\|(\mathbf{u}^m_H,\mathbf{B}^m_H)\|^2_{E,D^{j,k}}+\frac{1}{15}C_{\min}\|(\mathbf{u}^j_{mh},\mathbf{B}^j_{mh})\|^2_{E,D^{j,k}}\\
&\quad+\frac{15(\widehat{N})^2}{4C_{\min}}\|(\mathbf{u}^m_H,\mathbf{B}^m_H)\|^4_{E,D^{j,k}}+\frac{1}{15}C_{\min}\|(\mathbf{u}^j_{mh},\mathbf{B}^j_{mh})\|^2_{E,D^{j,k}}\\
&\quad+\frac{15d}{4C_{\min}}\|(p^m_H,r^m_H)\|^2_{B,D^{j,k}}+\frac{1}{15}C_{\min}\|(\mathbf{u}^j_{mh},\mathbf{B}^j_{mh})\|^2_{E,D^{j,k}}\\
&\quad+\frac{15d(C_{\max})^2}{4\beta^2C_{\min}}\|(\mathbf{u}^m_H,\mathbf{B}^m_H)\|^2_{E,D^{j,k}}+\frac{1}{15}C_{\min}\|(\mathbf{u}^j_{mh},\mathbf{B}^j_{mh})\|^2_{E,D^{j,k}}\\
&\quad+\frac{15d(\widehat{N})^2}{4\beta^2C_{\min}}\|(\mathbf{u}^m_H,\mathbf{B}^m_H)\|^4_{E,D^{j,k}}+\frac{1}{15}C_{\min}\|(\mathbf{u}^j_{mh},\mathbf{B}^j_{mh})\|^2_{E,D^{j,k}}\\
&\quad+\frac{1}{2C_{\min}}\|\mathbf{F}\|^2_{*,D^{j,k}}+\frac{d}{2\beta^2}C_{\min}\|(\mathbf{u}^m_H,\mathbf{B}^m_H)\|^2_{E,D^{j,k}}\\
&\quad+\frac{\sqrt{d}}{\beta}C_{\max}\|(\mathbf{u}^m_H,\mathbf{B}^m_H)\|^2_{E,D^{j,k}}+\frac{\sqrt{d}}{\beta}\widehat{N}\|(\mathbf{u}^m_H,\mathbf{B}^m_H)\|^3_{E,D^{j,k}}\\
&\quad+\frac{d}{4\beta^2}C_{\min}\|(\mathbf{u}^m_H,\mathbf{B}^m_H)\|^2_{E,D^{j,k}}+\frac{d}{C_{\min}}\|(p^m_H,r^m_H)\|^2_{B,D^{j,k}},
\end{aligned}
\tag{3.32}
$$

which can be rearranged as

$$
\begin{aligned}
&C_{\min}\|(\mathbf{u}^j_{mh},\mathbf{B}^j_{mh})\|^2_{E,D^{j,k}}\\
\le\ &C\bigg(\frac{1}{C_{\min}}\|\mathbf{F}\|^2_{*,D^{j,k}}+C_{R_e,R_{e_m},S_c}(C_{R_e,R_{e_m},S_c}+1)C_{\min}\|(\mathbf{u}^m_H,\mathbf{B}^m_H)\|^2_{E,D^{j,k}}\\
&\quad+\widehat{N}\|(\mathbf{u}^m_H,\mathbf{B}^m_H)\|^3_{E,D^{j,k}}+\frac{(\widehat{N})^2}{C_{\min}}\|(\mathbf{u}^m_H,\mathbf{B}^m_H)\|^4_{E,D^{j,k}}+\frac{1}{C_{\min}}\|(p^m_H,r^m_H)\|^2_{B,D^{j,k}}\bigg).
\end{aligned}
\tag{3.33}
$$

With the help of Theorem 3.2 and $0<\vartheta\le\frac{2}{5}$, it follows that

$$
\begin{aligned}
&C_{\min}\|(\mathbf{u}^j_{mh},\mathbf{B}^j_{mh})\|_{E,D^{j,k}}\\
\le\ &C\Big(\|\mathbf{F}\|_{*,D^{j,k}}+C_{R_e,R_{e_m},S_c}C_{\min}\|(\mathbf{u}^m_H,\mathbf{B}^m_H)\|_{E,D^{j,k}}+(\widehat{N}C_{\min})^{\frac12}\|(\mathbf{u}^m_H,\mathbf{B}^m_H)\|^{\frac32}_{E,D^{j,k}}\\
&\quad+\widehat{N}\|(\mathbf{u}^m_H,\mathbf{B}^m_H)\|^2_{E,D^{j,k}}+\|(p^m_H,r^m_H)\|_{B,D^{j,k}}\Big)\\
\le\ &C(C_{R_e,R_{e_m},S_c}+\vartheta+\vartheta^{\frac12})\|\mathbf{F}\|_{*,\Omega}\le CC_{R_e,R_{e_m},S_c}\|\mathbf{F}\|_{*,\Omega}.
\end{aligned}
\tag{3.34}
$$

An application of (3.30), (3.34) and Theorem 3.2 implies that

$$
\begin{aligned}
\|(p^j_{mh},r^j_{mh})\|_{B,D^{j,k}}\le\ &\frac{1}{\beta}\Big(C_{R_e,R_{e_m},S_c}(C_{R_e,R_{e_m},S_c}+1)\|\mathbf{F}\|_{*,\Omega}\\
&\quad+\vartheta\big(2C_{\min}\|(\mathbf{u}^j_{mh},\mathbf{B}^j_{mh})\|_{E,D^{j,k}}+\|(\mathbf{u}^m_H,\mathbf{B}^m_H)\|_{E,D^{j,k}}\big)\Big)\\
\le\ &C(C_{R_e,R_{e_m},S_c})^2\|\mathbf{F}\|_{*,D^{j,k}}.
\end{aligned}
\tag{3.35}
$$

Then, (3.28) follows by combining (3.34) and (3.35).

On the other hand, reorganizing the forms of Algorithm 3.1 with correction (3.9) and Correction II, we have

$$\begin{aligned} a_0((\mathbf{u}_H+\mathbf{u}_h^j,\mathbf{B}_H+\mathbf{B}_h^j),(\mathbf{v},\mathbf{C}))&+a_1((\mathbf{u}_H,\mathbf{B}_H),(\mathbf{u}_H+\mathbf{u}_h^j,\mathbf{B}_H+\mathbf{B}_h^j),(\mathbf{v},\mathbf{C}))\\ &+a_1((\mathbf{u}_H+\mathbf{u}_h^j,\mathbf{B}_H+\mathbf{B}_h^j),(\mathbf{u}_H,\mathbf{B}_H),(\mathbf{v},\mathbf{C}))-a_1((\mathbf{u}_H,\mathbf{B}_H),(\mathbf{u}_H,\mathbf{B}_H),(\mathbf{v},\mathbf{C}))\\ &-b(p_H+p_h^j,r_H+r_h^j;\mathbf{v},\mathbf{C})+b(q,s;\mathbf{u}_H+\mathbf{u}_h^j,\mathbf{B}_H+\mathbf{B}_h^j)=\mathbf{F}((\mathbf{v},\mathbf{C})), \end{aligned}\tag{3.36}$$

$$\begin{aligned} a_0((\mathbf{u}_H^m+\mathbf{u}_{mh}^j,\mathbf{B}_H^m+\mathbf{B}_{mh}^j),(\mathbf{v},\mathbf{C}))&+a_1((\mathbf{u}_H^m,\mathbf{B}_H^m),(\mathbf{u}_H^m+\mathbf{u}_{mh}^j,\mathbf{B}_H^m+\mathbf{B}_{mh}^j),(\mathbf{v},\mathbf{C}))\\ &+a_1((\mathbf{u}_H^m+\mathbf{u}_{mh}^j,\mathbf{B}_H^m+\mathbf{B}_{mh}^j),(\mathbf{u}_H^m,\mathbf{B}_H^m),(\mathbf{v},\mathbf{C}))-a_1((\mathbf{u}_H^m,\mathbf{B}_H^m),(\mathbf{u}_H^m,\mathbf{B}_H^m),(\mathbf{v},\mathbf{C}))\\ &-b(p_H^m+p_{mh}^j,r_H^m+r_{mh}^j;\mathbf{v},\mathbf{C})+b(q,s;\mathbf{u}_H^m+\mathbf{u}_{mh}^j,\mathbf{B}_H^m+\mathbf{B}_{mh}^j)=\mathbf{F}((\mathbf{v},\mathbf{C})) \end{aligned}\tag{3.37}$$

for all $(\mathbf{v},\mathbf{C},q,s)\in\mathbf{X}_0^h(D^{j,k})\times\mathbf{Y}_n^h(D^{j,k})\times S_0^h(D^{j,k})\times Q_0^h(D^{j,k})$. Setting $(\widetilde{\mathbf{e}}^j,\widetilde{\mathbf{H}}^j,\widetilde{\eta}^j,\widetilde{\xi}^j)\doteq(\mathbf{u}_h^j-\mathbf{u}_{mh}^j,\mathbf{B}_h^j-\mathbf{B}_{mh}^j,p_h^j-p_{mh}^j,r_h^j-r_{mh}^j)$ and subtracting (3.37) from (3.36) yield

