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Multi-Breather, Rogue Wave and Multi-Bright-Dark Soliton Interaction of the (2+1)-Dimensional Nonlocal Fokas System

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Abstract. We study the (2+1)-dimensional nonlocal Fokas system by using the Hirota's bilinear method. Firstly, a general tau-function of Kadomtsev-Petviashvili (KP) hierarchy satisfied with the bilinear equation under nonzero boundary condition is derived by considering differential relations and a variable transformation. Secondly, two Gram-type solutions are utilized to the construction of multi-breather, high-order rogue wave, and multi-bright-dark soliton solutions. Then the corresponding parameter restrictions of these solutions are given to satisfy with the complex conjugation symmetry. Furthermore, we find that if the parameter p_{iI} takes different values, the rogue wave solution can be classified as three types of states, such as dark-dark, four-peak and bright-bright high-order rogue wave. If the parameter c_i takes different values, the soliton solution can be classified as three type of states, including the multi-dark, multi-bright-dark and multi-bright solitons. By considering third-type of reduced tau-function to the Hirota's bilinear equations, we give the collisions between the high-order rogue wave and the multi-bright-dark solitons on constant (N is positive even) or periodic background (N is positive odd). In order to understand the dynamics behaviors of the obtained solutions better, the various rich patterns are theoretically and graphically analyzed in detail.

AMS subject classifications: 65M10, 78A48

Key words: (2+1)-dimensional nonlocal Fokas system, KP hierarchy reduction, multi-breather wave, high-order rogue wave, multi-bright-dark soliton.

1. Introduction

The parity ($\mathscr{P}$) and the time ($\mathscr{T}$) symmetries proposed by Bender and Boettcher [5] as the one type of important discrete symmetries were used to replace the Hermiticity of the Hamiltonians in quantum theory. In classic quantum theory, Hermiticity guarantees

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the real spectra of the Hamiltonians. The two researchers extended this condition to the non-Hermitian Hamiltonian operators with $\mathscr{P}\mathscr{T}$-symmetric, where the linear operator $\mathscr{P}$ satisfies $x \to -x$ and the antilinear operator $\mathscr{T}$ satisfies $\mathrm{i} \to -\mathrm{i}$. Replacing Hermiticity by a $\mathscr{P}\mathscr{T}$-symmetry maintains a necessary property in quantum field — i.e. unity of the time evolution [4]. Furthermore, the non-Hermitian Hamiltonian $\mathscr{P}\mathscr{T}$-symmetries have been verified to exist in many important fields, including the Bose-Einstein condensates [7], the electric circuits [40], the magnetics [20], and the population biology [30].

A lot of mathematical and physical properties have been found in nonlinear local models. Note that the nonlocal counterparts with $\mathscr{P}\mathscr{T}$ symmetric potentials may have potential application. As the most fundamental nonlocal model, a (1+1)-dimensional nonlocal NLS equation

$$\mathrm{i}u_t + u_{xx} + 2u^2u^*(-x,t) = 0 \tag{1.1}$$

has been proposed by Ablowitz and Musslimani. It is generated as a symmetry reduction of Ablowitz-Kaup-Newell-Segur hierarchy by considering the self-induced potential $V[u,x,t] = uu^*(-x,t)$ satisfies the $\mathscr{P}\mathscr{T}$ symmetric condition $V[u,x] = V^*[u,-x]$. Although it is only slightly different from the classic NLS equation, there exist both local and nonlocal solutions in the nonlocal model (1.1). More exactly, the evolution of the local model solution depends only on x, but in nonlocal models it also depends on $-x$. Ablowitz and Musslimani [1,2] used the inverse scattering method to obtain the soliton solutions of the Eq. (1.1). Feng *et al.* [13] applied the KP reduction method and constructed bright-dark soliton solutions using the tau-function of Gram- and Wronskian-type.

After the work of Ablowitz and Musslimani, Fokas [19] proposed two integrable nonlocal NLS equations, which are the multi-dimensional versions of the Eq. (1.1). The first equation is the nonlocal Davey-Stewartson (NDS) equation

$$\begin{aligned} &\mathrm{i}A_t = A_{xx} + \varepsilon A_{yy} + (\epsilon V - 2Q)A, \\ &Q_{xx} - \varepsilon Q_{yy} = \epsilon V_{xx}, \quad \epsilon = \pm 1, \end{aligned} \tag{1.2}$$

where $\varepsilon = 1$ corresponds to the NDSI equation, $\varepsilon = -1$ corresponds to the NDSII equation. Here $V[A,x,y,t] = AA^*(-x,-y,t)$ satisfies the $\mathscr{P}\mathscr{T}$ symmetric condition $V[A,x,y,t] = V^*[A,-x,-y,t]$. Hereafter, the equation was reported by Ablowitz and Musslimani [3]. There are many important nonlocal examples, such as the nonlocal sine-Gordon equation, the reverse space-time mKdV equation, and discrete version, multi component version of the nonlocal NLS equation [3]. Rao *et al.* [39] derived rational and semi-rational solutions for the Eq. (1.2), by using Hirota's bilinear method. Furthermore, Ohta *et al.* [33, 34] studied a local version of the Eq. (1.2) and reported the multi-fold or high-order rogue wave solutions by virtue of the KP hierarchy reduction method and Gram-type determinant.

In this work, we focus on the second equation, known as the (2+1)-dimensional nonlocal Fokas system — i.e.

$$\begin{aligned} &\mathrm{i}A_t + A_{xx} + AQ = 0, \\ &Q_y = V_x, \end{aligned} \tag{1.3}$$

where $V[A,x,y,t]=AA^*(-x,-y,t)$. It can also be written as the another form

$$iA_t+A_{xx}+A\int_{-\infty}^{y}\left[A(x,y',t)A^*(-x,-y',t)\right]_x dy'=0.$$

For this equation, Cao *et al.* [6] obtained the rational and semi-rational solutions with zero boundary condition, using the Hirota's bilinear method. Liu *et al.* [28] constructed bright-dark solitons by using the bilinear form with zero/nonzero boundary conditions and the KP hierarchy reduction. To the best of our knowledge, the breather, rogue wave and multi-bright-dark soliton interaction have not been reported so far. Thus, we consider the derivation of these analytical solutions by using two important methods — i.e. the Hirota's bilinear method and the KP hierarchy reduction method. The former was proposed by Hirota [22] in 2004. The latter was developed by Kyoto School. The original KP hierarchy is also known as A-type KP hierarchy. Their sub-hierarchies, including the B-type KP, C-type KP and D-type KP hierarchies have been classified by considering Lie algebra [23]. Based on that, there are so many important results obtained by considering the classic NLS equation [32], discrete NLS equation [35], the DS equation [33, 34, 39, 42], the Mel'nikov equation [26, 29], the Boussinesq-Burgers system [27], the long-wave-short-wave resonance system [8–10], and multi-component modified KP hierarchy [11, 24, 25]. In addition, our team has reported some important work [12, 21, 36, 43, 45–47]. Since Feng *et al.* used the KP reduction method to study the nonlocal NLS equations, the study on KP reduction of the nonlocal equations has attracted more and more attention [13].

The main goal of this paper is to derive breather, rogue wave and multi-bright-dark soliton interaction by using the Hirota's bilinear method and KP hierarchy reduction method. The objectives of our work are as follows:

(1) Under the bilinear KP hierarchy, the exact breather wave solution in terms of the corresponding tau-function will be derived. Under the complex conjugation symmetry conditions, the parameter restriction for this solution will be also provided.

(2) When some parameters of the given Gram-type tau-function are chosen as the special case, the rational solution reduces to three types of nonlinear behaviors, such as the high-order rogue wave, multi-bright-dark soliton and their combinations. They appear on plane or periodic wave background, when N is chosen as positive even or positive odd.

The remainder of this paper is arranged as follows. In Section 2, the (2+1)-dimensional nonlocal Fokas system admits the bilinear form by considering the dependent variable transformation. This bilinear form can be reduced to the KP hierarchy under a suitable general tau-function by using a C-type reduction method and an independent variable transformation. In Section 3, we use a tau-function to construct multi-breather wave solution and derive a class of parameter conditions. In Section 4, we give a general tau-function, which transfers to three types of reduced forms by setting the different parameter values. They corresponds to the high-order rogue wave, multi-bright-dark soliton and their interaction on the constant and periodic background. Finally, we summarize these results in the last section.

2. Bilinearization and KP Hierarchy Reduction

In this section, we use a lemma to present the bilinear form of the (2+1)-dimensional nonlocal Fokas system and give another lemma to present the relation of the bilinear form and KP hierarchy under the general tau-function and independent variable transformation.

Lemma 2.1. *Using the dependent variable transformation*

$$A=\sqrt{2}\mathrm{e}^{\mathrm{i}kt}\frac{g}{f},\quad A^*(-x,-y,t)=\sqrt{2}\mathrm{e}^{\mathrm{i}kt}\frac{h}{f},\quad Q=k+2(\log f)_{xx},$$

the Eq. (1.3) *converts to the bilinear form*

$$\left(D_x^2+\mathrm{i}D_t\right)g\cdot f=0, \tag{2.1}$$

$$(D_xD_y+2)f\cdot f=2gh \tag{2.2}$$

with the nonzero boundary condition $A=\sqrt{2}, Q=k$, where D denotes Hirota's bilinear differential operator [22], *i.e.*

$$D_x^nD_y^mf\cdot g=\left(\frac{\partial}{\partial x}-\frac{\partial}{\partial x'}\right)^n\left(\frac{\partial}{\partial y}-\frac{\partial}{\partial y'}\right)^m f(x,y)\cdot g(x',y')\bigg|_{x'=x,y'=y}.$$

