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Exact Meromorphic Stationary Solutions of the Cubic-Quintic Swift-Hohenberg Equation

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Abstract. In this paper, we study an ODE of the form

$$b_0u^{(4)}+b_1u''+b_2u+b_3u^3+b_4u^5=0, \quad '=\frac{d}{dz},$$

which includes, as a special case, the stationary case of the cubic-quintic Swift-Hohenberg equation. Based on Nevanlinna theory and Painlevé analysis, we first show that all its meromorphic solutions are elliptic or degenerate elliptic. Then we obtain them all explicitly by the method introduced in [7].

Key Words: Meromorphic solutions, Cubic-Quintic Swift-Hohenberg equation, Nevanlinna theory.

AMS Subject Classifications: 35Q53

1 Introduction

The real Swift-Hohenberg equation with a cubic-quintic nonlinearity

$$\partial_t u=au+bu^3-cu^5-d(q_0^2+\partial_x^2)^2u, \quad a,b,c,d,q_0\in\mathbb{R}, \tag{1.1}$$

has been extensively studied as a model equation to test the bifurcation of solutions of certain PDEs. For detailed results, see [12] and the references therein. Almost all the work concerning (1.1) is done by numerical method, few work has been undertaken on finding exact solutions of the stationary case of (1.1) in explicit form. Therefore the devotion of this paper to search for exact meromorphic solutions of (1.2) has both mathematical interests and physical significance. Here, meromorphic functions mean the functions

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meromorphic on the whole complex plane. For the stationary case, we have 0 on the l.h.s of (1.1) and it motivates the author to study a general ODE

$$b_0u^{(4)}+b_1u''+b_2u+b_3u^3+b_4u^5=0, \quad '=\frac{d}{dz}, \tag{1.2}$$

where $b_i\in\mathbb{C}$, $i=1,2,3,4,5$ and $b_0b_4\neq 0$. For $b_4=0$, which corresponds to real cubic Swift-Hohenberg (RCSH) equation, the meromorphic solutions of (1.2) have been studied in [6].

Recently, Kao and Knobloch [12] have studied the ODE (1.2) with two arbitrary constants b_2 and b_3. In our paper, we consider the ODE (1.2) with all the coefficients arbitrary. Compared with their work, the main differences are as follows. First we prove that (1.2) does not have any entire solutions and then we explicitly find *all* its meromorphic solutions with at least one pole on $\mathbb{C}$. In other words, we have found *all* the meromorphic solutions whether or not they have poles. In addition, by applying Proposition 2.1, it is shown that one can make use of the same method as we do in this paper to explicitly find all the (traveling wave) meromorphic solutions of many other ODEs and PDEs. In Section 4, we will present some new real solutions of the ODE (1.2) by choosing specific coefficients in (1.2).

Without loss of generality, we may assume $b_0=1$ and $b_4=-3/2$ by the transformation $u\mapsto ku$ with $k=\sqrt[4]{-3/2b_4}$. Multiplying (1.2) by u' and then integrating the resulting equation yield

$$4u'u'''-2(u'')^2+2b_1(u')^2+2b_2u^2+b_3u^4-u^6=c, \tag{1.3}$$

where $c\in\mathbb{C}$ is the integration constant.

The structure of this paper can now be explained. In Section 2, we will prove that all meromorphic solutions of the ODE (1.3) must belong to class W (like Weierstrass [9]), consisting of elliptic functions and their successive degeneracies, i.e., elliptic functions, rational functions of one exponential $\exp(kz)$, $k\in\mathbb{C}$ and rational functions of z. Here, W is chosen because Weierstrass proved that functions in class W are the only meromorphic functions satisfying an algebraic addition theorem [15, pp. 490]. The method involved here is a refinement [5] of Eremenko's method used in [8] as well as [9, 10], which is based on the local singularity analysis of meromorphic solutions of ODEs as well as the zero distribution and order of growth of meromorphic solutions. This is a very powerful method. For example, it has been used [9] to characterize all meromorphic traveling wave solutions of the Kuramoto-Sivashinsky (KS) equation. One key point of this method is that an upper bound on the number of poles of solutions to the ODEs being considered in the fundamental region $\mathcal{F}$ can be found. Here, the fundamental region $\mathcal{F}$ refers to $\mathbb{C}$, the period strip or the fundamental parallelogram corresponding to u rational, simply periodic or elliptic respectively. Then this allows us to construct explicitly all the meromorphic solutions of (1.3), as we shall do in Section 3. This can be done by either applying the subequation method [4, 14], or (as we will do in this paper) making use of the following result.