$$\begin{aligned} &a_0((\mathbf{u}_H-\mathbf{u}_H^m+\widetilde{\mathbf{e}}^j,\mathbf{B}_H-\mathbf{B}_H^m+\widetilde{\mathbf{H}}^j),(\mathbf{v},\mathbf{C}))\\ &\quad+a_1((\mathbf{u}_H,\mathbf{B}_H),(\mathbf{u}_H-\mathbf{u}_H^m+\widetilde{\mathbf{e}}^j,\mathbf{B}_H-\mathbf{B}_H^m+\widetilde{\mathbf{H}}^j),(\mathbf{v},\mathbf{C}))\\ &\quad+a_1((\mathbf{u}_H-\mathbf{u}_H^m+\widetilde{\mathbf{e}}^j,\mathbf{B}_H-\mathbf{B}_H^m+\widetilde{\mathbf{H}}^j),(\mathbf{u}_H,\mathbf{B}_H),(\mathbf{v},\mathbf{C}))\\ &\quad+b(q,s;\mathbf{u}_H-\mathbf{u}_H^m+\widetilde{\mathbf{e}}^j,\mathbf{B}_H-\mathbf{B}_H^m+\widetilde{\mathbf{H}}^j)-b(\widetilde{\eta}^j,\widetilde{\xi}^j;\mathbf{v},\mathbf{C})\\ =\;&\mathbf{G}((\mathbf{v},\mathbf{C})),\quad\forall(\mathbf{v},\mathbf{C},q,s)\in\mathbf{X}_0^h(D^{j,k})\times\mathbf{Y}_n^h(D^{j,k})\times S_0^h(D^{j,k})\times Q_0^h(D^{j,k}), \end{aligned}\tag{3.38}$$

where

$$\begin{aligned} \mathbf{G}((\mathbf{v},\mathbf{C}))=\;&b(p_H-p_H^m,r_H-r_H^m;\mathbf{v},\mathbf{C})\\ &-a_1((\mathbf{u}_H-\mathbf{u}_H^m,\mathbf{B}_H-\mathbf{B}_H^m),(\mathbf{u}_H^m+\mathbf{u}_{mh}^j,\mathbf{B}_H^m+\mathbf{B}_{mh}^j),(\mathbf{v},\mathbf{C}))\\ &-a_1((\mathbf{u}_H^m+\mathbf{u}_{mh}^j,\mathbf{B}_H^m+\mathbf{B}_{mh}^j),(\mathbf{u}_H-\mathbf{u}_H^m,\mathbf{B}_H-\mathbf{B}_H^m),(\mathbf{v},\mathbf{C}))\\ &+a_1((\mathbf{u}_H-\mathbf{u}_H^m,\mathbf{B}_H-\mathbf{B}_H^m),(\mathbf{u}_H,\mathbf{B}_H),(\mathbf{v},\mathbf{C}))\\ &+a_1((\mathbf{u}_H^m,\mathbf{B}_H^m),(\mathbf{u}_H-\mathbf{u}_H^m,\mathbf{B}_H-\mathbf{B}_H^m),(\mathbf{v},\mathbf{C})). \end{aligned}$$

Using (2.5), Theorem 3.2 and (3.28) we get

$$\begin{aligned} \|\mathbf{G}\|_{-1,D^{j,k}}&=\sup_{\mathbf{v}\in\mathbf{X}_0^h(D^{j,k}),\mathbf{C}\in\mathbf{Y}_n^h(D^{j,k})}\frac{\mathbf{G}((\mathbf{v},\mathbf{C}))}{\|(\mathbf{v},\mathbf{C})\|_{E,D^{j,k}}}\\ &\le\sqrt{d}\|(p_H-p_H^m,r_H-r_H^m)\|_{B,D^{j,k}}\\ &\quad+\widehat{N}\Big(3\|(\mathbf{u}_H^m,\mathbf{B}_H^m)\|_{E,D^{j,k}}+2\|(\mathbf{u}_{mh}^j,\mathbf{B}_{mh}^j)\|_{E,D^{j,k}}+\|(\mathbf{u}_H,\mathbf{B}_H)\|_{E,D^{j,k}}\Big)\\ &\quad\times\|(\mathbf{u}_H-\mathbf{u}_H^m,\mathbf{B}_H-\mathbf{B}_H^m)\|_{E,D^{j,k}}\\ &\le\sqrt{d}\|(p_H-p_H^m,r_H-r_H^m)\|_{B,D^{j,k}}+C\vartheta\big(1+C_{R_e,R_{e_m},S_c}\big)\\ &\quad\times C_{\min}\|(\mathbf{u}_H-\mathbf{u}_H^m,\mathbf{B}_H-\mathbf{B}_H^m)\|_{E,D^{j,k}}\\ &\le C\Big(\|(p_H-p_H^m,r_H-r_H^m)\|_{B,D^{j,k}}+C_{R_e,R_{e_m},S_c}C_{\min}\|(\mathbf{u}_H-\mathbf{u}_H^m,\mathbf{B}_H-\mathbf{B}_H^m)\|_{E,D^{j,k}}\Big). \end{aligned}\tag{3.39}$$

Thus, proceeding as in the proof of (3.30), we obtain by using (3.38) with $(q,s)=(0,0)$ and (2.12) that

$$\|(\widetilde{\eta}^j,\widetilde{\xi}^j)\|_{B,D^{j,k}}\le\frac{1}{\beta}\Big(C_{\max}\big(\|(\mathbf{u}_H-\mathbf{u}_H^m,\mathbf{B}_H-\mathbf{B}_H^m)\|_{E,D^{j,k}}+\|(\widetilde{\mathbf{e}}^j,\widetilde{\mathbf{H}}^j)\|_{E,D^{j,k}}\big)+\|\mathbf{G}\|_{-1,D^{j,k}}\Big)$$

$$
\begin{aligned}
&+\frac{2\widehat{N}}{\beta}\|(\mathbf{u}_H,\mathbf{B}_H)\|_{E,D^{j,k}}\big(\|(\widetilde{\mathbf{e}}^j,\widetilde{\mathbf{H}}^j)\|_{E,D^{j,k}}+\|(\mathbf{u}_H-\mathbf{u}_H^m,\mathbf{B}_H-\mathbf{B}_H^m)\|_{E,D^{j,k}}\big)\\
&\le C\Big(\|(p_H-p_H^m,r_H-r_H^m)\|_{B,D^{j,k}}+(\vartheta+C_{R_e,R_{e_m},S_c})C_{\min}\big(\|(\widetilde{\mathbf{e}}^j,\widetilde{\mathbf{H}}^j)\|_{E,D^{j,k}}\\
&\qquad+\|(\mathbf{u}_H-\mathbf{u}_H^m,\mathbf{B}_H-\mathbf{B}_H^m)\|_{E,D^{j,k}}\big)\Big). \qquad (3.40)
\end{aligned}
$$

Taking $(\mathbf{v},\mathbf{C},q,s)=(\widetilde{\mathbf{e}}^j,\widetilde{\mathbf{H}}^j,\widetilde{\eta}^j,\widetilde{\xi}^j)$ in (3.38), and combining with the fact that $(\mathbf{s},\mathbf{s}-\mathbf{t})=\frac{1}{2}(|\mathbf{s}|^2-|\mathbf{t}|^2+|\mathbf{s}-\mathbf{t}|^2)$ with $\mathbf{s}=\mathbf{u}_H-\mathbf{u}_H^m+\widetilde{\mathbf{e}}^j$ (or $\mathbf{B}_H-\mathbf{B}_H^m+\widetilde{\mathbf{H}}^j$), $\mathbf{t}=\mathbf{u}_H-\mathbf{u}_H^m$(or $\mathbf{B}_H-\mathbf{B}_H^m$), we conclude from (2.6) that