As was already mentioned, the functions $f(x,y,t)$ and $g(x,y,t)$ satisfy the complex conjugation symmetry

$$\frac{g^*(-x,-y,t)}{f^*(-x,-y,t)}=\frac{h(x,y,t)}{f(x,y,t)}, \tag{2.3}$$

which can be considered as the two cases that

$$h(x,y,t)=g^*(-x,-y,t),\quad f(x,y,t)=f^*(-x,-y,t)$$

or

$$h(x,y,t)=-g^*(-x,-y,t),\quad f(x,y,t)=-f^*(-x,-y,t).$$

Lemma 2.2. *Let $m_{ij}^{(n)}$ be associated with the functions $\psi^{(n)}(x_1,x_2,x_{-1}),\varphi^{(n)}(x_1,x_2,x_{-1})$ and satisfy the following differential relations:*

$$\begin{aligned}
&\partial_{x_1}m_{ij}^{(n)}=\psi_i^{(n)}\varphi_j^{(n)},\\
&\partial_{x_2}m_{ij}^{(n)}=\psi_i^{(n+1)}\varphi_j^{(n)}+\psi_i^{(n)}\varphi_j^{(n-1)},\\
&\partial_{x_{-1}}m_{ij}^{(n)}=-\psi_i^{(n-1)}\varphi_j^{(n+1)},\\
&m_{ij}^{(n+1)}=m_{ij}^{(n)}+\psi_i^{(n)}\varphi_j^{(n+1)},\\
&\partial_{x_\delta}\psi_i^{(n)}=\psi_i^{(n+\delta)},\ \partial_{x_\delta}\varphi_i^{(n)}=-\varphi_i^{(n-\delta)},
\end{aligned}$$

where $\delta=-1,1,2$. Then the tau-function

$$\tau_n=\det_{1\le i,j\le N}\left(m_{ij}^{(n)}\right)$$

satisfies the bilinear form

$$\left(D_{x_1}^2 - D_{x_2}\right)\tau_{n+1}\cdot\tau_n = 0, \tag{2.4}$$

$$(D_{x_1}D_{x_{-1}} - 2)\tau_n\cdot\tau_n + 2\tau_{n+1}\tau_{n-1} = 0. \tag{2.5}$$

Proof. We determine the derivatives of the tau-function as follows:

$$\begin{aligned}
&\partial_{x_1}\tau_n = \begin{vmatrix} m_{ij}^{(n)} & \psi_i^{(n)} \\ -\varphi_j^{(n)} & 0 \end{vmatrix}, \qquad \partial_{x_{-1}}\tau_n = \begin{vmatrix} m_{ij}^{(n)} & \psi_i^{(n-1)} \\ \varphi_j^{(n+1)} & 0 \end{vmatrix}, \\
&\partial_{x_{-1}}\partial_{x_1}\tau_n = \begin{vmatrix} m_{ij}^{(n)} & \psi_i^{(n-1)} \\ -\varphi_j^{(n)} & 0 \end{vmatrix} + \begin{vmatrix} m_{ij}^{(n)} & \psi_i^{(n)} \\ \varphi_j^{(n+1)} & 0 \end{vmatrix}, \\
&\partial_{x_1}^2\tau_n = \begin{vmatrix} m_{ij}^{(n)} & \psi_i^{(n+1)} \\ -\varphi_j^{(n)} & 0 \end{vmatrix} + \begin{vmatrix} m_{ij}^{(n)} & \psi_i^{(n)} \\ \varphi_j^{(n-1)} & 0 \end{vmatrix}, \\
&\partial_{x_2}\tau_n = \begin{vmatrix} m_{ij}^{(n)} & \psi_i^{(n+1)} \\ -\varphi_j^{(n)} & 0 \end{vmatrix} - \begin{vmatrix} m_{ij}^{(n)} & \psi_i^{(n)} \\ \varphi_j^{(n-1)} & 0 \end{vmatrix}, \\
&\partial_{x_1}^2\tau_{n+1} = \begin{vmatrix} m_{ij}^{(n)} & \psi_i^{(n+2)} \\ -\varphi_j^{(n+1)} & 0 \end{vmatrix} + \begin{vmatrix} m_{ij}^{(n)} & \psi_i^{(n)} & \psi_i^{(n+1)} \\ -\varphi_j^{(n)} & 0 & 0 \\ -\varphi_j^{(n+1)} & 1 & 0 \end{vmatrix}, \\
&\partial_{x_2}\tau_{n+1} = \begin{vmatrix} m_{ij}^{(n)} & \psi_i^{(n+2)} \\ -\varphi_j^{(n+1)} & 0 \end{vmatrix} - \begin{vmatrix} m_{ij}^{(n)} & \psi_i^{(n)} & \psi_i^{(n+1)} \\ -\varphi_j^{(n)} & 0 & 0 \\ -\varphi_j^{(n+1)} & 1 & 0 \end{vmatrix}, \\
&\tau_{n+1} = \begin{vmatrix} m_{ij}^{(n)} & \psi_i^{(n)} \\ -\varphi_j^{(n+1)} & 1 \end{vmatrix}, \qquad \tau_{n-1} = \begin{vmatrix} m_{ij}^{(n)} & \psi_i^{(n-1)} \\ \varphi_j^{(n)} & 1 \end{vmatrix}, \\
&\partial_{x_1}\tau_{n+1} = \begin{vmatrix} m_{ij}^{(n)} & \psi_i^{(n+1)} \\ -\varphi_j^{(n+1)} & 0 \end{vmatrix}, \quad \partial_{x_{-1}}\tau_{n+1} = \begin{vmatrix} m_{ij}^{(n)} & \psi_i^{(n-1)} \\ -\varphi_j^{(n+1)} & 0 \end{vmatrix}, \\
&\partial_{x_1}\partial_{x_{-1}}\tau_{n+1} = \begin{vmatrix} m_{ij}^{(n)} & \psi_i^{(n)} \\ -\varphi_j^{(n+1)} & 0 \end{vmatrix} + \begin{vmatrix} m_{ij}^{(n)} & \psi_i^{(n-1)} \\ \varphi_j^{(n)} & 0 \end{vmatrix}.
\end{aligned}$$

According to the definition of the Jacobi identity, these derivatives of tau-function satisfy the equation

$$\begin{aligned}
&\left[\left(\partial_{x_1}^2 - \partial_{x_2}\right)\tau_{n+1}\right]\tau_n + \tau_{n+1}\left(\partial_{x_1}^2 + \partial_{x_2}\right)\tau_n - 2\left(\partial_{x_1}\tau_{n+1}\right)\left(\partial_{x_1}\tau_n\right) = 0, \\
&\left(\partial_{x_1}\partial_{x_{-1}}\tau_{n+1}\right)\tau_n + \tau_{n+1}\left(\partial_{x_1}\partial_{x_{-1}}\tau_n\right) - \left(\partial_{x_{-1}}\tau_{n+1}\right)\left(\partial_{x_1}\tau_n\right) - \left(\partial_{x_1}\tau_{n+1}\right)\left(\partial_{x_{-1}}\tau_n\right) \\
&= 2\tau_n\tau_n - 2\tau_{n+1}\tau_{n-1}.
\end{aligned}$$

It is clear that the bilinear form (2.4)-(2.5) holds. □

We set the variable transformation $x_1 = x, x_2 = \mathrm{i}t, x_{-1} = -y$ to (2.4)-(2.5) and obtain

$$\left(D_x^2 + \mathrm{i}D_t\right)\tau_{n+1} \cdot \tau_n = 0, \tag{2.6}$$
$$(D_x D_y + 2)\tau_n \cdot \tau_n - 2\tau_{n+1}\tau_{n-1} = 0, \tag{2.7}$$

which converts to the bilinear form (2.1)-(2.2) with $g = \tau_1, f = \tau_0, h = \tau_{-1}$.

3. Multi-Breather Wave Solutions

In this section, based on the bilinear form (2.1)-(2.2), we derive multi-breather wave solution for the (2+1)-dimensional nonlocal Fokas system. Assuming a special tau-function of Gram-type, we establish the following theorem.

Theorem 3.1. *The Eq.* (1.3) *admits the multi-breather wave solution*

$$A = \sqrt{2}\mathrm{e}^{\mathrm{i}kt}\frac{g(x,y,t)}{f(x,y,t)}, \tag{3.1}$$

where $g = \tau_1, f = \tau_0$ and the tau-function τ_n, $n = 0,1$ is defined as

$$\begin{aligned}
\tau_n &= \det_{1\le i,j\le N}\left(m_{ij}^{(n)}\right), \qquad (3.2)\\
m_{ij}^{(n)} &= \sum_{\alpha,\beta=1}^{2}\frac{1}{p_{i\alpha}+q_{j\beta}}\left(-\frac{p_{i\alpha}}{q_{j\beta}}\right)^n \mathrm{e}^{\xi_{i\alpha}+\eta_{j\beta}},\\
\xi_{i\alpha} &= p_{i\alpha}x + \mathrm{i}p_{i\alpha}^2 t - \frac{1}{p_{i\alpha}}y + \xi_{i\alpha}^{(0)},\\
\eta_{j\beta} &= q_{j\beta}x - \mathrm{i}q_{j\beta}^2 t - \frac{1}{q_{j\beta}}y + \eta_{j\beta}^{(0)}.
\end{aligned}$$

We write the tau-function (3.2) as

$$\tilde{\tau}_n = C\tau_n,$$

where

$$\begin{aligned}
C &= \prod_{j=1}^{N}\mathrm{e}^{-\xi_{i1}-\eta_{j2}}, \quad \tilde{\tau}_n = \det\left(\tilde{m}_{ij}^{(n)}\right)_{1\le i,j\le N},\\
\tilde{m}_{ij}^{(n)} &= \sum_{\alpha,\beta=1}^{2}\frac{1}{p_{i\alpha}+q_{j\beta}}\left(-\frac{p_{i\alpha}}{q_{j\beta}}\right)^n \mathrm{e}^{\xi_{i\alpha}-\xi_{i1}+\eta_{j\beta}-\eta_{j2}}. \qquad (3.3)
\end{aligned}$$

Proposition 3.1. *Assume that the parameters of* (3.3) *satisfy the following relations:*

$$\begin{aligned}
&q_{j\beta} = p_{j\beta}^*, \qquad \eta_{j\beta}^{(0)} = \xi_{j\beta}^{(0)*}, \qquad p_{N_1+i,1} = -p_{i2},\\
&p_{N_1+i,2} = -p_{i1}, \quad p_{2N_1+1,1} = -p_{2N_1+1,2}, \qquad (3.4)\\
&\xi_{N_1+i,1}^{(0)} = \xi_{i2}^{(0)}, \quad \xi_{N_1+i,2}^{(0)} = \xi_{i1}^{(0)}, \quad \xi_{2N_1+1,1}^{(0)} = \xi_{2N_1+1,2}^{(0)},
\end{aligned}$$

where $i,j=1,2,\dots,N_1$. Then the tau-function $\tilde{\tau}_n$ satisfies the complex conjugation symmetry relation

$$\tilde{\tau}_{-n}(x,y,t)=\tilde{\tau}_n^*(-x,-y,t), \tag{3.5}$$
$$\tilde{\tau}_{-n}(x,y,t)=-\tilde{\tau}_n^*(-x,-y,t). \tag{3.6}$$

