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H^1-Estimates of the Littlewood-Paley and Lusin Functions for Jacobi Analysis II

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Abstract. Let $(\mathbb{R}_+,*,\Delta)$ be the Jacobi hypergroup. We introduce analogues of the Littlewood-Paley g function and the Lusin area function for the Jacobi hypergroup and consider their (H^1,L^1) boundedness. Although the g operator for $(\mathbb{R}_+,*,\Delta)$ possesses better property than the classical g operator, the Lusin area operator has an obstacle arisen from a second convolution. Hence, in order to obtain the (H^1,L^1) estimate for the Lusin area operator, a slight modification in its form is required.

Key Words: Jacobi analysis, Jacobi hypergroup, g function, area function, real Hardy space.

AMS Subject Classifications: 22E30, 43A30, 43A80

1 Introduction

One of main subjects of the so-called real method in classical harmonic analysis related to the Poisson integral $f*p_t$ is to investigate the Littlewood-Paley theory. For example, in the one dimensional setting, the following singular integral operators were respectively well-known as the Littlewod-Paley g function and the Lusin area function

$$g^{\mathbb{R}}(f)(x)=\Big(\int_0^\infty\Big|f*t\frac{\partial}{\partial t}p_t(x)\Big|^2\frac{dt}{t}\Big)^{1/2}, \tag{1.1a}$$

$$S^{\mathbb{R}}(f)(x)=\Big(\int_0^\infty\frac{1}{t}\chi_t*\Big|f*t\frac{\partial}{\partial t}p_t\Big|^2(x)\frac{dt}{t}\Big)^{1/2}, \tag{1.1b}$$

where χ_t is the characteristic function of $[-t,t]$. These operators satisfy the maximal theorem, that is, a weak type L^1 estimate and a strong type L^p estimate for $1<p\le\infty$. Moreover, they are bounded form H^1 into L^1 (cf. [10–12]). Our matter of concern is to extend these results to other topological spaces X. Roughly speaking, in some examples of X of homogeneous type (see [2]), Poisson integrals are generalized on X and analogous

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Littlewood-Paley theory has been developed (cf. [2,5,10]). On the other hand, if the space X is not of homogeneous type, we encounter difficulties. As an example of X of non homogeneous type with Poisson integrals, noncompact Riemannian symmetric spaces $X=G/K$ are well-known. Lohoue [9] and Anker [1] generalize the Littlewood-Paley g function and the Luzin area function to G/K and show that they satisfy the maximal theorem (see below). However, we know little or nothing whether they are bounded from H^1 into L^1, because we first have to find out a suitable definition of a real Hardy space on G/K. The aim of this paper is to introduce a real Hardy space $H^1(\Delta)$ and show that they are bounded from $H^1(\Delta)$ into $L^1(\Delta)$ for the Jacobi hypergroup $(\mathbb{R}_+,*,\Delta)$, which is a generalization of K-invariant setting on G/K of real rank one.

We briefly overview the Jacobi hypergroup $(\mathbb{R}_+,*,\Delta)$. We refer to [4] and [8] for a description of general context. For $\alpha\geq\beta\geq-\frac{1}{2}$ and $(\alpha,\beta)\neq(-\frac{1}{2},-\frac{1}{2})$ we define the weight function Δ on $\mathbf{R}_+$ as

$$\Delta(x)=(2\mathrm{sh}x)^{2\alpha+1}(2\mathrm{ch}x)^{2\beta+1}.$$

Clearly, it follows that

$$\Delta(x)\leq c\begin{cases} e^{2\rho x}, & x>1,\\ x^{2\gamma_0}, & x\leq 1,\end{cases}$$

where $\rho=\alpha+\beta+1$ and $\gamma_0=\alpha+\frac{1}{2}$. For $\lambda\in\mathbb{C}$ let ϕ_λ be the Jacobi function on $\mathbb{R}_+$ defined by

$$\phi_\lambda(x)={}_2F_1\Big(\frac{\rho+i\lambda}{2},\frac{\rho-i\lambda}{2};\alpha+1;-(\mathrm{sh}x)^2\Big),$$

where ${}_2F_1$ the hypergeometric function. Then the Jacobi transform $\hat{f}$ of a function f on $\mathbb{R}_+$ is defined by

$$\hat{f}(\lambda)=\frac{1}{\sqrt{2\pi}}\int_0^\infty f(x)\phi_\lambda(x)\Delta(x)dx.$$