$$
\begin{aligned}
&a_0((\mathbf{u}_H-\mathbf{u}_H^m+\widetilde{\mathbf{e}}^j,\mathbf{B}_H-\mathbf{B}_H^m+\widetilde{\mathbf{H}}^j),(\widetilde{\mathbf{e}}^j,\widetilde{\mathbf{H}}^j))\\
&=R_e^{-1}(\nabla(\mathbf{u}_H-\mathbf{u}_H^m+\widetilde{\mathbf{e}}^j),\nabla\widetilde{\mathbf{e}}^j)+S_cR_{e_m}^{-1}(\nabla\times(\mathbf{B}_H-\mathbf{B}_H^m+\widetilde{\mathbf{H}}^j),\nabla\times\widetilde{\mathbf{H}}^j)\\
&=\frac{1}{2}R_e^{-1}\big(\|\nabla(\mathbf{u}_H-\mathbf{u}_H^m+\widetilde{\mathbf{e}}^j)\|_{0,D^{j,k}}^2+\|\nabla\widetilde{\mathbf{e}}^j\|_{0,D^{j,k}}^2-\|\nabla(\mathbf{u}_H-\mathbf{u}_H^m)\|_{0,D^{j,k}}^2\big)\\
&\qquad+\frac{1}{2}S_cR_{e_m}^{-1}\big(\|\nabla\times(\mathbf{B}_H-\mathbf{B}_H^m+\widetilde{\mathbf{H}}^j)\|_{0,D^{j,k}}^2+\|\nabla\times\widetilde{\mathbf{H}}^j\|_{0,D^{j,k}}^2-\|\nabla\times(\mathbf{B}_H-\mathbf{B}_H^m)\|_{0,D^{j,k}}^2\big)\\
&\ge\frac{1}{2}C_{\min}\big(\|(\mathbf{u}^j-\mathbf{u}_m^j,\mathbf{B}^j-\mathbf{B}_m^j)\|_{E,D^{j,k}}^2+\|(\widetilde{\mathbf{e}}^j,\widetilde{\mathbf{H}}^j)\|_{E,D^{j,k}}^2\big)\\
&\qquad-\frac{1}{2}\big(R_e^{-1}\|\nabla(\mathbf{u}_H-\mathbf{u}_H^m)\|_{0,D^{j,k}}^2+S_cR_{e_m}^{-1}\|\nabla\times(\mathbf{B}_H-\mathbf{B}_H^m)\|_{0,D^{j,k}}^2\big). \qquad (3.41)
\end{aligned}
$$

Therefore, using (3.38), (3.41), (2.4), (2.5) and (2.12) leads to

$$
\begin{aligned}
&\frac{1}{2}C_{\min}\|(\mathbf{u}^j-\mathbf{u}_m^j,\mathbf{B}^j-\mathbf{B}_m^j)\|_{E,D^{j,k}}^2+\left(\frac{1}{2}-\vartheta\right)C_{\min}\|(\widetilde{\mathbf{e}}^j,\widetilde{\mathbf{H}}^j)\|_{E,D^{j,k}}^2\\
&\qquad+a_1((\mathbf{u}_H,\mathbf{B}_H),(\mathbf{u}_H-\mathbf{u}_H^m,\mathbf{B}_H-\mathbf{B}_H^m),(\widetilde{\mathbf{e}}^j,\widetilde{\mathbf{H}}^j))\\
&\qquad+a_1((\mathbf{u}_H-\mathbf{u}_H^m,\mathbf{B}_H-\mathbf{B}_H^m),(\mathbf{u}_H,\mathbf{B}_H),(\widetilde{\mathbf{e}}^j,\widetilde{\mathbf{H}}^j))\\
&\le\frac{1}{2}C_{\min}\|(\mathbf{u}^j-\mathbf{u}_m^j,\mathbf{B}^j-\mathbf{B}_m^j)\|_{E,D^{j,k}}^2+\frac{1}{2}C_{\min}\|(\widetilde{\mathbf{e}}^j,\widetilde{\mathbf{H}}^j)\|_{E,D^{j,k}}^2\\
&\qquad-\widehat{N}\|(\mathbf{u}_H,\mathbf{B}_H)\|_{E,D^{j,k}}\|(\widetilde{\mathbf{e}}^j,\widetilde{\mathbf{H}}^j)\|_{E,D^{j,k}}^2\\
&\qquad+a_1((\mathbf{u}_H-\mathbf{u}_H^m,\mathbf{B}_H-\mathbf{B}_H^m),(\mathbf{u}_H,\mathbf{B}_H),(\widetilde{\mathbf{e}}^j,\widetilde{\mathbf{H}}^j))\\
&\qquad+a_1((\mathbf{u}_H,\mathbf{B}_H),(\mathbf{u}_H-\mathbf{u}_H^m,\mathbf{B}_H-\mathbf{B}_H^m),(\widetilde{\mathbf{e}}^j,\widetilde{\mathbf{H}}^j))\\
&\le\frac{1}{2}\big(R_e^{-1}\|\nabla(\mathbf{u}_H-\mathbf{u}_H^m)\|_{0,D^{j,k}}^2+S_cR_{e_m}^{-1}\|\nabla\times(\mathbf{B}_H-\mathbf{B}_H^m)\|_{0,D^{j,k}}^2\big)\\
&\qquad-b(\widetilde{\eta}^j,\widetilde{\xi}^j;\mathbf{u}_H-\mathbf{u}_H^m,\mathbf{B}_H-\mathbf{B}_H^m)+\mathbf{G}((\widetilde{\mathbf{e}}^j,\widetilde{\mathbf{H}}^j)). \qquad (3.42)
\end{aligned}
$$

Due to (2.5), (2.12), (2.9), Young's inequality and (3.40), we have

$$
\begin{aligned}
&|a_1((\mathbf{u}_H,\mathbf{B}_H),(\mathbf{u}_H-\mathbf{u}_H^m,\mathbf{B}_H-\mathbf{B}_H^m),(\widetilde{\mathbf{e}}^j,\widetilde{\mathbf{H}}^j))|\\
&\qquad+|a_1((\mathbf{u}_H-\mathbf{u}_H^m,\mathbf{B}_H-\mathbf{B}_H^m),(\mathbf{u}_H,\mathbf{B}_H),(\widetilde{\mathbf{e}}^j,\widetilde{\mathbf{H}}^j))|\\
&\le 2\widehat{N}\|(\mathbf{u}_H,\mathbf{B}_H)\|_{E,D^{j,k}}\|(\mathbf{u}_H-\mathbf{u}_H^m,\mathbf{B}_H-\mathbf{B}_H^m)\|_{E,D^{j,k}}\|(\widetilde{\mathbf{e}}^j,\widetilde{\mathbf{H}}^j)\|_{E,D^{j,k}}\\
&\le 2\vartheta C_{\min}\|(\mathbf{u}_H-\mathbf{u}_H^m,\mathbf{B}_H-\mathbf{B}_H^m)\|_{E,D^{j,k}}\|(\widetilde{\mathbf{e}}^j,\widetilde{\mathbf{H}}^j)\|_{E,D^{j,k}}\\
&\le\frac{1}{60}C_{\min}\|(\widetilde{\mathbf{e}}^j,\widetilde{\mathbf{H}}^j)\|_{E,D^{j,k}}^2+60C_{\min}\|(\mathbf{u}_H-\mathbf{u}_H^m,\mathbf{B}_H-\mathbf{B}_H^m)\|_{E,D^{j,k}}^2, \qquad (3.43)\\
&|b(\widetilde{\eta}^j,\widetilde{\xi}^j;\mathbf{u}_H-\mathbf{u}_H^m,\mathbf{B}_H-\mathbf{B}_H^m)|\\
&\le\sqrt{d}\|(\mathbf{u}_H-\mathbf{u}_H^m,\mathbf{B}_H-\mathbf{B}_H^m)\|_{E,D^{j,k}}\|(\widetilde{\eta}^j,\widetilde{\xi}^j)\|_{B,D^{j,k}}
\end{aligned}
$$

$$
\begin{aligned}
&\le CC_{R_e,R_{e_m},S_c}C_{\min}\|(\mathbf{u}_H-\mathbf{u}_H^m,\mathbf{B}_H-\mathbf{B}_H^m)\|_{E,D^{j,k}}^2\\
&\quad +CC_{R_e,R_{e_m},S_c}C_{\min}\|(\mathbf{u}_H-\mathbf{u}_H^m,\mathbf{B}_H-\mathbf{B}_H^m)\|_{E,D^{j,k}}\|(\widetilde{\mathbf{e}}^j,\widetilde{\mathbf{H}}^j)\|_{E,D^{j,k}}\\
&\quad +C\|(\mathbf{u}_H-\mathbf{u}_H^m,\mathbf{B}_H-\mathbf{B}_H^m)\|_{E,D^{j,k}}\|(p_H-p_H^m,r_H-r_H^m)\|_{B,D^{j,k}}\\
&\le \frac{1}{60}C_{\min}\|(\widetilde{\mathbf{e}}^j,\widetilde{\mathbf{H}}^j)\|_{E,D^{j,k}}^2+C\Big((C_{R_e,R_{e_m},S_c})^2C_{\min}\|(\mathbf{u}_H-\mathbf{u}_H^m,\mathbf{B}_H-\mathbf{B}_H^m)\|_{E,D^{j,k}}^2\\
&\qquad +\frac{1}{C_{\min}}\|(p_H-p_H^m,r_H-r_H^m)\|_{B,D^{j,k}}^2\Big),
\end{aligned}
\tag{3.44}
$$

$$
|\mathbf{G}((\widetilde{\mathbf{e}}^j,\widetilde{\mathbf{H}}^j))|\le \frac{1}{60}C_{\min}\|(\widetilde{\mathbf{e}}^j,\widetilde{\mathbf{H}}^j)\|_{E,D^{j,k}}^2+\frac{15}{C_{\min}}\|\mathbf{G}\|_{-1,D^{j,k}}^2. \tag{3.45}
$$