Proof. It is easily checked that $C(x,y,t)=C^*(-x,-y,t)$. Simple calculations give

$$\xi_{j1}-\xi_{j2}=(p_{j1}-p_{j2})x+\mathrm{i}\left(p_{j1}^2-p_{j2}^2\right)t-\left(\frac{1}{p_{j1}}-\frac{1}{p_{j2}}\right)y+\left(\xi_{j1}^{(0)}-\xi_{j2}^{(0)}\right),$$
$$\eta_{j1}-\eta_{j2}=(q_{j1}-q_{j2})x-\mathrm{i}\left(q_{j1}^2-q_{j2}^2\right)t-\left(\frac{1}{q_{j1}}-\frac{1}{q_{j2}}\right)y+\left(\eta_{j1}^{(0)}-\eta_{j2}^{(0)}\right),$$

which indicates that

$$\begin{aligned}
&(\xi_{N_1+j,1}-\xi_{N_1+j,2})(-x,-y,t)=(\xi_{j2}-\xi_{j1})(x,y,t),\\
&(\eta_{N_1+j,1}-\eta_{N_1+j,2})^*(-x,-y,t)=(\xi_{j2}-\xi_{j1})(x,y,t),\\
&(\xi_{2N_1+1,1}-\xi_{2N_1+1,2})(-x,-y,t)=(\xi_{2N_1+1,2}-\xi_{2N_1+1,1})(x,y,t),\\
&(\eta_{2N_1+1,1}-\eta_{2N_1+1,2})^*(-x,-y,t)=(\xi_{2N_1+1,2}-\xi_{2N_1+1,1})(x,y,t).
\end{aligned}$$

It follows from the parameter restrictions (3.4) that

$$\begin{aligned}
&\tilde{m}_{N+i,N+j}^{(n)*}(-x,-y,t)\\
&=\frac{1}{p_{N+i,2}^*+q_{N+j,1}^*}\left(-\frac{p_{N+i,2}^*}{q_{N+j,1}^*}\right)^n \mathrm{e}^{(\xi_{N+i,2}^*-\xi_{N+i,1}^*+\eta_{N+j,1}^*-\eta_{N+j,2}^*)(-x,-y,t)}\\
&\quad+\frac{1}{p_{N+i,1}^*+q_{N+j,2}^*}\left(-\frac{p_{N+i,1}^*}{q_{N+j,2}^*}\right)^n\\
&\quad+\frac{1}{p_{N+i,2}^*+q_{N+j,2}^*}\left(-\frac{p_{N+i,2}^*}{q_{N+j,2}^*}\right)^n \mathrm{e}^{(\xi_{N+i,2}^*-\xi_{N+i,1}^*)(-x,-y,t)}\\
&\quad\times\frac{1}{p_{N+i,1}^*+q_{N+j,1}^*}\left(-\frac{p_{N+i,1}^*}{q_{N+j,1}^*}\right)^n \mathrm{e}^{(\eta_{N+j,1}^*-\eta_{N+j,2}^*)(-x,-y,t)}\\
&=\frac{-1}{p_{j1}+p_{i1}^*}\left(-\frac{p_{i1}^*}{p_{j1}}\right)^n \mathrm{e}^{(\xi_{i1}^*-\xi_{i2}^*)(x,y,t)}+\frac{-1}{p_{j2}+p_{i1}^*}\left(-\frac{p_{i1}^*}{p_{j2}}\right)^n \mathrm{e}^{(\xi_{j2}-\xi_{j1}+\xi_{i1}^*-\xi_{i2}^*)(x,y,t)}\\
&\quad+\frac{-1}{p_{j1}+p_{i2}^*}\left(-\frac{p_{i2}^*}{p_{j1}}\right)^n+\frac{-1}{p_{j2}+p_{i2}^*}\left(-\frac{p_{i2}^*}{p_{j2}}\right)^n \mathrm{e}^{(\xi_{j2}-\xi_{j1})(x,y,t)}\\
&=-\tilde{m}_{ji}^{(-n)}(x,y,t),
\end{aligned} \tag{3.7}$$

and

$$\tilde{m}_{2N_1+1,2N_1+1}^{(n)*}(-x,-y,t)=\frac{1}{p_{2N_1+1,1}^*+q_{2N_1+1,1}^*}\left(-\frac{p_{2N_1+1,1}^*}{q_{2N_1+1,1}^*}\right)^n \mathrm{e}^{(\eta_{2N_1+1,1}^*-\eta_{2N_1+1,2}^*)(-x,-y,t)}$$

$$
\begin{aligned}
&+\frac{1}{p^*_{2N_1+1,2}+q^*_{2N_1+1,1}}\left(-\frac{p^*_{2N_1+1,2}}{q^*_{2N_1+1,1}}\right)^n \mathrm{e}^{(\xi^*_{2N_1+1,2}-\xi^*_{2N_1+1,1}+\eta^*_{2N_1+1,1}-\eta^*_{2N_1+1,2})(-x,-y,t)}\\
&+\frac{1}{p^*_{2N_1+1,1}+q^*_{2N_1+1,2}}\left(-\frac{p^*_{2N_1+1,1}}{q^*_{2N_1+1,2}}\right)^n\\
&+\frac{1}{p^*_{2N_1+1,2}+q^*_{2N_1+1,2}}\left(-\frac{p^*_{2N_1+1,2}}{q^*_{2N_1+1,2}}\right)^n \mathrm{e}^{(\xi^*_{2N_1+1,2}-\xi^*_{2N_1+1,1})(-x,-y,t)}\\
=&\frac{-1}{p_{2N_1+1,2}+p^*_{2N_1+1,2}}\left(-\frac{p^*_{2N_1+1,2}}{p_{2N_1+1,2}}\right)^n \mathrm{e}^{(\xi_{2N_1+1,2}-\xi_{2N_1+1,1})(x,y,t)}\\
&+\frac{-1}{p_{2N_1+1,2}+p^*_{2N_1+1,1}}\left(-\frac{p^*_{2N_1+1,1}}{p_{2N_1+1,2}}\right)^n \mathrm{e}^{(\xi_{2N_1+1,2}-\xi_{2N_1+1,1}+\xi^*_{2N_1+1,1}-\xi^*_{2N_1+1,2})(x,y,t)}\\
&+\frac{-1}{p_{2N_1+1,1}+p^*_{2N_1+1,2}}\left(-\frac{p^*_{2N_1+1,2}}{p_{2N_1+1,1}}\right)^n\\
&+\frac{-1}{p_{2N_1+1,1}+p^*_{2N_1+1,1}}\left(-\frac{q_{2N_1+1,1}}{p_{2N_1+1,1}}\right)^n \mathrm{e}^{(\xi^*_{2N_1+1,1}-\xi^*_{2N_1+1,2})(x,y,t)}\\
=&-\tilde{m}^{(-n)}_{2N_1+1,2N_1+1}(x,y,t).
\end{aligned}
\tag{3.8}
$$

Similar considerations lead to the following equations:

$$\tilde{m}^{(n)*}_{N_1+i,j}(-x,-y,t)=-\tilde{m}^{(-n)}_{N_1+j,i}(x,y,t), \tag{3.9}$$

$$\tilde{m}^{(n)*}_{i,N_1+j}(-x,-y,t)=-\tilde{m}^{(-n)}_{j,N_1+i}(x,y,t), \tag{3.10}$$

$$\tilde{m}^{(n)*}_{2N_1+1,j}(-x,-y,t)=-\tilde{m}^{(-n)}_{N_1+j,2N_1+1}(x,y,t), \tag{3.11}$$

$$\tilde{m}^{(n)*}_{i,2N_1+1}(-x,-y,t)=-\tilde{m}^{(-n)}_{2N_1+1,N_1+i,}(x,y,t), \tag{3.12}$$

$$\tilde{m}^{(n)*}_{N_1+i,2N_1+1}(-x,-y,t)=-\tilde{m}^{(-n)}_{2N_1+1,i}(x,y,t), \tag{3.13}$$

$$\tilde{m}^{(n)*}_{2N_1+1,N_1+j}(-x,-y,t)=-\tilde{m}^{(-n)}_{j,2N_1+1}(x,y,t). \tag{3.14}$$

When $N=1$, by considering the complex symmetry relation (3.8), it is not difficult to find that

$$
\begin{aligned}
\tau^*_n(-x,-y,t)&=\mathrm{e}^{(\xi_{11}+\eta_{12})^*(-x,-y,t)}\tilde{m}^{(n)*}_{11}(-x,-y,t)\\
&=-\mathrm{e}^{(\xi_{11}+\eta_{12})(x,y,t)}\tilde{m}^{(-n)}_{11}(x,y,t)\\
&=-\tau_{-n}(x,y,t).
\end{aligned}
$$

When $N=2N_1$, by considering the complex symmetry relations (3.7), (3.9)-(3.10), we have

$$\tau^*_n(-x,-y,t)=C^*(-x,-y,t)\begin{vmatrix} \tilde{m}^{(n)*}_{ij}(-x,-y,t) & \tilde{m}^{(n)*}_{i,N_1+j}(-x,-y,t)\\ \tilde{m}^{(n)*}_{N_1+i,j}(-x,-y,t) & \tilde{m}^{(n)*}_{N_1+i,N_1+j}(-x,-y,t)\end{vmatrix}$$

$$
\begin{aligned}
&= C(x,y,t)\begin{vmatrix} -\tilde{m}^{(-n)}_{N_1+j,N_1+i}(x,y,t) & -\tilde{m}^{(-n)}_{j,N_1+i}(x,y,t) \\ -\tilde{m}^{(-n)}_{N_1+j,i}(x,y,t) & -\tilde{m}^{(-n)}_{j,i}(x,y,t) \end{vmatrix} \\
&= \tau_{-n}(x,y,t).
\end{aligned}
$$