Theorem 1.1 (see [7]). *Let $E[w(z)]=0$ be an autonomous algebraic ordinary differential equation, where $E[w(z)]$ is a polynomial in $w(z)$ and its derivatives and satisfies the following two conditions:*

I. For all the meromorphic solutions of $E[w(z)]=0$ with a pole at $z=0$, there are precisely N different Laurent series expansions around the pole $z=0$ given by

$$w^{(i)}(z)=\sum_{k=1}^{p_i}\frac{c_{-k}^{(i)}}{z^k}+\sum_{k=0}^{\infty}c_k^{(i)}z^k,\quad 0<|z|<\varepsilon_i,\quad i=1,2,\cdots,N, \tag{1.4}$$

II. Substituting $w(z)=kW(z)$ into equation $E[w(z)]=0$ yields an expression with only one term of the highest degree with respect to k.

Then any meromorphic solutions of $E[w(z)]=0$ must be of one of the following forms:

1) Elliptic solutions with periods w_1, w_2,

$$\begin{aligned}w(z)=&\Big\{\sum_{i\in I}\sum_{k=2}^{p_i}\frac{(-1)^k c_{-k}^{(i)}}{(k-1)!}\frac{d^{k-2}}{dz^{k-2}}\Big\}\Big(\frac{1}{4}\Big[\frac{\wp'(z)+B_i}{\wp(z)-A_i}\Big]^2-\wp(z)\Big)\\&+\sum_{i\in I}\frac{c_{-1}^{(i)}(\wp'(z)+B_i)}{2(\wp(z)-A_i)}+\Big\{\sum_{k=2}^{p_{i_0}}\frac{(-1)^k c_{-k}^{(i_0)}}{(k-1)!}\frac{d^{k-2}}{dz^{k-2}}\Big\}\wp(z)+h_0,\end{aligned}$$

where $\wp(z)=\wp(z;w_1,w_2)$, $B_i^2=4A_i^3-g_2A_i-g_3$. One necessary condition for the existence of elliptic solutions is $\sum_{i\in I}c_{-1}^{(i)}+c_{-1}^{(i_0)}=0$.

2) Simply periodic solutions with a period T,

$$\begin{aligned}w(z)=&\frac{\pi}{T}\Big\{\sum_{i\in I}\sum_{k=1}^{p_i}\frac{(-1)^{(k-1)}c_{-k}^{(i)}}{(k-1)!}\frac{d^{k-1}}{dz^{k-1}}\Big\}\frac{A_i\cot(\frac{\pi z}{T})+\frac{\pi}{T}}{A_i-\frac{\pi}{T}\cot(\frac{\pi z}{T})}\\&+\frac{\pi}{T}\Big\{\sum_{k=1}^{p_{i_0}}\frac{(-1)^{(k-1)}c_{-k}^{(i_0)}}{(k-1)!}\frac{d^{k-1}}{dz^{k-1}}\Big\}\cot\Big(\frac{\pi z}{T}\Big)+h_0.\end{aligned}$$

3) Rational solutions,

$$w(z)=\sum_{k=1}^{p_{i_0}}\frac{c_{-k}^{(i_0)}}{z^k}+\sum_{i\in I}\sum_{k=1}^{p_i}\frac{c_{-k}^{(i)}}{(z-a_i)^k}+\sum_{k=0}^{m}h_kz^k,\quad m\geq 0.$$

Here, $\{A_i\}$, $\{B_i\}$, $\{a_i\}$, $\{h_i\}$ are constants, p_i is the order of the pole, and $I=\emptyset$ or $I\subset\{1,2,\cdots,N\}\backslash\{i_0\}$, $1\leq i_0\leq N$.