Substituting (3.43)-(3.45) into (3.42) and tidying up the resultants, then employing (3.39) and $0<\vartheta\le\frac{2}{5}$, we deduce that

$$
\begin{aligned}
&C_{\min}\big(\|(\mathbf{u}^j-\mathbf{u}_m^j,\mathbf{B}^j-\mathbf{B}_m^j)\|_{E,D^{j,k}}+\|(\widetilde{\mathbf{e}}^j,\widetilde{\mathbf{H}}^j)\|_{E,D^{j,k}}\big)\\
&\le C\big(C_{R_e,R_{e_m},S_c}C_{\min}\|(\mathbf{u}_H-\mathbf{u}_H^m,\mathbf{B}_H-\mathbf{B}_H^m)\|_{E,D^{j,k}}+\|(p_H-p_H^m,r_H-r_H^m)\|_{B,D^{j,k}}\big).
\end{aligned}
\tag{3.46}
$$

It follows from (3.40), (3.46) and triangular inequality that

$$
\begin{aligned}
&\|(p^j-p_m^j,r^j-r_m^j)\|_{B,D^{j,k}}\\
&\le \|(p_H-p_H^m,r_H-r_H^m)\|_{B,D^{j,k}}+\|(\widetilde{\eta}^j,\widetilde{\xi}^j)\|_{B,D^{j,k}}\\
&\le CC_{R_e,R_{e_m},S_c}\big(C_{R_e,R_{e_m},S_c}C_{\min}\|(\mathbf{u}_H-\mathbf{u}_H^m,\mathbf{B}_H-\mathbf{B}_H^m)\|_{E,D^{j,k}}\\
&\qquad +\|(p_H-p_H^m,r_H-r_H^m)\|_{B,D^{j,k}}\big).
\end{aligned}
\tag{3.47}
$$

So, we can finish the proof of Part 1 by combining (3.46) with (3.47).

Part 2. This part is for the case of Iterations I, II and III with Corrections I and III. For the case of $(\mathbf{u}_{mh}^j,\mathbf{B}_{mh}^j,p_{mh}^j,r_{mh}^j)$ obtained by Iterations I, II and III with Correction III, to get (3.28) and (3.29), we just need to change the forms of (3.36) and (3.37) according to Algorithm 3.1 with correction (3.10) and Correction III, respectively. The rest of the proof is similar to the above procedure.

From Algorithm 3.1 with correction (3.8) and Correction I, we have

$$
\begin{aligned}
&a_0((\mathbf{u}_H+\mathbf{u}_h^j,\mathbf{B}_H+\mathbf{B}_h^j),(\mathbf{v},\mathbf{C}))+a_1((\mathbf{u}_H,\mathbf{B}_H),(\mathbf{u}_H,\mathbf{B}_H),(\mathbf{v},\mathbf{C}))\\
&\quad -b(p_H+p_h^j,r_H+r_h^j;\mathbf{v},\mathbf{C})+b(q,s;\mathbf{u}_H+\mathbf{u}_h^j,\mathbf{B}_H+\mathbf{B}_h^j)=\mathbf{F}((\mathbf{v},\mathbf{C})),
\end{aligned}
\tag{3.48}
$$

$$
\begin{aligned}
&a_0((\mathbf{u}_H^m+\mathbf{u}_{mh}^j,\mathbf{B}_H^m+\mathbf{B}_{mh}^j),(\mathbf{v},\mathbf{C}))+a_1((\mathbf{u}_H^m,\mathbf{B}_H^m),(\mathbf{u}_H^m,\mathbf{B}_H^m),(\mathbf{v},\mathbf{C}))\\
&\quad -b(p_H^m+p_{mh}^j,r_H^m+r_{mh}^j;\mathbf{v},\mathbf{C})+b(q,s;\mathbf{u}_H^m+\mathbf{u}_{mh}^j,\mathbf{B}_H^m+\mathbf{B}_{mh}^j)=\mathbf{F}((\mathbf{v},\mathbf{C}))
\end{aligned}
\tag{3.49}
$$

for all $(\mathbf{v},\mathbf{C},q,s)\in\mathbf{X}_0^h(D^{j,k})\times\mathbf{Y}_n^h(D^{j,k})\times S_0^h(D^{j,k})\times Q_0^h(D^{j,k})$. In this case, two trilinear terms disappear in (3.48) and (3.49) respectively compared to (3.36) and (3.37). Therefore, the results of (3.28) and (3.29) for the case of $(\mathbf{u}_{mh}^j,\mathbf{B}_{mh}^j,p_{mh}^j,r_{mh}^j)$ obtained by Iterations I, II and III with Correction I can be regarded as a natural result of Part 1. This ends the proof.□

Remark 3.2. It is shown in (3.42) that if $0<\vartheta<\frac{1}{2}$, $(\mathbf{u}_m^j,\mathbf{B}_m^j,p_m^j,r_m^j)$ provided by the Correction II in Step II is convergent to $(\mathbf{u}^j,\mathbf{B}^j,p^j,r^j)$. From Part 2, we see that the terms $-\widehat{N}\|(\mathbf{u}_H,\mathbf{B}_H)\|_{E,\Omega}\cdot\|(\widetilde{\mathbf{e}}^j,\widetilde{\mathbf{H}}^j)\|_{E,D^{j,k}}^2$ and $-\vartheta C_{\min}\|(\widetilde{\mathbf{e}}^j,\widetilde{\mathbf{H}}^j)\|_{E,D^{j,k}}^2$ vanish in (3.42) and $\frac{1}{2}C_{\min}\cdot\|(\widetilde{\mathbf{e}}^j,\widetilde{\mathbf{H}}^j)\|_{E,D^{j,k}}^2$ stands alone on the left hand. So $(\mathbf{u}_m^j,\mathbf{B}_m^j,p_m^j,r_m^j)$ derived by the Corrections I and III converges to $(\mathbf{u}^j,\mathbf{B}^j,p^j,r^j)$ without the restriction of $0<\vartheta<\frac{1}{2}$. This is also the basis for our discussion in Subsections 3.4 and 3.5.

Theorem 3.3. *Under the conditions of Lemma* 3.1, $(\mathbf{u},\mathbf{B},p,r)$ *and* $(\mathbf{u}_h,\mathbf{B}_h,p_h,r_h)$ *are solutions of* (2.1) *and* (2.11), *respectively. If* $(\mathbf{u}_m^h,\mathbf{B}_m^h,p_m^h,r_m^h)$ *is obtained from Iteration I with Corrections I, II and III, then there holds*

$$\begin{aligned}&\|(\mathbf{u}-\mathbf{u}_m^h,\mathbf{B}-\mathbf{B}_m^h)\|_{E,\Omega}+\|(p-p_m^h,r-r_m^h)\|_{B,\Omega}\\ &\le C\left[h^\tau+H^{2\tau}+C_{R_e,R_{e_m},S_c}\left(\frac{11}{5}\vartheta\right)^m\|\mathbf{F}\|_{*,\Omega}\right].\end{aligned}\tag{3.50}$$

If $(\mathbf{u}_m^h,\mathbf{B}_m^h,p_m^h,r_m^h)$ *is obtained from Iteration II with Corrections I, II and III, then there holds*

$$\begin{aligned}&\|(\mathbf{u}-\mathbf{u}_m^h,\mathbf{B}-\mathbf{B}_m^h)\|_{E,\Omega}+\|(p-p_m^h,r-r_m^h)\|_{B,\Omega}\\ &\le C\left[h^\tau+H^{2\tau}+C_{R_e,R_{e_m},S_c}\left(\frac{15}{13}\vartheta\right)^{2^m-1}\|\mathbf{F}\|_{*,\Omega}\right].\end{aligned}\tag{3.51}$$

If $(\mathbf{u}_m^h,\mathbf{B}_m^h,p_m^h,r_m^h)$ *is obtained from Iteration III with Corrections I, II and III, then there holds*

$$\begin{aligned}&\|(\mathbf{u}-\mathbf{u}_m^h,\mathbf{B}-\mathbf{B}_m^h)\|_{E,\Omega}+\|(p-p_m^h,r-r_m^h)\|_{B,\Omega}\\ &\le C\big(h^\tau+H^{2\tau}+C_{R_e,R_{e_m},S_c}\vartheta^m\|\mathbf{F}\|_{*,\Omega}\big).\end{aligned}\tag{3.52}$$