When $N = 2N_1 + 1$, using the complex symmetry relations (3.7)-(3.14), we can obtain

$$
\begin{aligned}
&\tau_n^*(-x,-y,t) \\
&= C^*(-x,-y,t)\begin{vmatrix} \tilde{m}^{(n)*}_{ij}(-x,-y,t) & \tilde{m}^{(n)*}_{i,N_1+j}(-x,-y,t) & \tilde{m}^{(n)*}_{i,2N_1+1}(-x,-y,t) \\ \tilde{m}^{(n)*}_{N_1+i,j}(-x,-y,t) & \tilde{m}^{(n)*}_{N_1+i,N_1+j}(-x,-y,t) & \tilde{m}^{(n)*}_{N_1+i,2N_1+1}(-x,-y,t) \\ \tilde{m}^{(n)*}_{2N_1+1,j}(-x,-y,t) & \tilde{m}^{(n)*}_{2N_1+1,N_1+j}(-x,-y,t) & \tilde{m}^{(n)*}_{2N_1+1,2N_1+1}(-x,-y,t) \end{vmatrix} \\
&= C(x,y,t)\begin{vmatrix} -\tilde{m}^{(-n)}_{N_1+j,N_1+i}(x,y,t) & -\tilde{m}^{(-n)}_{j,N_1+i}(x,y,t) & -\tilde{m}^{(-n)}_{2N_1+1,N_1+i}(x,y,t) \\ -\tilde{m}^{(-n)}_{N_1+j,i}(x,y,t) & -\tilde{m}^{(-n)}_{ji}(x,y,t) & -\tilde{m}^{(-n)}_{2N_1+1,i}(x,y,t) \\ -\tilde{m}^{(-n)}_{N_1+j,2N_1+1}(x,y,t) & -\tilde{m}^{(-n)}_{j,2N_1+1}(x,y,t) & -\tilde{m}^{(-n)}_{2N_1+1,2N_1+1}(x,y,t) \end{vmatrix} \\
&= -C(x,y,t)\begin{vmatrix} \tilde{m}^{(-n)}_{i,j}(x,y,t) & \tilde{m}^{(-n)}_{i,N_1+j}(x,y,t) & \tilde{m}^{(-n)}_{i,2N_1+1}(x,y,t) \\ \tilde{m}^{(-n)}_{N_1+i,j}(x,y,t) & \tilde{m}^{(-n)}_{N_1+i,N_1+j}(x,y,t) & \tilde{m}^{(-n)}_{N_1+i,2N_1+1}(x,y,t) \\ \tilde{m}^{(-n)}_{2N_1+1,j}(x,y,t) & \tilde{m}^{(-n)}_{2N_1+1,N_1+j}(x,y,t) & \tilde{m}^{(-n)}_{2N_1+1,2N_1+1}(x,y,t) \end{vmatrix} \\
&= -\tau_{-n}(x,y,t).
\end{aligned}
$$

Thus (3.5) is satisfied with N being positive even, whereas (3.6) is satisfied with N being positive odd. □

3.1. One-breather wave solution

In this part, one-breather wave solution is obtained by choosing the parameters $N = 1, p_{11} = -p_{12}, q_{11} = p_{11}^*, q_{12} = p_{12}^*$ in (3.2). Then the exact one-breather wave solution is given as

$$
A = \sqrt{2}\mathrm{e}^{\mathrm{i}kt}\frac{p_{11}(\mathrm{i}p_{11R}G_1 + p_{11I}G_2 - p_{11I}G_3 - \mathrm{i}p_{11R})}{p_{11}^*(\mathrm{i}p_{11R}G_1 - p_{11I}G_2 + p_{11I}G_3 - \mathrm{i}p_{11R})},
$$

where

$$
\begin{aligned}
G_1 &= \cos\left(\frac{4p_{11I}(|p_{11}|^2x + y)}{|p_{11}|^2}\right) + \sinh\left(\frac{-4\mathrm{i}p_{11I}(|p_{11}|^2x + y)}{|p_{11}|^2}\right), \\
G_2 &= \cos\left(-\frac{2\mathrm{i}(p_{11}^2x - y)}{p_{11}}\right) - \sinh\left(\frac{2p_{11}^2x - 2y}{p_{11}}\right), \\
G_3 &= \cos\left(\frac{2\mathrm{i}(p_{11}^{*2}x - y)}{p_{11}^*}\right) + \sinh\left(\frac{2p_{11}^{*2}x - y}{p_{11}^*}\right).
\end{aligned}
$$

3.2. Multi-fold breather wave solution

In order to study the dynamics of multi-breather wave solution, we firstly assume $N=2$, $p_{21}=-p_{12}, p_{22}=-p_{11}, q_{11}=p_{11}^*, q_{12}=p_{12}^*, q_{21}=p_{21}^*, q_{22}=p_{22}^*$. Then two-breather wave solution can be expressed as

$$A=\sqrt{2}\mathrm{e}^{\mathrm{i}kt}\frac{g(x,y,t)}{f(x,y,t)} \tag{3.15}$$

with the corresponding functions

$$f(x,y,t)=\begin{vmatrix} \tilde{m}_{11}^{(0)} & \tilde{m}_{12}^{(0)} \\ \tilde{m}_{21}^{(0)} & \tilde{m}_{22}^{(0)} \end{vmatrix}, \quad g(x,y,t)=\begin{vmatrix} \tilde{m}_{11}^{(1)} & \tilde{m}_{12}^{(1)} \\ \tilde{m}_{21}^{(1)} & \tilde{m}_{22}^{(1)} \end{vmatrix}, \tag{3.16}$$

where $\tilde{m}_{ij}^{(0)}, \tilde{m}_{ij}^{(1)}$ are given by

$$\begin{aligned} \tilde{m}_{ij}^{(0)} &= \frac{\mathrm{e}^{\xi_{i2}-\xi_{i1}+\xi_{j1}^*-\xi_{j2}^*}}{p_{i2}+p_{j1}^*}+\frac{\mathrm{e}^{\xi_{j1}^*-\xi_{j2}^*}}{p_{i1}+p_{j1}^*}+\frac{\mathrm{e}^{\xi_{i2}-\xi_{i1}}}{p_{i2}+p_{j2}^*}+\frac{1}{p_{i1}+p_{j2}^*}, \\ \tilde{m}_{ij}^{(1)} &= \frac{-p_{i2}\mathrm{e}^{\xi_{i2}-\xi_{i1}+\xi_{j1}^*-\xi_{j2}^*}}{p_{j1}^*(p_{i2}+p_{j1}^*)}+\frac{-p_{i1}\mathrm{e}^{\xi_{j1}^*-\xi_{j2}^*}}{p_{j1}^*(p_{i1}+p_{j1}^*)}+\frac{-p_{i2}\mathrm{e}^{\xi_{i2}-\xi_{i1}}}{p_{j2}^*(p_{i2}+p_{j2}^*)}+\frac{-p_{i1}}{p_{j2}^*(p_{i1}+p_{j2}^*)}, \\ \xi_{i1}-\xi_{i2} &= (p_{i1}-p_{i2})x+\mathrm{i}\left(p_{i1}^2-p_{i2}^2\right)t-\left(\frac{1}{p_{i1}}-\frac{1}{p_{i2}}\right)y+\xi_{i1}^{(0)}-\xi_{i2}^{(0)}, \quad i,j=1,2. \end{aligned} \tag{3.17}$$

As shown in Fig. 1, we display the two-breather wave in xy-pane by choosing suitable parameters to the solution (3.15) with the corresponding functions (3.16). If we choose $p_{11}=1.2+\mathrm{i}, p_{12}=(1.2+0.125\mathrm{i}, 2.2+0.125\mathrm{i}, 1.4+0.125\mathrm{i})$, there are x-periodic two-breather wave, y-periodic two-breather wave and declining two-breather wave, respectively. The upper row denotes stereoscopic pattern and the lower row denotes corresponding projection pattern. These two-breather wave phenomena can be seen as the superposition of two one-breather waves with same velocity and amplitude.

By calculating the function $\tilde{m}_{ij}^{(n)}$ under the complex conjugation symmetry restriction (3.6), we find that when $p_{31}=-p_{32}$ is hold and other parameters are chosen the same relations as the two-breather wave solution, it leads to three-breather solution with the functions

$$f(x,y,t)=\begin{vmatrix} m_{11}^{(0)} & m_{12}^{(0)} & m_{13}^{(0)} \\ m_{21}^{(0)} & m_{22}^{(0)} & m_{23}^{(0)} \\ m_{31}^{(0)} & m_{32}^{(0)} & m_{33}^{(0)} \end{vmatrix}, \quad g(x,y,t)=\begin{vmatrix} m_{11}^{(1)} & m_{12}^{(1)} & m_{13}^{(1)} \\ m_{21}^{(1)} & m_{22}^{(1)} & m_{23}^{(1)} \\ m_{31}^{(1)} & m_{32}^{(1)} & m_{33}^{(1)} \end{vmatrix},$$

where the functions $\tilde{m}_{ij}^{(0)}, \tilde{m}_{ij}^{(1)}$ are given by (3.17) with $i,j=1,2,3$. In order to display the three-breather wave behaviors in detail, we set the same parameter values as Fig. 1 except $p_{32}=1+0.5\mathrm{i}$. Then the corresponding phenomena are presented in Figs. 2(a)-2(c), from which we find an inclined single breather crossing the two-breather wave. Furthermore, we also present four-breather wave behaviors. The parameters p_{11} and p_{12} are chosen the same

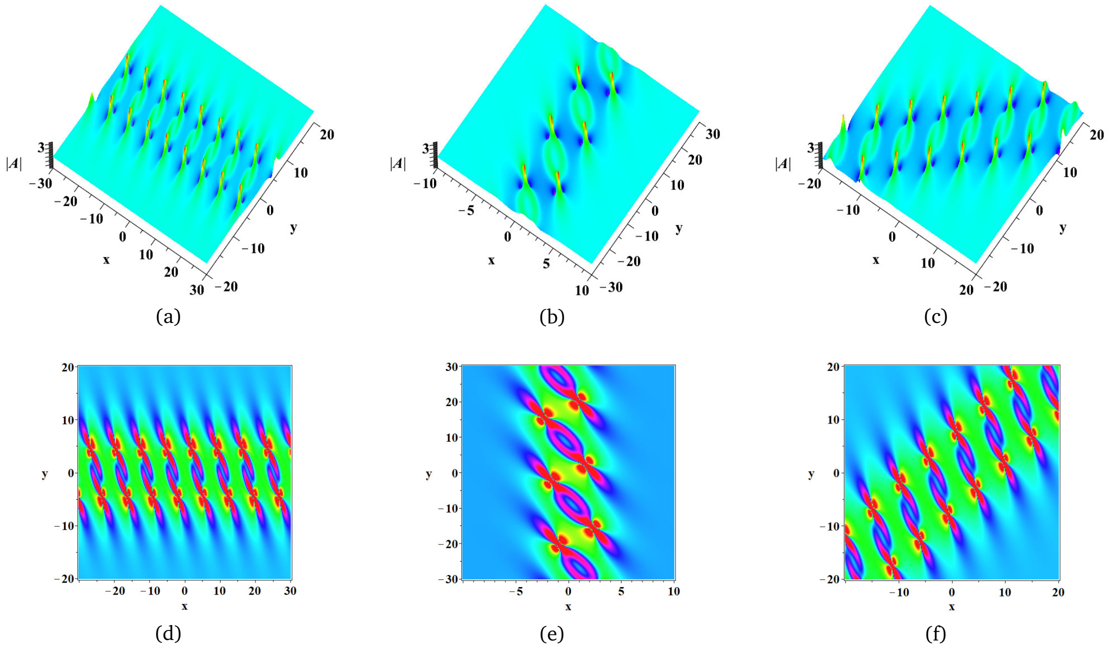


Figure 1: Two-breather wave, $N=2, \xi_{ij}^{(0)}=\eta_{ij}^{(0)}=0,\ 1\le i,j\le 2$. (a): $p_{11}=1.2+\mathrm{i}, p_{12}=1.2+0.125\mathrm{i}$. (b): $p_{11}=1.2+\mathrm{i}, p_{12}=2.2+0.125\mathrm{i}$. (c): $p_{11}=1.2+\mathrm{i}, p_{12}=1.4+0.125\mathrm{i}$.