In the last section, for physical interest, we present some real solutions of (1.2) by choosing some particular parameters and using the knowledge on the theory of elliptic functions.

2 Classification of meromorphic solutions

We may use the local singularity analysis to study the global properties of meromorphic solutions to (1.3). Here, we introduce some basic knowledge on Painlevé Test [2,3].

Suppose $u(z)=\sum_{n=0}^{+\infty}u_n(z-z_0)^{n+p}$ ($u_0\neq 0$, $p<0$, $p\in\mathbb{Z}$) with a pole at $z=z_0$ is a meromorphic solution of $H(f,f',\cdots,f^{(N)})=0$, where H is a polynomial in f and its derivatives with constant coefficients. Then if we plug $u(z)$ into H, we will get an expression of the form $E=\sum_{j=0}^{+\infty}E_j\chi^{j+q}=0$, where q is the smallest integer among the list of leading powers. p, q are determined by several terms in H, which are defined as **dominant term**.

Since $u(z)$ is a solution to $H=0$, we should have $E_j=0$, for all $j\in\mathbb{N}$. For $j=1,2,\cdots$, we can express $E_j=0$ as:

$$E_j\equiv P(u_0;j)u_j+Q_j(\{u_l|l<j\})=0. \tag{2.1}$$

For each j, the above equation is linear in u_j. For such equation to vanish identically, we may have for each j, either

1. u_j is uniquely determined by $P(u_0;j)$ and Q_j, *or*
2. Both $P(u_0;j)$ and Q_j vanish,

otherwise there is no meromorphic function satisfying $H=0$.

Therefore if $P(u_0;j)$ does not have nonnegative integer zeros, then each u_j is uniquely determined by $P(u_0;j)$ and Q_j. The zeros of $P(u_0;j)$, which are the so-called **Fuchs indices**, can be determined by the following indicial equation

$$P(i)=\lim_{\chi\to 0}\chi^{-i-q}\hat{E}'(x,u_0\chi^p)\chi^{i+p}=0, \tag{2.2}$$

where $\hat{E}'(x,u)$ is defined by

$$\hat{E}'(x,u)v\equiv\lim_{\lambda\to 0}\frac{\hat{E}(x,u+\lambda v)-\hat{E}(x,u)}{\lambda}. \tag{2.3}$$

Proposition 2.1. Let H be the same polynomial as above and suppose it has the following dominant term:

$$u^{(r)}(z)+u^k(z),$$

where $r,k\in\mathbb{N}^+$, $k>1$. Then $i=r-2p$ is the only nonnegative integer Fuchs index of $H=0$ if r is even, and there does not exist any nonnegative integer Fuchs index for $H=0$ if r is odd.

Proof. One necessary condition for the existence of meromorphic solutions of $H=0$ with at least one pole $z=z_0$ is that there exists $p(<0)\in\mathbb{Z}$ such that $p-r=kp$. Suppose $u(z)=\sum_{n=0}^{\infty}u_n(z-z_0)^{n+p}$, $u_0\neq 0$ is the solution of $H=0$, then the leading coefficient u_0 should satisfy $u_0p(p-1)\cdots(p-r+1)+u_0^k=0$, which gives $u_0^{k-1}=-p(p-1)\cdots(p-r+1)$.

The indicial equation is then given by

$$\begin{aligned}P(i)&=ku_0^{k-1}+(i+p)(i+p-1)\cdots(i+p-r+1)\\&=-(p-1)\cdots(p-r+1)(p-r)+(i+p)(i+p-1)\cdots(i+p-r+1).\end{aligned}$$

For r even, let $i=r-2p$, then

$$\begin{aligned}P(i)&=-(p-1)\cdots(p-r+1)(p-r)+(r-p)(r-p-1)\cdots(-p+1)\\&=-(p-1)\cdots(p-r+1)(p-r)+(-1)^r(p-r)(p+1-r)\cdots(p-1)=0.\end{aligned}$$