Proof. We just give the proof of the case of $(\mathbf{u}_m^h,\mathbf{B}_m^h,p_m^h,r_m^h)$ obtained from Iterations I, II and III with Correction II, the other two cases can be proved similarly. For $(\mathbf{u}^h,\mathbf{B}^h,p^h,r^h)$ got by Algorithm 3.1, we have

$$\begin{aligned}&\|(\mathbf{u}^h-\mathbf{u}_m^h,\mathbf{B}^h-\mathbf{B}_m^h)\|_{E,\Omega}+\|(p^h-p_m^h,r^h-r_m^h)\||_{B,\Omega}\\ =&\left\|\sum_{j=1}^J\phi_j(\mathbf{u}^j-\mathbf{u}_m^j,\mathbf{B}^j-\mathbf{B}_m^j)\right\|_{E,\Omega}+\left\|\sum_{j=1}^J\phi_j(p^j-p_m^j,r^j-r_m^j)\right\|_{B,\Omega}\\ \le& C\left(\sum_{j=1}^J\big(\|\phi_j(\mathbf{u}^j-\mathbf{u}_m^j,\mathbf{B}^j-\mathbf{B}_m^j)\|_{E,D^j}^2+\|\phi_j(p^j-p_m^j,r^j-r_m^j)\|_{B,D^j}^2\big)\right)^{\frac12}\\ \le& C\left[\sum_{j=1}^J\Big(C_p^2\big(\|(\mathbf{u}^j-\mathbf{u}_m^j,\mathbf{B}^j-\mathbf{B}_m^j)\|_{E,D^j}^2+\|(p^j-p_m^j,r^j-r_m^j)\|_{B,D^j}^2\big)\right.\\ &\left.+C_p^2H_p^{-2}\big(\|(\mathbf{u}^j-\mathbf{u}_m^j,\mathbf{B}^j-\mathbf{B}_m^j)\|_{0,D^j}^2+\|(p^j-p_m^j,r^j-r_m^j)\|_{0,D^j}^2\big)\Big)\right]^{\frac12}.\end{aligned}\tag{3.53}$$

Using triangular inequality derives

$$\begin{aligned}&\|(\mathbf{u}_h-\mathbf{u}_m^h,\mathbf{B}_h-\mathbf{B}_m^h)\|_{E,\Omega}+\|(p_h-p_m^h,r_h-r_m^h)\|_{B,\Omega}\\ \le&\|(\mathbf{u}_h-\mathbf{u}^h,\mathbf{B}_h-\mathbf{B}^h)\|_{E,\Omega}+\|(p_h-p^h,r_h-r^h)\|_{B,\Omega}\\ &+\|(\mathbf{u}^h-\mathbf{u}_m^h,\mathbf{B}^h-\mathbf{B}_m^h)\|_{E,\Omega}+\|(p^h-p_m^h,r^h-r_m^h)\|_{B,\Omega}.\end{aligned}\tag{3.54}$$

Since H_p is a fixed and independent of H,h, the proof ends with (3.53), (3.54), Lemma 3.1, and Theorems 2.2, 3.12 and 3.2. □

Remark 3.3. From Lemma 3.1, we may see that $(\mathbf{u}_m^h,\mathbf{B}_m^h,p_m^h,r_m^h)$ derived by Iteration i with Correction j $(i,j=I,II,III)$ is stable and convergent when $0<\vartheta\le\frac25$. Furthermore, the LPFEIA consisting of Iteration II and Correction II is suggested to be a preferable way to solve the stationary MHD problem according to Theorems 3.2 and 3.3.

3.3. LPFEIAs based on PUM with $2/5 < \vartheta \leq 5/11$

When $\frac{2}{5} < \vartheta \leq \frac{5}{11}$, Iterations II and III are stable. This subsection is devoted to giving six kinds of LPFEIAs consisting of the iterative solution $(\mathbf{u}_H^m, \mathbf{B}_H^m, p_H^m, r_H^m)$ derived by Iterations II and III on $T_H(\Omega)$ and Corrections I, II and III in parallel on $T_h(D^{j,k}), j = 1, \ldots, J$. The stability and convergence of these algorithms are analyzed. The LPFEIAs are presented as

Step I. Find a coarse solution $(\mathbf{u}_H^m, \mathbf{B}_H^m, p_H^m, r_H^m) \in \mathbf{X}_0^H(\Omega) \times \mathbf{Y}_n^H(\Omega) \times S_0^H(\Omega) \times Q_0^H(\Omega)$ by Iterations II and III.

Step II. Solve local corrections $(\mathbf{u}_{mh}^j, \mathbf{H}_{mh}^j, p_{mh}^j, r_{mh}^j) \in \mathbf{X}_0^h(D^{j,k}) \times \mathbf{Y}_n^h(D^{j,k}) \times S_0^h(D^{j,k}) \times Q_0^h(D^{j,k})$ in parallel by Corrections I, II and III.

Step III. Set $(\mathbf{u}_m^j, \mathbf{B}_m^j, p_m^j, r_m^j) = (\mathbf{u}_H^m + \mathbf{u}_{mh}^j, \mathbf{B}_H^m + \mathbf{B}_{mh}^j, p_H^m + p_{mh}^j, r_H^m + r_{mh}^j)$ in $D^{j,k}$ $(j = 1, \ldots, J)$.

Step IV. Assemble the solution

$$\mathbf{u}_m^h = \sum_{j=1}^{J} \phi_j \mathbf{u}_m^j, \quad \mathbf{B}_m^h = \sum_{j=1}^{J} \phi_j \mathbf{B}_m^j, \quad p_m^h = \sum_{j=1}^{J} \phi_j p_m^j, \quad r_m^h = \sum_{j=1}^{J} \phi_j r_m^j.$$

Lemma 3.2. *Suppose that the conditions of Theorem* 2.2 *hold and* $\frac{2}{5} < \vartheta \leq \frac{5}{11}$. *Then* $(\mathbf{u}_{mh}^j, \mathbf{B}_{mh}^j, p_{mh}^j, r_{mh}^j)$ *derived by Iterations II and III with Corrections I, II and III satisfies*

$$\begin{aligned}
&C_{\min} C_{R_e,R_{e_m},S_c} \|(\mathbf{u}_{mh}^j, \mathbf{B}_{mh}^j)\|_{E,D^{j,k}} + \|(p_{mh}^j, r_{mh}^j)\|_{B,D^{j,k}} \\
&\leq C(C_{R_e,R_{e_m},S_c})^2 \|\mathbf{F}\|_{*,\Omega}, \qquad (3.55) \\
&C_{\min}\|(\mathbf{u}^j - \mathbf{u}_m^j, \mathbf{B}^j - \mathbf{B}_m^j)\|_{E,D^{j,k}} + \|(p^j - p_m^j, r^j - r_m^j)\|_{B,D^{j,k}} \\
&\leq C\big(C_{\min}\|(\mathbf{u}_H - \mathbf{u}_H^m, \mathbf{B}_H - \mathbf{B}_H^m)\|_{E,D^{j,k}} + \|(p_H - p_H^m, r_H - r_H^m)\|_{B,D^{j,k}}\big), \qquad (3.56)
\end{aligned}$$

where $(\mathbf{u}_H, \mathbf{B}_H, p_H, r_H)$ *and* $(\mathbf{u}^j, \mathbf{B}^j, p^j, r^j)$ *are obtained by Algorithm* 3.1, *and* $(\mathbf{u}_H^m, \mathbf{B}_H^m, p_H^m, r_H^m)$ *is from Iterations II-III.*

Proof. For the case of $(\mathbf{u}_{mh}^j, \mathbf{B}_{mh}^j, p_{mh}^j, r_{mh}^j)$ obtained by Iterations II and III with Correction II when $\frac{2}{5} < \vartheta \leq \frac{5}{11}$, the process of proof of (3.55) and (3.56) is the same as the ones of (3.28) and (3.29) for the case of $(\mathbf{u}_{mh}^j, \mathbf{B}_{mh}^j, p_{mh}^j, r_{mh}^j)$ obtained by Iterations I, II and III with Correction II when $0 < \vartheta \leq \frac{2}{5}$. We just need to change some coefficients in the process, for example, change $\frac{7}{15}$ and $1 - \max\{\frac{6}{5}, \frac{4}{3}, 1\}\vartheta$ to $\frac{13}{33}$ and $1 - \max\{\frac{4}{3}, 1\}\vartheta$ in (3.31), respectively; replace $\frac{7}{15}$, $\frac{1}{15}$ and $\frac{15}{4}$ by $\frac{13}{33}$, $\frac{2}{33}$ and $\frac{33}{8}$ in (3.32), respectively; replace $\frac{1}{60}$ and 15 by $\frac{1}{132}$ and 44 in (3.43)-(3.45), respectively. (3.55) and (3.56) in the case of $(\mathbf{u}_{mh}^j, \mathbf{B}_{mh}^j, p_{mh}^j, r_{mh}^j)$ obtained by Iterations II and III with Correction I when $\frac{2}{5} < \vartheta \leq \frac{5}{11}$ can be derived naturally by the one in the case of Correction II.