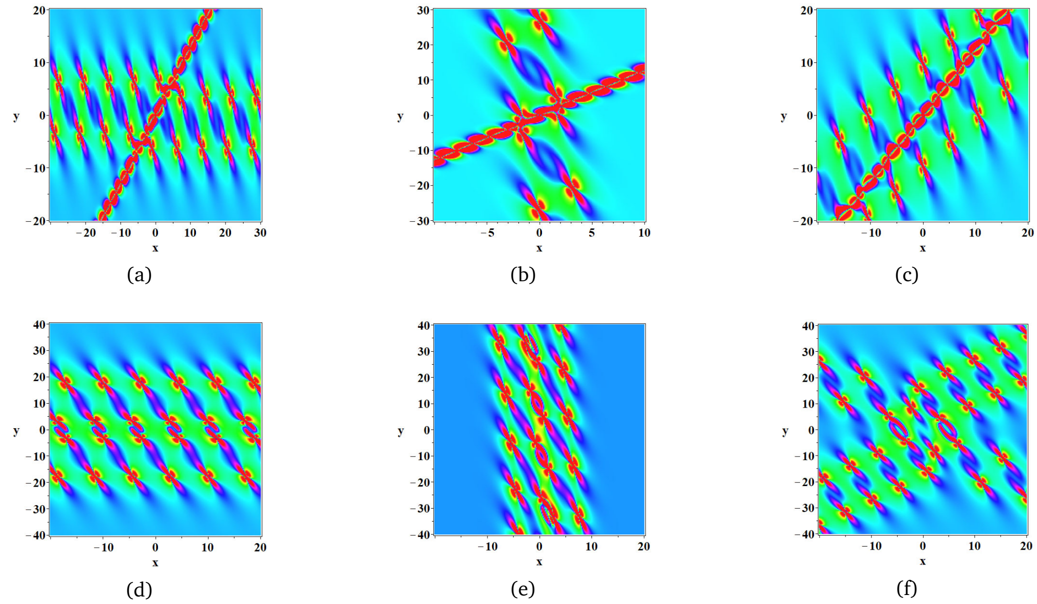


Figure 2: Three-breather wave, $N=3, \xi_{ij}^{(0)}=\eta_{ij}^{(0)}=0,\ 1\le i,j\le 3$. (a): $p_{11}=1.2+\mathrm{i}, p_{12}=1.2+0.125\mathrm{i}, p_{32}=1+0.5\mathrm{i}$. (b): $p_{11}=1.2+\mathrm{i}, p_{12}=2.2+0.125\mathrm{i}, p_{32}=1+0.5\mathrm{i}$. (c): $p_{11}=1.2+\mathrm{i}, p_{12}=1.4+0.125\mathrm{i}, p_{32}=1+0.5\mathrm{i}$. Four-breather wave, $N=4, \xi_{ij}^{(0)}=\eta_{ij}^{(0)}=0,\ 1\le i,j\le 4$. (d): $p_{21}=1.4+\mathrm{i}, p_{22}=1.4+0.125\mathrm{i}$. (e): $p_{21}=1.3+\mathrm{i}, p_{22}=2.5+0.125\mathrm{i}$. (f): $p_{21}=1.5+\mathrm{i}, p_{22}=1.3+0.125\mathrm{i}$.

values as Fig. 1. We choose suitable p_{21} and p_{22} in the four-breather wave solution. Then the four-breather superposition phenomenon is also obtained. Actually, the four-breather wave is superimposed on the basis of the two-breather in Fig. 1 with another two-breather wave, which is distributed on outside of initial two-breather wave for Figs. 2(d)-2(e) and corresponds to the wave sloping to the left for Fig. 2(f).

4. High-Order Rogue Wave and Multi-Bright-Dark Soliton Interaction

We have constructed the multi-breather wave solutions in above. In this section, we prepare to derive high-order rogue wave, multi-bright-dark soliton and their coexistence behavior. To this end, we firstly give the following lemma and introduce a general tau-function.

Lemma 4.1. *Assume that the tau-function of the bilinear form* (2.6)-(2.7) *has the form of*

$$\tau_n = \det_{1\le i,j\le N}\left(m_{ij}^{(n)}\right) \tag{4.1}$$

with

$$m_{ij}^{(n)} = c_j\delta_{i,j} + \frac{1}{p_i+q_j}\left(-\frac{p_i}{q_j}\right)^n \mathrm{e}^{\xi_i+\eta_j},$$

where

$$\xi_i = p_i x + \mathrm{i}p_i^2 t - \frac{1}{p_i}y + \xi_i^{(0)},$$
$$\eta_j = q_j x - \mathrm{i}q_j^2 t - \frac{1}{q_j}y + \eta_j^{(0)}.$$

Introducing a derivative operator to this tau-function yields that

$$\begin{aligned}A_i^{(n_i)}(p_i)B_j^{(n_j)}(q_j)m_{ij}^{(n)} = {} & c_j\delta_{i,j} + \left(-\frac{p_i}{q_j}\right)^n \mathrm{e}^{\xi_i+\eta_j}\sum_{k=0}^{n_i} a_{ik}(p_i)(p_i\partial p_i + n + \tilde{\xi}_i)^{n_i-k} \\ & \times\sum_{l=0}^{n_j} b_{jl}(q_j)(q_j\partial q_j - n + \tilde{\eta}_j)^{n_j-l}\frac{1}{p_i+q_j},\end{aligned}$$

where

$$A_i^{(n_i)}(p) = \sum_{k=0}^{n_i} a_{ik}(p)(p\partial p)^{n_i-k}, \quad B_j^{(n_j)}(q) = \sum_{l=0}^{n_j} b_{jl}(q)(q\partial q)^{n_j-l},$$
$$\tilde{\xi}_i = p_i x + 2\mathrm{i}p_i^2 t + \frac{1}{p_i}y + \xi_i^{(0)}, \quad \tilde{\eta}_j = q_j x - 2\mathrm{i}q_j^2 t + \frac{1}{q_j}y + \eta_j^{(0)}.$$

For a given reduction condition

$$\tau_n = C\tilde{\tau}_n, \quad C = \prod_{j=1}^{N}\mathrm{e}^{\xi_j+\eta_j}, \tag{4.2}$$

we derive a reduced tau-function.

Theorem 4.1. *The Eq.* (1.3) *has the general solution*

$$A=\sqrt{2}e^{ikt}\frac{g(x,y,t)}{f(x,y,t)}, \tag{4.3}$$

where

$$f(x,y,t)=\tilde{\tau}_0(x,y,t),$$
$$g(x,y,t)=\tilde{\tau}_1(x,y,t).$$

Under the reduction condition (4.2)*, the reduced tau-function is given by*

$$\tilde{\tau}_n=\det_{1\le i,j\le N}\left(\tilde{m}_{ij}^{(n)}\right), \tag{4.4}$$

and

$$\begin{aligned}\tilde{m}_{ij}^{(n)}=&c_j\delta_{ij}e^{-\xi_i-\eta_j}+\left(-\frac{p_i}{q_j}\right)^n\sum_{k=0}^{n_i}a_{ik}(p_i)(p_i\partial p_i+n+\tilde{\xi}_i)^{n_i-k}\\&\times\sum_{l=0}^{n_j}b_{jl}(q_j)(q_j\partial q_j-n+\tilde{\eta}_j)^{n_j-l}\frac{1}{p_i+q_j}.\end{aligned} \tag{4.5}$$

Proposition 4.1. *By considering the complex conjugation symmetry conditions* (2.3)*, the constant parameters in the Eq.* (4.5) *satisfy with the relations*

$$\begin{array}{llll}\xi_i^{(0)}+\eta_i^{(0)}=real, & r_{N_1+i}=r_i, & c_{N_1+j}=-c_j^*, & a_{N_1+i,k}=b_{ik}^*,\\ b_{N_1+j,l}=a_{jl}^*, & p_{N_1+i}=-q_i^*, & q_{N_1+j}=-p_j^*, & i,j=1,2,\dots,N_1,\\ \xi_i^{(0)}=\eta_i^{(0)}, & a_{i,k}=b_{i,k}^*, & b_{j,l}=a_{j,l}^*, & \\ c_j=-c_j^*, & p_i=q_i^*, & i,j=2N_1+1. & \end{array}$$

Proof. The proof is similar to Proposition 3.1. □

Remark 4.1. General solution (4.3) with (4.1) is nonsingular for any constant column vector $v=(v_1,v_2,\dots,v_n)^T$, if $p_i=q_i^*$ such that the tau-function τ_0 is positive definite, if $c_i>0,p_{iR}>0$; the tau-function τ_0 is negative definite, if $c_i<0,p_{iR}<0$; the tau-function τ_0 is nonzero, if $c_{iI}\neq 0,p_{iR}\neq 0$.