Suppose r is odd and $i\in N$ is a zero of $P(i)$, then

$$(i+p)(i+p-1)\cdots(i+p-r+1)=(p-1)\cdots(p-r+1)(p-r). \tag{2.4}$$

Since the r.h.s of (2.4) is negative, we must have $i\le r-p-1$ otherwise the l.h.s of (2.4) is positive. It is easy to see that the r.h.s of (2.4) is nonzero, so $i\neq -p,-p+1,-p+r-1$. Hence $i\le -p-1$, but $|(i+p)(i+p-1)\cdots(i+p-r+1)|<|(p-1)\cdots(p-r+1)(p-r)|$, which shows that the module of l.h.s of (2.4) is strictly less than the module of r.h.s of (2.4). This contradicts to our assumption that i is the zero of $P(i)$.

By making use of the same argument, we can prove that if r is even, then $H=0$ does not have other nonnegative integer Fuchs index except $i=r-2p$. $\square$

Remark 2.1. If $H=0$ has the following dominant term

$$[u^{(r)}(z)]^{i_1}u(z)^{j_1}+[u^{(r)}(z)]^{i_2}u(z)^{j_2},$$

where $r,k\in\mathbb{N}^+$, $k>1$, $i_1,j_1,i_2,j_2\in\mathbb{N}^+\cup\{0\}$, $i_1^2+i_2^2\neq 0$, $(i_1,j_1)\neq(i_2,j_2)$, then similar argument gives the same conclusion, namely, $H=0$ has only one nonnegative integer Fuchs index $i=r-2p$ when r is even and no nonnegative integer Fuchs index when r is odd.

Remark 2.2. If r is odd, then each u_j, $j=1,2,\cdots$, can be uniquely determined by the leading coefficient u_0. If r is even and one can integrate $H=0$ once, for example by multiplying the factor u', then one can move the arbitrary coefficient u_{i-2p} from $H=0$ to the integration constant and therefore all the coefficients u_n, $n=1,2,\cdots$ of each meromorphic solution to the resulting equation are determined by its leading coefficients and the integration constant.

Now we are ready to prove our main result.

Theorem 2.1. *If the ODE* (1.3) *possesses a particular meromorphic solution $u(z)$, then it belongs to the class W.*

For the convenience of readers, here we include the proof of Theorem 2.1, which is mainly Eremenko's method [9] with slight refinement [5]. We shall assume that the readers are familiar with the standard terminologies and basic results on Nevanlinna theory. The standard references of this theory are [11] and Laine's book [13] which contains its applications to complex differential equations. The argument used in the proof of Theorem 2.1 makes use of the following version of Clunie's Lemma (see [13], Lemma 2.4.2).

Lemma 2.1. *Let f be a transcendental meromorphic solution of*

$$f^n P(z,f) = Q(z,f),$$

where $P(z,f)$ and $Q(z,f)$ are polynomials in f and its derivatives with meromorphic coefficients $\{a_\lambda | \lambda \in I\}$ such that $m(r,a_\lambda) = S(r,f)$ for all $\lambda \in I$. If the total degree of $Q(z,f)$ as a polynomial in f and its derivatives is less than or equal to n, then

$$m(r,P(r,f)) = S(r,f).$$

Here $S(r,f)$ is called the "small" function, which denotes all the quantities with growth $o(T(r,f))$ as $r \to \infty$ outside a possible exceptional set of finite linear measure.

Proof. The proof of Theorem 2.1 involves two steps:

1. Prove that for fixed $z_0 \in \mathbb{C}$, there are finitely many Laurent series expansions around $z = z_0$ for all the possible meromorphic solutions $u(z)$ of (1.3).
2. Show that any transcendental meromorphic solution $u(z)$ of (1.3) must have infinitely many poles.