We define a generalized translation on $\mathbb{R}_+$ by using the kernel form of the product formula of Jacobi functions: For $x,y\in\mathbb{R}_+$,

$$\phi_\lambda(x)\phi_\lambda(y)=\int_0^\infty\phi_\lambda(z)K(x,y,z)\Delta(z)dx.$$

The kernel $K(x,y,z)$ is non-negative and symmetric in the tree variables. Then the generalized translation T_x of f is defined as

$$T_xf(y)=\int_0^\infty f(z)K(x,y,z)\Delta(z)dz$$

and the convolution of f, g is given by

$$f*g(x)=\int_0^\infty f(y)T_xg(y)\Delta(y)dy.$$

Since $T_xf(y)=T_yf(x)$ and $\widehat{T_xf}(\lambda)=\phi_\lambda(x)\hat{f}(\lambda)$, it follows that $f*g=g*f$ and $\widehat{f*g}(\lambda)=\hat{f}(\lambda)\cdot\hat{g}(\lambda)$. We call $(\mathbb{R}_+,*,\Delta)$ the Jacobi hypergroup and the associated harmonic analysis is called by Jacobi analysis. The Jacobi hypergroup is not a space of homogeneous type, because $\Delta(x)$ has an exponential growth order $e^{2\rho x}$ when x goes to ∞.

In Jacobi analysis, the Poisson kernel $p_t(x)$, $t>0$, is defined as the function such that

$$\widehat{p_t}(\lambda)=e^{-t\sqrt{\lambda^2+\rho^2}}.$$

Then, as analogue of the classical case, we introduce a generalized Littlewood-Paley g function $g_\sigma(f)$ and a generalized Lusin area function $S_{a,h}(f)$, which are respectively defined by

$$g_\sigma(f)(x)=\Big(\int_0^\infty e^{2\sigma t}\Big|f*t\frac{\partial}{\partial t}p_t(x)\Big|^2\frac{dt}{t}\Big)^{1/2}, \tag{1.2a}$$

$$S_{a,h}(f)(x)=\frac{1}{h(x)}\Big(\int_0^\infty \tilde{\chi}_{B(at)}*\Big|h\cdot f*t\frac{\partial}{\partial t}p_t\Big|^2(x)\frac{dt}{t}\Big)^{1/2}, \tag{1.2b}$$

where $\sigma,a\geq 0$, $h(x)$ is a positive function on $\mathbb{R}_+$ and

$$\tilde{\chi}_{B(at)}=\frac{1}{|B(t)|}\chi_{B(at)}.$$

Here $\chi_{B(t)}$ is the characteristic function of $B(t)=[0,t]$ and $|B(t)|$ the volume of $B(t)$ with respect to $\Delta(x)dx$. Similarly as in the case of non-compact Riemannian symmetric spaces (see [1, 9]), g_σ is strongly bounded on $L^p(\Delta)$ for $\sigma<2\rho/\sqrt{pp'}$, where $\frac{1}{p}+\frac{1}{p'}=1$ and g_0 satisfies a weak type L^1 estimate with respect to $\Delta(x)dx$. In the previous paper [7], the author introduces a real Hardy space $H^1(\Delta)$ and shows that g_0 is bounded form $H^1(\Delta)$ into $L^1(\Delta)$. As for $S_{a,h}$, the strong type L^p estimate of $S_{a,1}$ for $p>1$ is essentially obtained in [9]. However, whether $S_{a,1}$ is bounded from $H^1(\Delta)$ into $L^1(\Delta)$ is still an open question. In [7], Section 7, we obtained a partial answer for a modified operator of $S_{a,1}$ with $a\leq\frac{1}{3}$. In this paper we refine this result and extend it to a more general area operator $S_{a,h}$.

This paper is organized as follows. Basic notations are given in Section 2. Especially we recall the definition of the Hardy space $H^1(\Delta)$ and give a relation with Euclidean weighted Hardy spaces $H^1_w(\mathbb{R})$. In Section 3 we prove key lemmas on generalized translations. Finally, in Section 4 and Section 5 we consider $(L^2(\Delta),L^2(\Delta))$ and $(H^1(\Delta),L^1(\Delta))$ boundedness of g_σ and $S_{a,h}$ respectively.