For the case of $(\mathbf{u}_{mh}^j, \mathbf{B}_{mh}^j, p_{mh}^j, r_{mh}^j)$ obtained by Iterations II and III with Correction III when $\frac{2}{5} < \vartheta \leq \frac{5}{11}$, the proof of (3.55) and (3.56) can be deduced similarly. The proof ends. □

Similar to the proof of Theorem 3.3, we conclude from Lemma 3.2 that

Theorem 3.4. *Assume that the conditions of Lemma* 3.2 *hold,* $(\mathbf{u}, \mathbf{B}, p, r)$ *and* $(\mathbf{u}_h, \mathbf{B}_h, p_h, r_h)$ *are the solutions to* (2.1) *and* (2.11), *respectively. If* $(\mathbf{u}_m^h, \mathbf{B}_m^h, p_m^h, r_m^h)$ *is obtained from Iteration II with Corrections I, II and III, then there holds*

$$\|(\mathbf{u} - \mathbf{u}_m^h, \mathbf{B} - \mathbf{B}_m^h)\|_{E,\Omega} + \|(p - p_m^h, r - r_m^h)\|_{B,\Omega}$$

$$\leq C\left[h^{\tau}+H^{2\tau}+C_{R_e,R_{e_m},S_c}\left(\frac{15}{13}\vartheta\right)^{2^m-1}\|\mathbf{F}\|_{*,\Omega}\right]. \tag{3.57}$$

If $(\mathbf{u}_m^h, \mathbf{B}_m^h, p_m^h, r_m^h)$ is obtained from Iteration III with Corrections I, II and III, then there holds

$$\begin{aligned}&\|(\mathbf{u}-\mathbf{u}_m^h, \mathbf{B}-\mathbf{B}_m^h)\|_{E,\Omega}+\|(p-p_m^h, r-r_m^h)\|_{B,\Omega}\\ \leq\ & C\big(h^{\tau}+H^{2\tau}+C_{R_e,R_{e_m},S_c}\vartheta^m\|\mathbf{F}\|_{*,\Omega}\big).\end{aligned} \tag{3.58}$$

Remark 3.4. From Lemma 3.2, we realize that the LPFEIA formed by Iteration i and Correction j ($i = II, III, j = I, II, III$) is valid for slightly high ϑ ($\frac{2}{5} < \vartheta < \frac{5}{11}$). Moreover, the combination of Iteration II with Correction II is a priority to numerically solving the stationary MHD problem.

3.4. LPFEIAs based on PUM with $\frac{5}{11} < \vartheta \leq \frac{1}{2} - \epsilon_0$

According to Theorem 3.2, when $\frac{5}{11} < \vartheta$, only Iteration III is stable and convergent. Due to the explicit expression of the constant with respect to $\frac{1}{2} - \vartheta$ in (3.42), for $\frac{5}{11} < \vartheta \leq \frac{1}{2} - \epsilon_0$ (ϵ_0 is a fixed constant can be chosen from 0 to $\frac{1}{22}$), we consider three kinds of LPFEIAs by combining the iterative solution $(\mathbf{u}_H^m, \mathbf{B}_H^m, p_H^m, r_H^m)$ obtained by Iteration III on $T_H(\Omega)$ with Corrections I, II and III in parallel on $T_h(D^{j,k}), j = 1, \ldots, J$. The stability and convergence of the three algorithms are proved. The LPFEIAs are

Step I. Find a coarse solution $(\mathbf{u}_H^m, \mathbf{B}_H^m, p_H^m, r_H^m) \in \mathbf{X}_0^H(\Omega) \times \mathbf{Y}_n^H(\Omega) \times S_0^H(\Omega) \times Q_0^H(\Omega)$ by Iteration III.

Step II. Solve local corrections $(\mathbf{u}_{mh}^j, \mathbf{B}_{mh}^j, p_{mh}^j, r_{mh}^j) \in \mathbf{X}_0^h(D^{j,k}) \times \mathbf{Y}_n^h(D^{j,k}) \times S_0^h(D^{j,k}) \times Q_0^h(D^{j,k})$ in parallel by Corrections I, II and III.

Step III. Set $(\mathbf{u}_m^j, \mathbf{B}_m^j, p_m^j, r_m^j) = (\mathbf{u}_H^m + \mathbf{u}_{mh}^j, \mathbf{B}_H^m + \mathbf{B}_{mh}^j, p_H^m + p_{mh}^j, r_H^m + r_{mh}^j)$ in $D^{j,k}$ ($j = 1, 2, \ldots, J$).

Step IV. Assemble the solution

$$\mathbf{u}_m^h = \sum_{j=1}^J \phi_j \mathbf{u}_m^j, \quad \mathbf{B}_m^h = \sum_{j=1}^J \phi_j \mathbf{B}_m^j, \quad p_m^h = \sum_{j=1}^J \phi_j p_m^j, \quad r_m^h = \sum_{j=1}^J \phi_j r_m^j.$$

Lemma 3.3. *Assume that the conditions of Theorem 2.2 hold and $\frac{5}{11} < \vartheta \leq \frac{1}{2} - \epsilon_0$. Then $(\mathbf{u}_{mh}^j, \mathbf{B}_{mh}^j, p_{mh}^j, r_{mh}^j)$ provided by Iteration III with Corrections I, II and III satisfies*

$$\begin{aligned}&C_{\min}C_{R_e,R_{e_m},S_c}\|(\mathbf{u}_{mh}^j, \mathbf{B}_{mh}^j)\|_{E,D^{j,k}}+\|(p_{mh}^j, r_{mh}^j)\|_{B,D^{j,k}}\\ \leq\ & C(C_{R_e,R_{e_m},S_c})^2\|\mathbf{F}\|_{*,\Omega},\end{aligned} \tag{3.59}$$

$$\begin{aligned}&C_{\min}\|(\mathbf{u}^j-\mathbf{u}_m^j, \mathbf{B}^j-\mathbf{B}_m^j)\|_{E,D^{j,k}}+\|(p^j-p_m^j, r^j-r_m^j)\|_{B,D^{j,k}}\\ \leq\ & C\big(C_{\min}\|(\mathbf{u}_H-\mathbf{u}_H^m, \mathbf{B}_H-\mathbf{B}_H^m)\|_{E,D^{j,k}}+\|(p_H-p_H^m, r_H-r_H^m)\|_{B,D^{j,k}}\big),\end{aligned} \tag{3.60}$$

where $(\mathbf{u}_H, \mathbf{B}_H, p_H, r_H)$ and $(\mathbf{u}^j, \mathbf{B}^j, p^j, r^j)$ are derived by Algorithm 3.1, and $(\mathbf{u}_H^m, \mathbf{B}_H^m, p_H^m, r_H^m)$ is from Iteration III.

Proof. For the case of $(\mathbf{u}_{mh}^j, \mathbf{B}_{mh}^j, p_{mh}^j, r_{mh}^j)$ obtained by the combination of Iterations II and III with Correction II when $\frac{5}{11} < \vartheta \leq \frac{1}{2} - \epsilon_0$, the process of proof of (3.59) and (3.60) is the same as the ones of (3.28) and (3.29) for the case of $(\mathbf{u}^j_{mh}, \mathbf{B}^j_{mh}, p^j_{mh}, r^j_{mh})$ obtained by Iterations I, II and III with Correction II when $0 < \vartheta \le \frac{2}{5}$. We just need to change some coefficients in the process, for example, change $\frac{7}{15}$ and $1-\max\{\frac{6}{5},\frac{4}{3},1\}\vartheta$ to $\frac{1}{2}+\epsilon_0$ and $1-\vartheta$ in (3.31), respectively; replace $\frac{7}{15}$, $\frac{1}{15}$ and $\frac{15}{4}$ by $\frac{1}{2}$, $\frac{1}{24}$ and 6 in (3.32), respectively; substitute $\frac{1}{60}$ and 15 by $\frac{\epsilon_0}{8}$ and $\frac{2}{\epsilon_0}$ in (3.43)-(3.45), respectively. Similarly, (3.59) and (3.60) for the case of $(\mathbf{u}^j_{mh}, \mathbf{B}^j_{mh}, p^j_{mh}, r^j_{mh})$ obtained by Iterations II and III with Correction I when $\frac{5}{11} < \vartheta \le \frac{1}{2}-\epsilon_0$ are naturally valid.

For the case of $(\mathbf{u}^j_{mh}, \mathbf{B}^j_{mh}, p^j_{mh}, r^j_{mh})$ obtained by Iterations II and III with Correction III when $\frac{5}{11} < \vartheta \le \frac{1}{2}-\epsilon_0$, the proof of (3.59) and (3.60) can be deduced similarly. The proof is complete. $\square$

Based on Lemma 3.3, similar to the proof of Theorem 3.3, we have the following error estimates:

Theorem 3.5. *Suppose that the conditions of Lemma* 3.3 *hold,* $(\mathbf{u}, \mathbf{B}, p, r)$ *and* $(\mathbf{u}_h, \mathbf{B}_h, p_h, r_h)$ *are the solutions to* (2.1) *and* (2.11), *respectively.* $(\mathbf{u}^h_m, \mathbf{B}^h_m, p^h_m, r^h_m)$ *is obtained from Iteration III with Corrections I, II and III, then there holds*

$$\begin{aligned}
&\|(\mathbf{u}-\mathbf{u}^h_m, \mathbf{B}-\mathbf{B}^h_m)\|_{E,\Omega} + \|(p-p^h_m, r-r^h_m)\|_{B,\Omega} \\
&\le C\big(h^\tau + H^{2\tau} + C_{R_e,R_{e_m},S_c}\vartheta^m \|\mathbf{F}\|_{*,\Omega}\big).
\end{aligned} \tag{3.61}$$

Remark 3.5. From Lemma 3.3, we realize that the combination of Iteration III with Correction j in parallel $(j = I, II, III)$ is valid for slightly high ϑ $(\frac{5}{11} < \vartheta \le \frac{1}{2}-\epsilon_0)$.