Step 1 Let u be a meromorphic solution of the ODE (1.3), then it has at least one pole, say at $z = z_0$, which will be proved in Step 2. Substitute $u = \sum_{n=0}^{+\infty} u_n (z-z_0)^{n+p} (u_0 \neq 0,\ p < 0,\ p \in \mathbb{Z})$ into (1.3), then one deduces that $p = -1$ and $u_0 = 2\exp(ik\pi/2)$, $k = 0,1,2,3$. By applying Proposition 2.1, we see that (1.2) has only one nonnegative integer Fuchs index. Therefore after integration, the only arbitrary coefficient has been moved to the integration constant c and so (1.3) does not have any nonnegative integer Fuchs index. Hence, all other coefficients u_n, $n = 1,2,\cdots$, are uniquely determined [2, pp. 90] by u_0 and c. Hence, there are at most four types of Laurent series expansions around $z = z_0$ for meromorphic solutions satisfying the ODE (1.3) with a pole at $z = z_0$. The first few terms are given by

$$\begin{aligned} u(z) = &\frac{u_0}{z-z_0} + \frac{u_0}{120}(b_3 u_0^2 + 2b_1)(z-z_0) \\ &+ \frac{1}{7200} u_0 (8b_3^2 - 4b_1^2 + 60b_2 - b_3 b_1 u_0^2)(z-z_0)^3 + o(|z-z_0|^3). \end{aligned} \tag{2.5}$$

Step 2 u is rational, then we are done. Suppose u is a transcendental meromorphic solution of the ODE (1.3), then we have

$$u^6 = 4u'u''' - 2(u'')^2 + 2b_1(u')^2 + 2b_2 u^2 + b_3 u^4 - c. \tag{2.6}$$

Take $f = u$, $P = u$, $n = 5$ and apply Lemma 2.1 to the above equation, we conclude that $m(r,u) = S(r,u)$ and hence $(1 - o(1))T(r,u) = N(r,u)$. We claim that u must have infinitely many poles on $\mathbb{C}$, otherwise $N(r,u) = \mathcal{O}(\log r)$. Therefore, $T(r,u) = \mathcal{O}(\log r)$ which is impossible since u is transcendental. Specifically, u is not a transcendental entire function. Also, one can easily see that u cannot be a non-constant polynomial.

To show *all* the transcendental meromorphic solutions of (1.3) belong to class W, the argument is the same as that given in [5], so we omit it here.

Thus, we complete the proof of the lemma. $\square$

3 Explicit solutions in the class W

In Section 2, we have proved that all the meromorphic solutions of the ODE (1.3) belong to the class W. Besides that, the proof also gives the upper bound, which is 4, on the number of poles of the solutions of the ODE (1.3) in the fundamental region $\mathcal{F}$. As it is shown in [7], this allows us to find all of them explicitly by applying Theorem 1.1. From now on, by considering $u(z+z_0)$ instead of $u(z)$, we can always assume one pole of $u(z)$ is at $z=0$.

3.1 Rational solutions

According to Theorem 1.1, since $u_0=2\exp(ik\pi/2)$, $k=0,1,2,3$, any rational solution u of (1.3) can have at most four poles in $\mathbb{C}$. If u has one pole only, then we have

$$u(z)=\frac{u_0}{z}+\sum_{k=0}^{m}h_k z^k. \tag{3.1}$$

Substituting (3.1) into (1.2) gives four families of rational solutions with the correlations on the parameters

$$\begin{aligned}
&u_1(z)=\pm\frac{2}{z-z_0}, && 2b_3+b_1=b_2=0, && z_0\in\mathbb{C},\\
&u_2(z)=\pm\frac{2i}{z-z_0}, && 2b_3-b_1=b_2=0, && z_0\in\mathbb{C}.
\end{aligned}$$

3.2 Simply periodic solutions

Let us first construct exact simply periodic meromorphic solutions with one pole in $\mathcal{F}$. According to Theorem 1.1, they are of the form

$$u(z)=\frac{\pi}{T}c_{-1}\cot\left(\frac{\pi z}{T}\right)+h_0=\sqrt{L}c_{-1}\cot(\sqrt{L}z)+h_0, \tag{3.2}$$

where $L=\pi^2/T^2\neq 0$, $c_{-1}^4=16$.

Comparing the Laurent series of $u(z)$ around $z=0$ with (2.5) gives the solutions

$$u_3(z)=\sqrt{-\frac{b_3c_{-1}^2+2b_1}{40}}c_{-1}\cot\left[\sqrt{-\frac{b_3c_{-1}^2+2b_1}{40}}(z-z_0)\right],\quad z_0\in\mathbb{C}, \tag{3.3}$$

provided that $-16b_3^2+6b_1^2-100b_2+b_3b_1c_{-1}^2=0$ and $b_3c_{-1}^2+2b_1\neq 0$, where $c_{-1}^4=16$.