Proof. In order to verify $f=\tau_0\neq 0$, we note $M=(A_iB_jm_{ij}^{(0)})$. When $c_i>0,p_{iR}>0$ is hold, then we have

$$\begin{aligned}v^\dagger Mv=&\sum_{i,j=1}^{2N_1}v_i^*\left(A_iB_jm_{ij}^{(0)}\right)v_j\\=&\sum_{i,j=1}^{2N_1}v_i^*\left(c_i\delta_{ij}+A_iB_j\frac{1}{p_i+q_j}e^{\xi_i+\eta_j}\right)v_j\end{aligned}$$

$$
\begin{aligned}
&= \sum_{i=1}^{2N_1} c_i v_i^* \delta_{ij} v_j + \sum_{i,j=1}^{2N_1} v_i^* v_j A_i B_j \frac{1}{p_i+q_j} \mathrm{e}^{\xi_i+\eta_j} \\
&= \sum_{i=1}^{2N_1} c_i |v_i|^2 + \sum_{i,j=1}^{2N_1} v_i^* v_j A_i B_j \frac{1}{p_i+q_j} \mathrm{e}^{\xi_i+\eta_j} \\
&= \sum_{i=1}^{2N_1} c_i |v_i|^2 + \int_{-\infty}^{x} \sum_{i,j=1}^{2N_1} v_i^* v_j A_i B_j \mathrm{e}^{\xi_i+\eta_j} dx \\
&= \sum_{i=1}^{2N_1} c_i |v_i|^2 + \int_{-\infty}^{x} \left| \sum_{i=1}^{2N_1} v_i^* A_i \mathrm{e}^{\xi_i} \right|^2 dx > 0.
\end{aligned}
$$

Assuming that $c_i < 0, p_{iR} < 0$, we have

$$
\begin{aligned}
v^\dagger M v &= \sum_{i,j=1}^{2N_1} v_i^* \left(A_i B_j m_{ij}^{(0)} \right) v_j \\
&= \sum_{i,j=1}^{2N_1} v_i^* \left(c_i \delta_{ij} + A_i B_j \frac{1}{p_i+q_j} \mathrm{e}^{\xi_i+\eta_j} \right) v_j \\
&= \sum_{i=1}^{2N_1} c_i |v_i|^2 - \int_{x}^{+\infty} \sum_{i,j=1}^{2N_1} v_i^* v_j A_i B_j \mathrm{e}^{\xi_i+\eta_j} dx \\
&= \sum_{i=1}^{2N_1} c_i |v_i|^2 - \int_{x}^{+\infty} \left| \sum_{i=1}^{2N_1} v_i^* A_i \mathrm{e}^{\xi_i} \right|^2 dx < 0.
\end{aligned}
$$

Assuming that $c_{iI} \neq 0, p_{iR} \neq 0$, we have

$$
\begin{aligned}
v^\dagger M v &= \sum_{i,j=1}^{2N_1} v_i^* \left(A_i B_j m_{ij}^{(0)} \right) v_j \\
&= \sum_{i,j=1}^{2N_1} v_i^* \left(c_i \delta_{ij} + A_i B_j \frac{1}{p_i+q_j} \mathrm{e}^{\xi_i+\eta_j} \right) v_j \\
&= \sum_{i=1}^{2N_1} c_i v_i^* \delta_{ij} v_j + \sum_{i,j=1}^{2N_1} v_i^* v_j A_i B_j \frac{1}{p_i+q_j} \mathrm{e}^{\xi_i+\eta_j} \\
&= \sum_{i=1}^{2N_1} c_i |v_i|^2 + \sum_{i,j=1}^{2N_1} v_i^* v_j A_i B_j \frac{1}{p_i+q_j} \mathrm{e}^{\xi_i+\eta_j}.
\end{aligned}
$$

It is obvious that the second term of last expression is real, which leads to $v^\dagger M v \neq 0$, when $c_{iI} \neq 0$. □

Function $\tilde{m}_{ij}$ contains exponential term and rational polynomial with respective to the variables x, y, t. Then we can obtain a purely polynomial function (rational), a purely exponential function (soliton solution) and a mixed function (semi-rational solution) by choosing the different values of c_i, n_i. We consider three special cases of this and present their dynamics behaviors.

Case I. If we choose $c_i = 0$, $n_i = 1$, $i = 1,2,\ldots,2N_1$, $c_{2N_1+1} \neq 0, n_{2N_1+1} = 0$ in (4.5), there exists rogue wave solution with the reduced function

$$\tilde{m}_{ij}^{(n)} = \begin{cases} \left(-\dfrac{p_i}{q_j}\right)^n \left[\dfrac{p_i q_j}{(p_i+q_j)^2} + P_{ij}Q_{ij}\right]\dfrac{1}{p_i+q_j}, & i,j = 1,2,\ldots,2N_1, \\ \left(-\dfrac{p_i}{q_j}\right)^n P_{ij}\dfrac{1}{p_i+q_j}, & i = 1,2,\ldots,2N_1, \quad j = 2N_1+1, \\ \left(-\dfrac{p_i}{q_j}\right)^n Q_{ij}\dfrac{1}{p_i+q_j}, & j = 1,2,\ldots,2N_1, \quad i = 2N_1+1, \\ c_j e^{-\xi_i-\eta_j} + \left(-\dfrac{p_i}{q_j}\right)^n \dfrac{1}{p_i+q_j}, & i,j = 2N_1+1, \\ P_{ij} = \dfrac{-p_i}{p_i+q_j} + \tilde{\xi}_i + n + a_{i1}, \quad Q_{ij} = \dfrac{-q_i}{p_i+q_j} + \tilde{\eta}_j - n + b_{j1}. \end{cases} \tag{4.6}$$

If we let $N = 2$ in Theorem 4.1, the functions $\tilde{m}_{ij}^{(n)}$ in (4.5) reduce to (4.6). This solution presents the second-order rogue wave on constant background. Then its expression can be given as

$$A = \sqrt{2}e^{ikt}\frac{\vartheta_{11}\vartheta_{21} + \vartheta_{31}\vartheta_{41}}{\vartheta_{10}\vartheta_{20} + \vartheta_{30}\vartheta_{40}},$$

where

$$\begin{aligned} \vartheta_{1n} = {} & \frac{p_1q_1}{(p_1+q_1)^3} + \frac{1}{p_1+q_1}\left(\frac{-p_1}{p_1+q_1} + p_1x + 2ip_1^2t + \frac{1}{p_1}y + n + a_{11}\right) \\ & \times\left(\frac{-q_1}{p_1+q_1} + q_1x - 2iq_1^2t + \frac{1}{q_1}y - n + b_{11}\right), \\ \vartheta_{2n} = {} & \frac{q_1^*p_1^*}{(q_1^*+p_1^*)^3} + \frac{1}{q_1^*+p_1^*}\left(\frac{-q_1^*}{q_1^*+p_1^*} - q_1^*x + 2iq_1^{*2}t - \frac{1}{q_1^*}y + n + b_{11}^*\right) \\ & \times\left(\frac{-p_1^*}{q_1^*+p_1^*} - p_1^*x - 2ip_1^{*2}t - \frac{1}{p_1^*}y - n + a_{11}\right), \\ \vartheta_{3n} = {} & \frac{-p_1p_1^*}{(p_1-p_1^*)^3} + \frac{1}{p_1-p_1^*}\left(\frac{-p_1}{p_1-p_1^*} + p_1x + 2ip_1^2t + \frac{1}{p_1}y + n + a_{11}\right) \\ & \times\left(\frac{p_1^*}{p_1-p_1^*} - p_1^*x - 2ip_1^{*2}t - \frac{1}{p_1^*}y - n + a_{11}^*\right), \\ \vartheta_{4n} = {} & \frac{-q_1q_1^*}{(q_1-q_1^*)^3} + \frac{1}{q_1-q_1^*}\left(\frac{q_1^*}{q_1-q_1^*} - q_1^*x + 2iq_1^{*2}t - \frac{1}{q_1^*}y + n + b_{11}^*\right) \\ & \times\left(\frac{-q_1}{q_1-q_1^*} + q_1x - 2iq_1^2t + \frac{1}{q_1}y - n + b_{11}\right). \end{aligned}$$

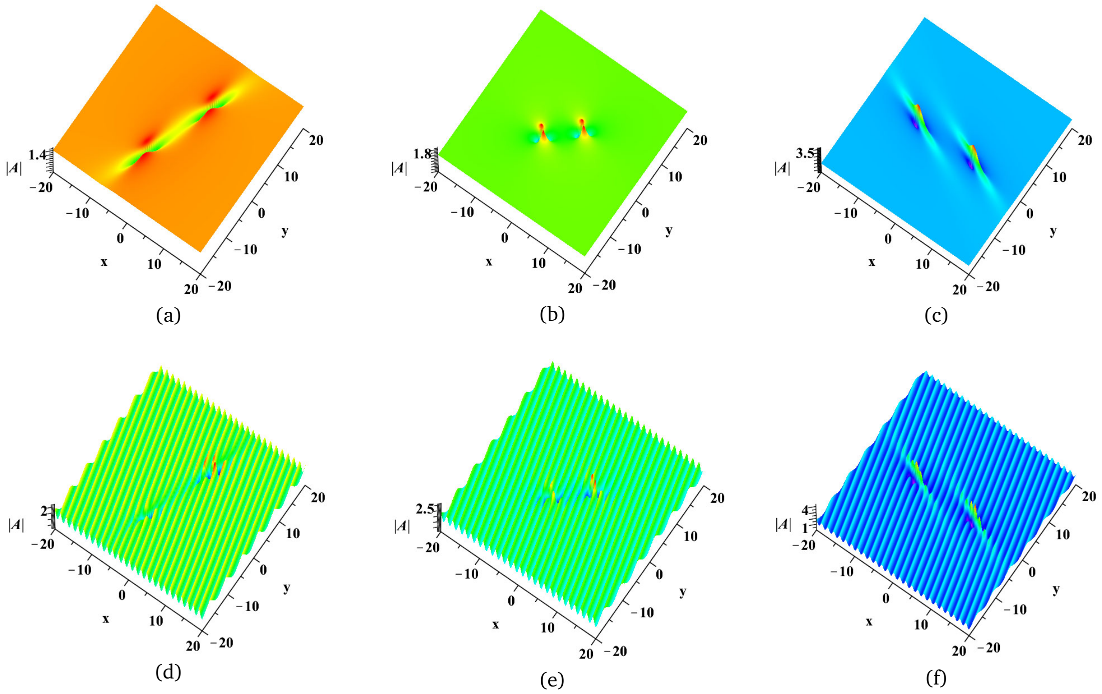


Figure 3: Dark, four-petal, bright rogue waves presented by the solution (4.3)-(4.4) with (4.6) on constant background (a)-(c): $N=2, p_1=2/3+(4/3)\mathrm{i}$ (a), $2/3+(2/3)\mathrm{i}$ (b), $2/3+(1/5)\mathrm{i}$ (c), $\xi_1^{(0)}=\xi_2^{(0)}=0, a_{11}=b_{11}=5$ and periodic background (d)-(f): $N=3, \xi_3^{(0)}=0, a_{31}=0, p_3=c_3=2\mathrm{i}$.