3.5. LPFEIAs with $1/2-\epsilon_0 < \vartheta < 1$

From Remark 3.2, for $\vartheta \ge \frac{1}{2}$, we see that $(\mathbf{u}^j_m, \mathbf{B}^j_m, p^j_m, r^j_m)$ obtained by Iterations I, II and III with Correction II in parallel may not converge to $(\mathbf{u}^j, \mathbf{B}^j, p^j, r^j)$ of Algorithm 3.1. So, for $\frac{1}{2}-\epsilon_0 < \vartheta < 1$, we investigate two kinds of LPFEIAs formed by $(\mathbf{u}^m_H, \mathbf{B}^m_H, p^m_H, r^m_H)$ obtained by Iteration III and Corrections I and III in parallel. The stability and convergence of the two algorithms are given. The LPFEIAs are described as follows:

Step I. Find a coarse solution $(\mathbf{u}^m_H, \mathbf{B}^m_H, p^m_H, r^m_H) \in \mathbf{X}^H_0(\Omega) \times \mathbf{Y}^H_n(\Omega) \times S^H_0(\Omega) \times Q^H_0(\Omega)$ by Iteration III.

Step II. Solve local corrections $(\mathbf{u}^j_{mh}, \mathbf{B}^j_{mh}, p^j_{mh}, r^j_{mh}) \in \mathbf{X}^h_0(D^{j,k}) \times \mathbf{Y}^h_n(D^{j,k}) \times S^h_0(D^{j,k}) \times Q^h_0(D^{j,k})$ $(j = 1, 2, \ldots, J)$ in parallel by Corrections I and III.

Step III. Set $(\mathbf{u}^j_m, \mathbf{B}^j_m, p^j_m, r^j_m) = (\mathbf{u}^m_H + \mathbf{u}^j_{mh}, \mathbf{B}^m_H + \mathbf{B}^j_{mh}, p^m_H + p^j_{mh}, r^m_H + r^j_{mh})$ in $D^{j,k}$.

Step IV. Assemble the solution

$$\mathbf{u}^h_m = \sum_{j=1}^J \phi_j \mathbf{u}^j_m, \quad \mathbf{B}^h_m = \sum_{j=1}^J \phi_j \mathbf{B}^j_m, \quad p^h_m = \sum_{j=1}^J \phi_j p^j_m, \quad r^h_m = \sum_{j=1}^J \phi_j r^j_m.$$

Lemma 3.4. *Suppose that the conditions of Theorem* 2.2 *hold and* $\frac{1}{2}-\epsilon_0 < \vartheta < 1$. *Then* $(\mathbf{u}^j_{mh}, \mathbf{B}^j_{mh}, p^j_{mh}, r^j_{mh})$ *provided by Iteration III with Corrections I and III satisfies*

$$\begin{aligned}
&C_{\min} C_{R_e,R_{e_m},S_c} \|(\mathbf{u}^j_{mh}, \mathbf{B}^j_{mh})\|_{E,D^{j,k}} + \|(p^j_{mh}, r^j_{mh})\|_{B,D^{j,k}} \\
&\le C(C_{R_e,R_{e_m},S_c})^2 \|\mathbf{F}\|_{*,\Omega},
\end{aligned} \tag{3.62}$$

$$\begin{aligned}&C_{\min}\|(\mathbf{u}^j-\mathbf{u}_m^j,\mathbf{B}^j-\mathbf{B}_m^j)\|_{E,D^{j,k}}+\|(p^j-p_m^j,r^j-r_m^j)\|_{B,D^{j,k}}\\ \le{}& C\big(C_{\min}\|(\mathbf{u}_H-\mathbf{u}_H^m,\mathbf{B}_H-\mathbf{B}_H^m)\|_{E,D^{j,k}}+\|(p_H-p_H^m,r_H-r_H^m)\|_{B,D^{j,k}}\big),\end{aligned}\tag{3.63}$$

where $(\mathbf{u}_H,\mathbf{B}_H,p_H,r_H)$ *and* $(\mathbf{u}^j,\mathbf{B}^j,p^j,r^j)$ *are derived by Algorithm* 3.1*, and* $(\mathbf{u}_H^m,\mathbf{B}_H^m,p_H^m,r_H^m)$ *is from Iteration III.*

Proof. The proof of (3.62) and (3.63) can be derived naturally by previous techniques. □

Based on Lemma 3.4, similar to the proof of Theorem 3.3, we have the following error estimates:

Theorem 3.6. *Suppose that the conditions of Lemma* 3.4 *hold,* $(\mathbf{u},\mathbf{B},p,r)$ *and* $(\mathbf{u}_h,\mathbf{B}_h,p_h,r_h)$ *are the solutions to* (2.1) *and* (2.11)*, respectively.* $(\mathbf{u}_m^h,\mathbf{B}_m^h,p_m^h,r_m^h)$ *is got by Iteration III with Corrections I and III, then there holds*

$$\begin{aligned}&\|(\mathbf{u}-\mathbf{u}_m^h,\mathbf{B}-\mathbf{B}_m^h)\|_{E,\Omega}+\|(p-p_m^h,r-r_m^h)\|_{B,\Omega}\\ \le{}& C\big(h^\tau+H^{2\tau}+C_{R_e,R_{e_m},S_c}\vartheta^m\|\mathbf{F}\|_{*,\Omega}\big).\end{aligned}\tag{3.64}$$

Remark 3.6. From Lemma 3.4, we see that the LPFEIA consisting of Iteration III and Correction j $(j=I,III)$ is valid for large $\frac{1}{2}-\epsilon_0<\vartheta<1$. ϑ is proportional to the physical parameters R_e and R_{e_m}, this means that LPFEIA by Iteration III and Corrections I and III is a better way to efficiently solve MHD problem at high Reynolds numbers.

4. Numerical Investigations

In this section, the FreeFEM++ software [40] is used for the numerical test. We consider the MHD form of the classical Poiseuille flows, called Hartmann flow [3]. We show some test results to assess the numerical efficiency of our parallel algorithms for the MHD equations (1.1)-(1.5) under different physics parameters. The Mini element (P_1^b,P_1) [39] is used for the hydrodynamic subproblem, the lowest order Nédélec's element for the magnetic subproblem. To treat the nonlinear problem in the step I, the iterative tolerance is chosen as $1.0e-12$. Denote Algorithm $\mathrm{M}_i\mathrm{C}_j$ $(i,j=1,2,3)$ by Iteration i with Correction j $(i,j=I,II,III)$.

Hartmann flow describes the changes of a steady unidirectional incompressible flow in the channel $\Omega=[0,6]\times[-1,1]$ under an external transverse magnetic field $\mathbf{B}_d=(0,1)$. Let $\mathrm{Ha}=\sqrt{R_eR_mS_c}$ be the Hartmann number. Take $\mathbf{f}=\mathbf{g}=\mathbf{0}$ and the solutions are

$$\begin{aligned}&\mathbf{u}(x,y)=(u(y),0),\quad \mathbf{B}(x,y)=(b(y),1),\\ &p(x,y)=-Gx-\frac{1}{2}S_cb^2(y)+p_0\end{aligned}\tag{4.1}$$

with

$$u(y)=\frac{GR_e}{\mathrm{Ha}\cdot\tanh(\mathrm{Ha})}\left(1-\frac{\cosh(y\mathrm{Ha})}{\cosh(\mathrm{Ha})}\right),\quad b(y)=\frac{G}{S_c}\left(\frac{\sinh(y\mathrm{Ha})}{\sinh(\mathrm{Ha})}-y\right),$$

which has the boundary conditions

$$\begin{aligned}&\mathbf{u}=\mathbf{0} && \text{on } y=\pm1,\\ &(p\mathbf{I}-Re^{-1}\nabla\mathbf{u})\mathbf{n}=p_d\mathbf{n} && \text{on } x=0 \quad\text{and}\quad x=6,\\ &\mathbf{n}\times\mathbf{B}=\mathbf{n}\times\mathbf{B}_d && \text{on } \partial\Omega,\end{aligned}$$

where $p_d(x,y)=p(x,y)$, p_0 is a constant and $\mathbf{I}$ is identity matrix.

Set $Re = 1.0, R_{e_m} = 0.1, Sc = 1.0$. Let $T_\mu(\Omega)$ $(\mu = H_p, H, h)$ be the unstructured triangulation of Ω. We employ the PUM based on P_1 on $T_{H_p}(\Omega)$ with $H_p = \frac{1}{6}, J = 536$, and 1 layer oversampling.