For simply periodic solutions of (1.3) which have two poles in $\mathcal{F}$, by Theorem 1.1, they should be of the form

$$u(z)=r_k\sqrt{L}\cot(\sqrt{L}z)+r_l\sqrt{L}\cot(\sqrt{L}(z-a))+h_0, \tag{3.4}$$

where $r_k^4=r_l^4=16$, $r_k\neq r_l$, $a(\neq 0)\in\mathbb{C}$.

By comparing the Laurent series of $u(z)$ around z=0 with (2.5), we obtain two families of such type of solutions

$$u_4(z)=\pm[2\sqrt{L}\cot(\sqrt{L}(z-z_0)+2i\sqrt{L}\tan(\sqrt{L}(z-z_0))], \tag{3.5}$$

where $z_0\in\mathbb{C}$, $L=-b_1/20$, provided that $b_3=-3ib_1/2$, $b_1^2=25b_2/9$.

$$u_5(z)=r\sqrt{L}\big[\cot(\sqrt{L}(z-z_0))+\tan(\sqrt{L}(z-z_0))\big] \ \text{ with } \ L=\frac{1}{80}(r^2b_3+2b_1), \ \ r^4=16, \tag{3.6}$$

provided that one of the following holds

$$\begin{cases} 16b_3b_1+9b_1^2-4(b_3^2+25b_2)=0, & \text{if } r^2=4, \\ 16b_3b_1-9b_1^2+4(b_3^2+25b_2)=0, & \text{if } r^2=-4. \end{cases}$$

3.3 Elliptic solutions

First of all, let us recall the definition of *elliptic order* of an elliptic function. It is the number of poles inside the fundamental parallelogram, counting multiplicity. From the theory of elliptic functions [1, 15], we know that the sum of Residues of any elliptic function in the fundamental parallelogram is zero. Therefore there are at most two types of elliptic solutions of (1.2) in the sense that they are of elliptic order either 2 or 4.

For elliptic solutions of (1.3) with elliptic order 2, by Theorem 1.1, they are of the form

$$u(z)=r_k\zeta(z)+r_l\zeta(z-a)+\tilde{h}_0=\frac{r_l(\wp'(z)+B)}{2(\wp(z)-A)}+h_0, \tag{3.7}$$

where $r_k=-r_l$, $r_k^4=16$, $A=\wp(a)$ and $B=\wp'(a)$, $a(\neq 0)\in\mathbb{C}$.

By comparing the Laurent series expansions of (3.7) around 0 and a with the series (2.5), we obtain the following family of elliptic solutions

$$u_6(z)=-\frac{r\wp'(z;g_2,g_3)}{2\wp(z;g_2,g_3)-\frac{1}{60}(b_3r^2+2b_1)}, \quad r^4=16, \tag{3.8}$$

where $\Delta=g_2^3-27g_3^2\neq 0$ and

$$\begin{cases} g_2=\dfrac{1}{240}(2b_1^2-20b_2+b_3b_1r_k^2), \\ g_3=\dfrac{1}{108000}(4b_3^3r_k^2-36b_3^2b_1-12b_3r_k^2b_1^2-13b_1^3+75b_3r_k^2b_2+150b_1b_2). \end{cases}$$

Finally, we construct elliptic solutions of (1.3) with elliptic order 4, according to Theorem 1.1, we may assume they are of the form

$$\begin{aligned} u(z) &= 2\zeta(z)-2\zeta(z-a_1)+2i\zeta(z-a_2)-2i\zeta(z-a_3)+\tilde{h}_0 \\ &= \frac{-(\wp'(z)+B_1)}{\wp(z)-A_1}+\frac{i(\wp'(z)+B_2)}{\wp(z)-A_2}+\frac{-i(\wp'(z)+B_3)}{\wp(z)-A_3}+h_0, \end{aligned} \tag{3.9}$$

where $A_i=\wp(a_i;g_2,g_3)$, $B_i=\wp'(a_i;g_2,g_3)$, $i=1,2,3$ and $a_1,a_2,a_3\in\mathbb{C}\backslash\{0\}$ are distinct.