Letting $p_1=(2/3+(4/3)\mathrm{i}, 2/3+(2/3)\mathrm{i}, 2/3+(1/5)\mathrm{i})$ in the second-order rogue wave solution, Figs. 3(a)-3(c) present three types of different states, which are dark-dark, four-petal and bright-bright rogue wave, respectively. If set $p_3=c_3=2\mathrm{i}$ in third-order solution with $N=3$, other parameters are the same as second-order rogue wave solution. As shown in Figs. 3(d)-3(f), there is no higher-order rogue wave, but the initial double rogue wave appearing on the periodic background. Due to the complexity of the solution with $N=3$, we do not present its expression.

Case II. If we choose $c_i\neq 0, n_i=0,\ i=1,2,\ldots,N$, the tau-function of soliton solution possesses the element

$$\tilde{m}_{ij}^{(n)}=c_j\delta_{i,j}\mathrm{e}^{-\xi_i-\eta_j}+\left(-\frac{p_i}{q_j}\right)^n\frac{1}{p_i+q_j},\quad i,j=1,2,\ldots,N. \tag{4.7}$$

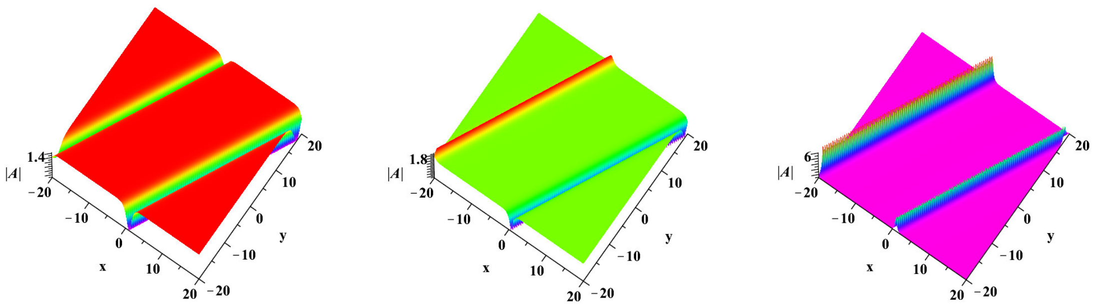
Choosing the parameter $N=2$ leads to the two-soliton solution

$$A=\frac{\sqrt{2}\mathrm{e}^{\mathrm{i}kt}}{p_1^*q_1}\frac{p_1^*q_1\varphi_1\mathrm{e}^{-\xi_1-\eta_1-\xi_2-\eta_2}-|q_1|^2\varphi_2\mathrm{e}^{-\xi_1-\eta_1}-|p_1|^2\varphi_3\mathrm{e}^{-\xi_2-\eta_2}+p_1q_1^*\varphi_4}{\varphi_1\mathrm{e}^{-\xi_1-\eta_1-\xi_2-\eta_2}+\varphi_2\mathrm{e}^{-\xi_1-\eta_1}+\varphi_3\mathrm{e}^{-\xi_2-\eta_2}+\varphi_4},$$

where

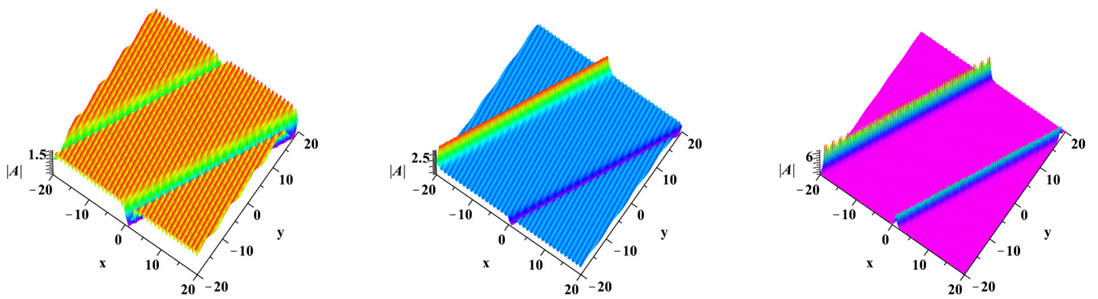
$$
\begin{aligned}
\varphi_1 = \Big(& q_1^{*2}p_1^*p_1 + q_1^{*2}p_1^*q_1 - q_1^{*2}p_1^2 - q_1^{*2}p_1q_1 + q_1^*p_1^{*2}p_1 + q_1^*p_1^{*2}q_1 - q_1^*p_1^*p_1^2 - q_1^*p_1^*q_1^2 \\
& - 2q_1^*p_1^*p_1q_1 + q_1^*p_1^2q_1 + q_1^*p_1q_1^2 - p_1^{*2}p_1q_1 - p_1^{*2}q_1^2 + p_1^*p_1^2q_1 + p_1^*p_1q_1^2\Big)|c_1|^2, \\
\varphi_2 = \big(& q_1^*p_1^*p_1 + q_1^*p_1^*q_1 - q_1^*p_1^2 - q_1^*p_1q_1 - p_1^*p_1q_1 - p_1^*q_1^2 + p_1^2q_1 + p_1q_1^2\big)c_1, \\
\varphi_3 = \big(& q_1^{*2}p_1^* - q_1^{*2}p_1 + q_1^*p_1^{*2} - q_1^*p_1^*p_1 - q_1^*p_1^*q_1 + q_1^*p_1q_1 - p_1^{*2}q_1 + p_1^*p_1q_1\big)c_1^*, \\
\varphi_4 = \;& q_1^*p_1^* + q_1q_1^* + p_1p_1^* + p_1q_1.
\end{aligned}
$$

In the second case, we suppose $c_i \neq 0, n_i = 0$, that is to say the tau-function contains the exponential function without the partial derivative operators of p_i, q_j. We obtain the multi-bright-dark soliton solution. In order to display the dynamics behavior of soliton solution, we set $p_1 = 1+\mathrm{i}, p_2 = 1-\mathrm{i}$. As displayed in Fig. 4, the shape of soliton changes with the value of c_1. When $c_1 = (3-\mathrm{i}, -1.6\mathrm{i}, -3-1.2\mathrm{i})$, they corresponds to the dark-dark, bright-dark and bright-bright solitons, respectively. Furthermore, these multi-solitons happen to collide with each other and move to the opposite direction. Figs. 4-5 exhibit the propagation of



(a) (b) (c)

Figure 4: Dark-dark soliton, bright-dark soliton, bright-bright soliton presented by the solution (4.3)-(4.4) with parameters: $N=2, \xi_1^{(0)} = \xi_2^{(0)} = 0, p_1 = 1+\mathrm{i}, p_2 = 1-\mathrm{i}, c_1 = 3-\mathrm{i}$ (a), $-1.6\mathrm{i}$ (b), $-3-1.2\mathrm{i}$ (c).



(a) (b) (c)

Figure 5: Periodic-type dark-dark soliton, bright-dark soliton, bright-bright soliton in the solution (4.3) presented by the solution (4.3)-(4.4), $N=3, \xi_3^{(0)} = 0, p_3 = 3\mathrm{i}, c_3 = -5\mathrm{i}$.

two-soliton at the time $t=-5$. Let $p_3=3\mathrm{i}, c_3=-5\mathrm{i}$ in the solution (4.7) with $N=3$. Other parameters are the same as Fig. 4. As shown in Fig. 5, it displays periodic-type dark-dark, bright-dark and bright-bright solitons. The periodicity of the soliton background increases followed by increasing $|p_3|$.

Case III. For $c_i \neq 0$, $n_i=1$, $i=1,2,\ldots,N$, the interaction solution admits the function

$$\tilde{m}_{ij}^{(n)}=\begin{cases} c_j\delta_{i,j}\mathrm{e}^{-\xi_i-\eta_j}+\left(-\dfrac{p_i}{q_j}\right)^n\left[\dfrac{p_iq_j}{(p_i+q_j)^2}+P_{ij}Q_{ij}\right]\dfrac{1}{p_i+q_j}, & i,j=1,2,\ldots,2N_1,\\ \left(-\dfrac{p_i}{q_j}\right)^n P_{ij}\dfrac{1}{p_i+q_j}, & i=1,2,\ldots,2N_1,\quad j=2N_1+1,\\ \left(-\dfrac{p_i}{q_j}\right)^n Q_{ij}\dfrac{1}{p_i+q_j}, & j=1,2,\ldots,2N_1,\quad i=2N_1+1,\\ c_j\mathrm{e}^{-\xi_i-\eta_j}+\left(-\dfrac{p_i}{q_j}\right)^n\dfrac{1}{p_i+q_j}, & i,j=2N_1+1,\\ P_{ij}=\dfrac{-p_i}{p_i+q_j}+\tilde{\xi}_i+n+a_{i1},\quad Q_{ij}=\dfrac{-q_i}{p_i+q_j}+\tilde{\eta}_j-n+b_{j1}. \end{cases}$$