According to Theorems 3.3-3.6, set the mesh size $H = \mathcal{O}(h^{\frac{1}{2}})$ to obtain

$$\|(\mathbf{u}-\mathbf{u}_m^h, \mathbf{B}-\mathbf{B}_m^h)\|_{E,\Omega} + \|(p-p_m^h, r-r_m^h)\|_{B,\Omega} \leq C(h + I_{error}),$$

where I_{error} is the iterative errors with respect to m. If I_{error} is higher order infinitesimal quantity compared to h as $h \to 0$, then this error estimate can be transformed to

$$\|(\mathbf{u}-\mathbf{u}_m^h, \mathbf{B}-\mathbf{B}_m^h)\|_{E,\Omega} + \|p-p_m^h\|_{0,\Omega} \approx \mathcal{O}(h).$$

It should be noted that in the numerical experiment, as m increases, I_{error} decreases significantly.

The results in Table 4.1 display the errors of standard finite element method (SFEM) for this problem. From Tables 4.2-4.4, we observe that the Algorithms $\mathrm{M}_i\mathrm{C}_j$ $(i, j = 1, 2, 3)$ can compute well, the rates of $\|(\mathbf{u}-\mathbf{u}_m^h, \mathbf{B}-\mathbf{B}_m^h)\|_{E,\Omega}$ and $\|p-p_m^h\|_{0,\Omega}$ have optimal order, $\|\nabla(r-r_m^h)\|_{0,\Omega}$ nearly equal to zero, and the Algorithms $\mathrm{M}_2\mathrm{C}_j$ save lots of CPU time. In a word, compared to Table 4.1, the data in Tables 4.2-4.4 show the correctness and effectiveness of the proposed Algorithms $\mathrm{M}_i\mathrm{C}_j$ $(i, j = 1, 2, 3)$.

Table 4.1: The errors of SFEM.

h	$\|(\mathbf{u}-\mathbf{u}_h, \mathbf{B}-\mathbf{B}_h)\|_{E,\Omega}$	rate	$\|p-p_h\|_{0,\Omega}$	rate	$\|\nabla(r-r_h)\|_{0,\Omega}$	CPU
1/36	2.02E-03	/	5.22E-04	/	2.01E-11	31
1/64	1.14E-03	9.95E-01	3.06E-04	9.27E-01	6.50E-11	146
1/144	5.01E-04	1.01E+00	1.18E-04	1.18E+00	1.33E-10	311
1/256	–	–	–	–	–	–
1/400	–	–	–	–	–	–

Table 4.2: The errors of Algorithms $\mathrm{M}_1\mathrm{C}_j$ $(j = 1, 2, 3)$.

	H	h	$\|(\mathbf{u}-\mathbf{u}_m^h, \mathbf{B}-\mathbf{B}_m^h)\|_{E,\Omega}$	rate	$\|p-p_m^h\|_{0,\Omega}$	rate	$\|\nabla(r-r_m^h)\|_{0,\Omega}$	CPU
	1/8	1/64	9.32E-03	/	1.34E-02	/	2.46E-13	40
	1/12	1/144	4.19E-03	9.86E-01	5.60E-03	1.08E+00	3.87E-13	192
$\mathrm{M}_1\mathrm{C}_1$	1/16	1/256	2.25E-03	1.08E+00	3.16E-03	9.94E-01	5.29E-13	599
	1/20	1/400	1.43E-03	1.02E+00	2.05E-03	9.70E-01	6.42E-13	1167
	1/24	1/576	9.97E-04	9.89E-01	1.44E-03	9.69E-01	1.91E-13	3435
	1/8	1/64	9.30E-03	/	1.29E-02	/	2.47E-13	31
	1/12	1/144	4.18E-03	9.86E-01	5.57E-03	1.04E+00	3.89E-13	192
$\mathrm{M}_1\mathrm{C}_2$	1/16	1/256	2.25E-03	1.08E+00	3.16E-03	9.85E-01	5.31E-13	600
	1/20	1/400	1.43E-03	1.01E+00	2.05E-03	9.70E-01	6.78E-13	1161
	1/24	1/576	9.92E-04	1.00E+00	1.43E-03	9.88E-01	1.34E-13	3431
	1/8	1/64	9.30E-03	/	1.29E-02	/	2.82E-13	31
	1/12	1/144	4.18E-03	9.86E-01	5.57E-03	1.04E+00	3.66E-13	191
$\mathrm{M}_1\mathrm{C}_3$	1/16	1/256	2.25E-03	1.08E+00	3.16E-03	9.85E-01	5.30E-13	600
	1/20	1/400	1.43E-03	1.01E+00	2.05E-03	9.70E-01	6.94E-13	1162
	1/24	1/576	9.92E-04	1.00E+00	1.43E-03	9.88E-01	1.12E-13	3436

Table 4.3: The errors of Algorithms $\mathrm{M}_2\mathrm{C}_j$ $(j = 1, 2, 3)$.

	H	h	$\|(\mathbf{u}-\mathbf{u}_m^h, \mathbf{B}-\mathbf{B}_m^h)\|_{E,\Omega}$	rate	$\|p-p_m^h\|_{0,\Omega}$	rate	$\|\nabla(r-r_m^h)\|_{0,\Omega}$	CPU
	1/8	1/64	9.29E-03	/	1.21E-02	/	7.50E-14	8
	1/12	1/144	4.16E-03	9.90E-01	5.55E-03	9.61E-01	9.83E-14	34
$\mathrm{M}_2\mathrm{C}_1$	1/16	1/256	2.25E-03	1.07E+00	3.15E-03	9.84E-01	1.30E-13	103
	1/20	1/400	1.43E-03	1.02E+00	2.05E-03	9.63E-01	1.41E-13	269
	1/24	1/576	9.96E-04	9.92E-01	1.44E-03	9.69E-01	2.11E-13	688
	1/8	1/64	9.27E-03	/	1.18E-02	/	7.49E-14	8
	1/12	1/144	4.16E-03	9.88E-01	5.55E-03	9.30E-01	9.90E-14	43
$\mathrm{M}_2\mathrm{C}_2$	1/16	1/256	2.25E-03	1.07E+00	3.15E-03	9.84E-01	1.33E-13	106
	1/20	1/400	1.41E-03	1.05E+00	2.05E-03	9.63E-01	1.45E-13	255
	1/24	1/576	9.92E-04	9.64E-01	1.41E-03	1.03E+00	2.11E-13	676
	1/8	1/64	9.27E-03	/	1.18E-02	/	7.50E-14	9
	1/12	1/144	4.16E-03	9.88E-01	5.55E-03	9.30E-01	9.81E-14	41
$\mathrm{M}_2\mathrm{C}_3$	1/16	1/256	2.25E-03	1.07E+00	3.15E-03	9.84E-01	1.32E-13	117
	1/20	1/400	1.41E-03	1.05E+00	2.05E-03	9.63E-01	1.45E-13	289
	1/24	1/576	9.92E-04	9.64E-01	1.41E-03	1.03E+00	2.11E-13	629

Table 4.4: The errors of Algorithms $\mathrm{M}_3\mathrm{C}_j$ $(j = 1, 2, 3)$.

	H	h	$\|(\mathbf{u}-\mathbf{u}_m^h, \mathbf{B}-\mathbf{B}_m^h)\|_{E,\Omega}$	rate	$\|p-p_m^h\|_{0,\Omega}$	rate	$\|\nabla(r-r_m^h)\|_{0,\Omega}$	CPU
	1/8	1/64	9.29E-03	/	1.21E-02	/	7.23E-14	10
	1/12	1/144	4.16E-03	9.90E-01	5.55E-03	9.61E-01	8.65E-14	52
$\mathrm{M}_3\mathrm{C}_1$	1/16	1/256	2.25E-03	1.07E+00	3.15E-03	9.84E-01	1.14E-13	152
	1/20	1/400	1.43E-03	1.02E+00	2.05E-03	9.63E-01	1.47E-13	348
	1/24	1/576	9.96E-04	9.92E-01	1.44E-03	9.69E-01	1.96E-13	785
	1/8	1/64	9.27E-03	/	1.18E-02	/	8.53E-14	10
	1/12	1/144	4.16E-03	9.88E-01	5.55E-03	9.30E-01	6.22E-14	56
$\mathrm{M}_3\mathrm{C}_2$	1/16	1/256	2.25E-03	1.07E+00	3.15E-03	9.84E-01	1.14E-13	153
	1/20	1/400	1.41E-03	1.05E+00	2.05E-03	9.63E-01	1.62E-13	404
	1/24	1/576	9.92E-04	9.64E-01	1.41E-03	1.03E+00	1.96E-13	927
	1/8	1/64	9.27E-03	/	1.18E-02	/	7.25E-14	11
	1/12	1/144	4.16E-03	9.88E-01	5.55E-03	9.30E-01	6.21E-14	46
$\mathrm{M}_3\mathrm{C}_3$	1/16	1/256	2.25E-03	1.07E+00	3.15E-03	9.84E-01	1.14E-13	144
	1/20	1/400	1.41E-03	1.05E+00	2.05E-03	9.63E-01	1.62E-13	387
	1/24	1/576	9.92E-04	9.64E-01	1.41E-03	1.03E+00	1.96E-13	825

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