By comparing the Laurent series expansions of (3.7) with the series (2.5), we obtain the following family of elliptic solutions

$$u_7(z)=\frac{-\wp'(z-z_0)}{\wp(z-z_0)-\frac{b_1}{60}}+\frac{i\wp'(z-z_0)}{\wp(z-z_0)-\frac{1}{120}(2ib_3-b_1)}+\frac{-i\wp'(z-z_0)}{\wp(z-z_0)-\frac{1}{120}(-2ib_3-b_1)}, \tag{3.10a}$$

$$b_3\neq 0,\quad b_1\neq\pm\frac{2i}{3}b_3,\quad b_2=-\frac{3}{100}(4b_3^2-3b_1^2), \tag{3.10b}$$

where $z_0\in\mathbb{C}$ and the two invariants of $\wp(z;g_2,g_3)$ are given by

$$\begin{cases} A_1=\dfrac{1}{360}\Big(4b_3+2b_1\pm i\sqrt{2}\sqrt{-44b_3^2+16b_3b_1+19b_1^2-300b_2}\Big), \\ A_2=\dfrac{1}{120}(2ib_3-(60+60i)A_1+ib_1), \\ A_3=-\dfrac{i}{120}(2b_3-(60+60i)A_1+b_1), \\ g_2=4(A_1^2+A_1A_2+A_2^2)=-4(A_1A_2+A_1A_3+A_2A_3), \\ g_3=4A_1A_2A_3. \end{cases} \tag{3.11}$$

Remark 3.1. For meromorphic solutions of other types, the corresponding algebraic systems are inconsistent and thus we have found all the meromorphic solutions as listed above. Also, one may check that all of them satisfy the ODE (1.3) by direct substitution.

4 Real valued solutions

In many situations from physics or other fields, real solutions are much more interesting and applicable. In this section, we shall present some real-valued solutions of the ODE (1.2) which are obtained by choosing particular parameters from the solutions found in the previous section.

4.1 Real elliptic solutions

First of all, let us recall a theorem on the theory of elliptic functions.

Theorem 4.1 (see [15]). *If $\wp(z|w_1,w_2)$ has two real invariants g_2, g_3, then either*

- $\Delta<0$, *and* $w_1=\overline{w}_2$; *or*
- $\Delta>0$, *one of* w_1,w_2 *is real and the other is purely imaginary.*

Consider elliptic solutions with two poles in the fundamental parallelogram. If $r^2=2$, $b_1,b_2,b_3\in\mathbb{R}$, then $\wp(z)$ in the elliptic solution (3.8) has the two real invariants

$$\begin{cases} g_2=\dfrac{1}{120}(b_1^2-10b_2+2b_3b_1),\\ g_3=\dfrac{1}{108000}(b_1+2b_3)(-13b_1^2-22b_1b_3+150b_2+8b_3^2), \end{cases}$$

and the corresponding discriminant is given by $\Delta=g_2^3-27g_3^2=-(8b_3^2-2b_3b_1-3b_1^2+50b_2)^2(-16b_1b_3-9b_1^2+4b_3^2+100b_2)\neq 0$. Hence, if $-16b_1b_3-9b_1^2+4b_3^2+100b_2<0$, then by Theorem 4.1, we conclude that one period of $\wp(z;w_1,w_2)$ is real and the other is purely imaginary. Without loss of generality, we may assume w_1 is real and so it implies that (3.8) is a real-valued function on $\mathbb{R}$ with period w_1. On the contrary, if $-16b_1b_3-9b_1^2+4b_3^2+100b_2>0$, then $\wp(z;w_1,w_2)$ has two conjugated periods $w_1,w_2=\overline{w}_1$, and so it implies that both $\wp(z;w_1,w_2)$ and (3.8) are real-valued function on $\mathbb{R}$ with period $2|\Re w_1|$.