Choosing $N=2$, we obtain the second-order interaction solution

$$A=\sqrt{2}\mathrm{e}^{\mathrm{i}kt}\frac{\left(c_1\mathrm{e}^{-\xi_1-\eta_1}-(p_1/q_1)\theta_{11}\right)\left(c_1^*\mathrm{e}^{-\xi_2-\eta_2}-(q_1^*/p_1^*)\theta_{21}\right)-((p_1q_1^*)/(q_1p_1^*))\theta_{31}\theta_{41}}{\left(c_1\mathrm{e}^{-\xi_1-\eta_1}-\theta_{10}\right)\left(c_1^*\mathrm{e}^{-\xi_2-\eta_2}-\theta_{20}\right)-\theta_{30}\theta_{40}},$$

where

$$\begin{aligned}\theta_{1n}=&\frac{p_1q_1}{(p_1+q_1)^3}+\frac{1}{p_1+q_1}\left(\frac{-p_1}{p_1+q_1}+p_1x+2\mathrm{i}p_1^2t+n+\frac{1}{p_1}y+a_{11}\right)\\ &\times\left(\frac{-q_1}{p_1+q_1}+q_1x-2\mathrm{i}q_1^2t-n+\frac{1}{q_1}y+b_{11}\right),\\ \theta_{2n}=&\frac{q_1^*p_1^*}{(p_1^*+q_1^*)^3}-\frac{1}{p_1^*+q_1^*}\left(\frac{-q_1^*}{q_1^*+p_1^*}-q_1^*x+2\mathrm{i}q_1^{*2}t+n-\frac{1}{q_1}y^*+b_{11}^*\right)\\ &\times\left(\frac{p_1^*}{q_1^*+p_1^*}+p_1^*x+2\mathrm{i}p_1^{*2}t+n+\frac{1}{p_1}y^*-a_{11}^*\right),\\ \theta_{3n}=&-\frac{p_1p_1^*}{(p_1-p_1^*)^3}-\frac{1}{p_1-p_1^*}\left(\frac{p_1}{p_1^*-p_1}+p_1x+2\mathrm{i}p_1^2t+n+\frac{1}{p_1}y+a_{11}\right)\\ &\times\left(\frac{p_1^*}{p_1^*-p_1}+p_1^*x+2\mathrm{i}p_1^{*2}t+n+\frac{1}{p_1^*}y-a_{11}^*\right),\\ \theta_{4n}=&-\frac{q_1^*q_1}{(q_1-q_1^*)^3}+\frac{1}{q_1-q_1^*}\left(\frac{q_1^*}{q_1-q_1^*}-q_1^*x+2\mathrm{i}q_1^{*2}t+n-\frac{1}{q_1}y^*+b_{11}^*\right)\\ &\times\left(\frac{q_1}{q_1^*-q_1}+q_1x-2\mathrm{i}q_1^2t-n+\frac{1}{q_1}y+b_{11}\right).\end{aligned}$$

Choosing $c_i \neq 0, n_i = 1$, we obtain an interaction solution in the Case III. As shown in Fig. 6, the interaction of rogue wave and bright, dark solitons is exhibited. From that, we find that the three different rogue waves coexist with dark-dark soliton, bright-dark soliton and bright-bright soliton. Fig. 6 presents the state at the time $t = -5$. When the rogue wave approaches to the soliton, the latter happens to deform. Especially, the rogue wave exists in the middle of two individual solitons, while the rogue waves are distributed to both sides of the solitons. Similar to the phenomena in Fig. 6, if $N = 3$ is hold in the solution (4.4) and the other parameters remain unchanged. Thus, an interesting coexistence phenomenon of rogue waves with multi-bright-dark soliton on periodic background is displayed in Fig. 7.

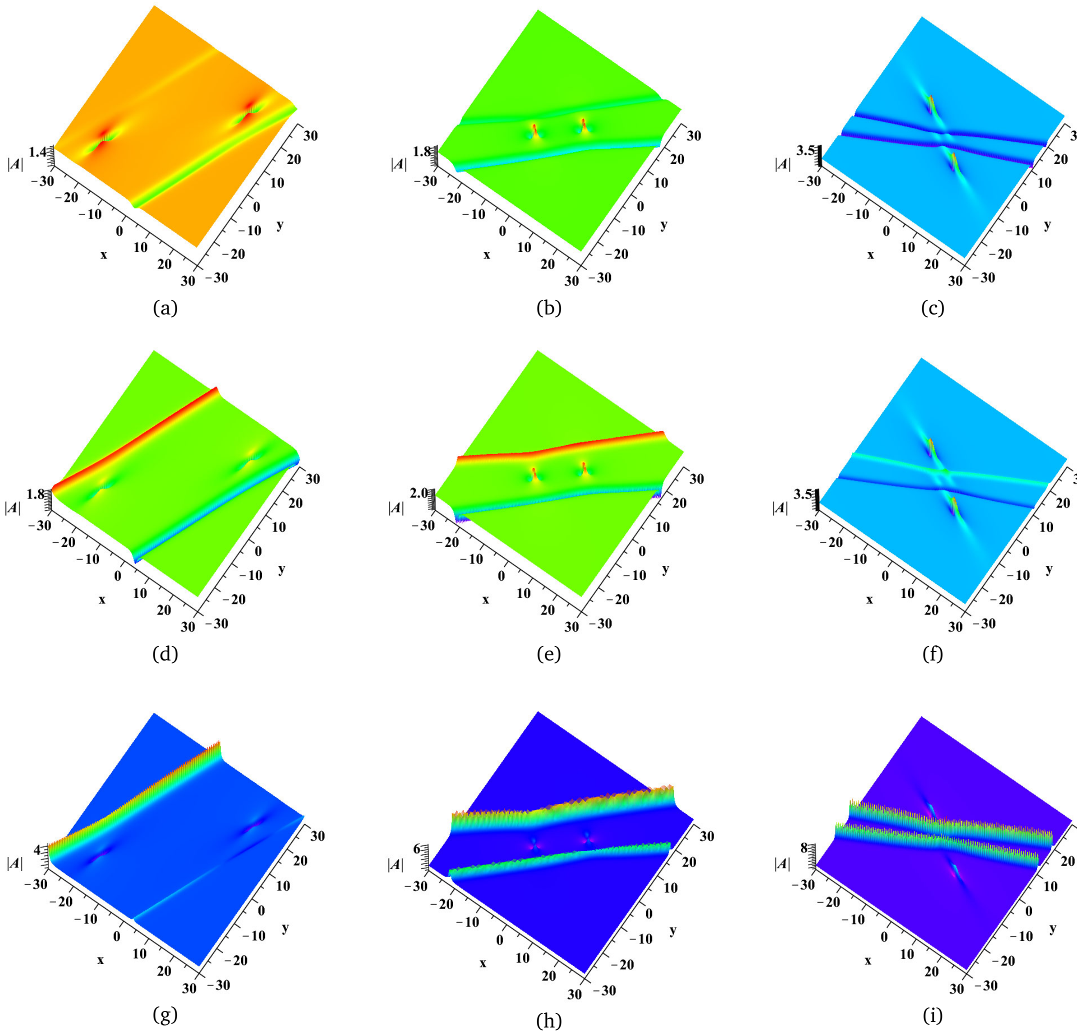
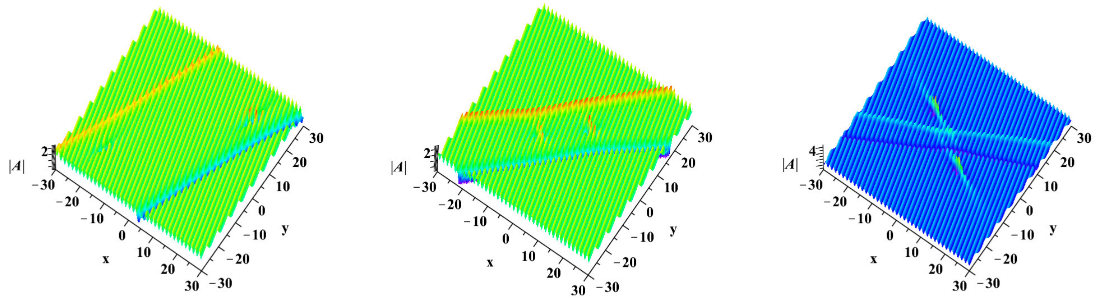


Figure 6: Interactions of dark-dark, four-petal, bright-bright rogue wave and dark-dark, bright-dark and bright-bright solitons presented by the solution (4.3)-(4.4), $N = 2, \xi_1^{(0)} = \xi_2^{(0)} = 0, a_{11} = b_{11} = 0, p_1 = 2/3 + (4/3)\mathrm{i}, 2/3 + (2/3)\mathrm{i}, 2/3 + (1/5)\mathrm{i},\ c_1 = (3 - \mathrm{i}, -1.6\mathrm{i}, -3 - 1.2\mathrm{i})$.



(a) (b) (c)

Figure 7: Interactions of dark-dark, four-petal, bright-bright rogue wave and bright-dark soliton on periodic background presented by the solution (4.3)-(4.4), $N=3, \xi_3^{(0)}=0,\ p_3=c_3=2\mathrm{i}, c_1=-1.6\mathrm{i}, a_{31}=0$.

5. Conclusion

In this paper, the (2+1)-dimensional nonlocal Fokas system has been systematically investigated by means of the Hirota's bilinear method and the KP hierarchy reduction method. Based on a given bilinear KP hierarchy, we have derived their various analytic solutions, including a multi-breather wave solution, a rogue wave solution and a multi-bright-dark soliton solution. These solutions are constructed by considering two classes of determinant functions of Gram-type. Furthermore, the complex conjugation symmetry condition is used to derive parameter restrictions in two Gram-type tau functions for the bilinear form. Firstly, we have derived a multi-breather solution in Theorem 3.1. The dynamics behaviors of multi-breather waves have been displayed via suitable parameters. Theorem 4.1 provides a general solution with the tau-function (4.4)-(4.5). If c_i and n_i are chosen as the different special values, respectively, then there exist three types of nonlinear phenomena, such as the high-order rogue wave (Case I), multi-bright-dark soliton (Case II) and interaction of rogue wave and soliton (Case III). Actually, these phenomena appear on constant or periodic background if N is positive even or odd.

The combination of the Hirota's bilinear method and the KP reduction method is a very effective tool to the study of soliton, breather wave and rogue wave solutions even discretization for nonlinear integrable systems. Ohta and Yang derived the high-order rogue wave solution for the NLS equation and the Ablowitz-Ladik equation [32, 35]. The work of Feng *et al.* mainly includes the construction of a general solution and an integrable discretizations for the Yajima-Oikawa system [8–10], Degasperis-Procesi equation [15–17] and classic Cammasa-Holm equation [14,31] and modified Cammasa-Holm equation [41]. He *et al.* have study the DS equation [38] and its nonlocal models [37]. Wu constructed bilinear forms and obtained interesting analytical solutions of the Sasa-Satsuma equation [18, 44] and the nonlocal Mel'nikov equation [26]. Although known work has been extended to the multi-component couple models, nonlocal models, there exist few works to the study of the high-order, high-dimensional even more complicated integrable systems. For their bilinear forms and the relations with the KP hierarchy, there are still many questions need to deal with.

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