For elliptic solutions of *order* 4, suppose $b_3,b_1,z_0\in\mathbb{R}$ satisfies the correlations in (3.10b), then $\wp(z)$ in the solution (3.10a) has two real invariants

$$\begin{cases} g_2=\dfrac{-4b_3^2+3b_1^2}{3600},\\ g_3=\dfrac{4b_3^2b_1+b_1^3}{216000}, \end{cases}$$

and the discriminant is given by $\Delta=g_2^3-27g_3^2=-(4b_3^3+9b_3b_1^2)^2/11664000000<0$. Again, Theorem 4.1 shows that $\wp(z;w_1,w_2)$ has two conjugated periods $w_1,w_2=\overline{w}_1$ and $\wp(z;w_1,w_2)|_{\mathbb{R}}$ is a real-valued function with period $2|\Re w_1|$. Also

$$\overline{\left(\frac{i\wp'(z)}{\wp(z)-\frac{1}{120}(2ib_3-b_1)}\right)}=\frac{-i\wp'(z)}{\wp(z)-\frac{1}{120}(-2ib_3-b_1)}\quad \text{for } z\in\mathbb{R}, \tag{4.1}$$

and so it implies that (3.10a) is a real-valued function on $\mathbb{R}$.

4.2 Real simply periodic solutions

For simply periodic solutions of (1.2) with one pole in the fundamental region, suppose $b_1,b_2,b_3\in\mathbb{R}$ satisfy the constraints in (3.3), and let $c_{-1}=2$ or -2, then we have two cases

1. If $2b_3+b_1>0$, then we have the solution

$$u(z)=\pm 2iK\cot[iK(z-z_0)]=\pm\frac{2K(e^{2Kz}+e^{2Kz_0})}{e^{2Kz}-e^{2Kz_0}},$$

   where $z_0\in\mathbb{C}$, $K^2=(2b_3+b_1)/20$.

i) If we choose z_0 such that $2Kz_0 = t+i\pi$, $t\in\mathbb{R}$, then $u(z)$ is real-valued and smooth on $\mathbb{R}$.

ii) If we choose z_0 such that $2Kz_0\in\mathbb{R}$, then $u(z)$ is real-valued and smooth on $\mathbb{R}$ except at countably many points at which $u(z)$ blows up.

2. If $2b_3+b_1<0$, then for $z_0\in\mathbb{R}$, the solution

$$u(z) \quad = \quad \pm 2\sqrt{-\frac{2b_3+b_1}{20}}\cot\left[\sqrt{-\frac{2b_3+b_1}{20}}(z-z_0)\right], \quad z_0\in\mathbb{C},$$

is real-valued solution on $\mathbb{R}$ except at countably many points at which $u(z)$ blows up.

For simply periodic solutions of (1.2) with two poles in the fundamental region, suppose $b_1,b_2,b_3\in\mathbb{R}$ satisfy the constraints in (3.6), and let $r=2$ or -2, then we have two cases:

1. If $2b_3+b_1<0$, then we have the solution

$$u(z)=\pm 2iK\big(\tan(iK(z-z_0))+\cot(iK(z-z_0))\big)=\pm\frac{8Ke^{2K(z-z_0)}}{e^{4K(z-z_0)}-1},$$

where $z_0\in\mathbb{C},K^2=-(2b_3+b_1)/40$.

If we choose z_0 such that $2Kz_0\in\mathbb{R}$ or $2Kz_0=t+i\pi$, $t\in\mathbb{R}$, then $u(z)$ is real-valued on $\mathbb{R}$ except at countably many points at which $u(z)$ blows up.

2. If $2b_3+b_1>0$, then for $z_0\in\mathbb{R}$, the solution

$$u(z)=\pm 2\sqrt{L}\Big[\cot(\sqrt{L}(z-z_0))+\tan(\sqrt{L}(z-z_0))\Big], \quad L=\frac{1}{40}(2b_3+b_1),$$

is real-valued solution on $\mathbb{R}$ except at countably many points at which $u(z)$ blows up.

Remark 4.1. To the best knowledge of the author, the real elliptic solutions of *order* 2 and real simply periodic solutions with one pole in $\mathcal{F}$ are not known before.

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