2 Notations

Let $L^p(\Delta)$ denote the space of functions f on $\mathbb{R}_+$ with finite L^p-norm:

$$\|f\|^p_{L^p(\Delta)}=\frac{1}{\sqrt{2\pi}}\int_0^\infty|f(x)|^p\Delta(x)dx,$$

and $L^1_{\rm loc}(\Delta)$ the space of locally integrable functions on $\mathbb{R}_+$. We may regard these functions on $\mathbb{R}_+$ as even function on $\mathbb{R}$. Let C_c^∞ be the space of compactly supported C^∞ even functions on $\mathbb{R}$. For $f\in C_c^\infty$ the Jacobi transform $\hat{f}$ is well-defined and the Paley-Wiener theorem holds: The map $f\to\hat{f}$ is a bijection of C_c^∞ onto the space of entire holomorphic even functions of exponential type on $\mathbb{R}$. The inverse transform is given as

$$f(x)=\frac{1}{\sqrt{2\pi}}\int_0^\infty \hat{f}(\lambda)\phi_\lambda(x)|C(\lambda)|^{-2}d\lambda,$$

where $C(\lambda)$ is Harish-Chandra's C-function. Furthermore, the map $f\to\hat{f}$ extends to an isometry of $L^2(\Delta)$ onto $L^2(\mathbb{R}_+,|C(\lambda)|^{-2}d\lambda)$:

$$\|f\|_{L^2(\Delta)}=\|\hat{f}\|_{L^2(\mathbb{R}_+,|C(\lambda)|^{-2}d\lambda)}$$

(see [4, Section 2] and [8, Theorem 3.1, Remark 3]). Let $f\in L^1(\Delta)$. Since ϕ_λ is bounded by 1 for $|\Im\lambda|\le\rho$ (see [4, (2.17)]), $\hat{f}$ has a holomorphic extension on $|\Im\lambda|\le\rho$ and $|\hat{f}(\lambda)|\le\|f\|_{L^1(\Delta)}$. We recall that, as a function of λ, $\phi_\lambda(x)$ is the Fourier Cosine transform of a function $A(x,y)$ supported on $[0,x]$:

$$\Delta(x)\phi_\lambda(x)=\int_0^x\cos\lambda y A(x,y)dy$$

(see [8, (2.16)]). Hence, if we define the Abel transform $W^0_+(f)$ of f by

$$W^0_+(f)(x)=\int_x^\infty f(y)A(x,y)dy,$$

then we see that

$$\hat{f}(\lambda)=c\mathcal{F}(W^0_+(f))(\lambda),$$

where $W^0_+(f)$ is extended as an even function on $\mathbb{R}$ and $\mathcal{F}$ is the Euclidean Fourier transform on $\mathbb{R}$. We put

$$W^s_+(f)(x)=e^{s\rho x}W^0_+(f)(x).$$

Since $|A(x,y)|\le ce^{\rho y}(\mathrm{th}y)^{2\alpha}$ by the explicit form (see [8], (2.18)), it follows that

$$\|W^s_+(f)\|_{L^1(\mathbb{R}_+)}\le c\|f\|_{L^1(\Delta)}\quad\text{for }\ |s|\le 1,$$

and for $\lambda\in\mathbb{R}$,

$$\hat{f}(\lambda+i\rho s)=c\mathcal{F}(W^s_+(f))(\lambda).$$

Especially, we have

$$W^s_+(f*g)=W^s_+(f)\otimes W^s_+(g),$$

where $\otimes$ denotes the Euclidean convolution on $\mathbb{R}$. As shown in [8], Section 3, W^0_+ is of the form:

$$W^0_+(f)=cW^1_{\alpha-\beta}\circ W^2_{\beta+1/2}(f),$$

where W_μ^σ is the generalized Weyl type fractional operators on $\mathbb{R}_+$; for $n=0,1,2,\cdots$, $\Re\mu>-n$ and $\sigma\in\mathbb{R}$,

$$W_\mu^\sigma(f)(s)=\frac{(-1)^n}{\Gamma(1+n)}\int_s^\infty \frac{d^n}{d(\mathrm{ch}\sigma t)^n}f(t)\cdot(\mathrm{ch}\sigma t-\mathrm{ch}\sigma s)^{\mu+n-1}d(\mathrm{ch}\sigma t).$$

Since the inverse of W_μ^σ is given by $W_{-\mu}^\sigma$, the inverse operator W_-^s of W_+^s is given by $W_-^s(f)=W^2_{-(\beta+1/2)}\circ W^1_{-(\alpha-\beta)}(e^{-s\rho x}f)$. The following formula is obtained in [7, Corollary 3.7]. For $f\in L^1(\Delta)$, let $F=W_+^1(f)$. Then there exist finite sets Γ_0,Γ_1 in $\mathbb{R}_+$ for which

$$\begin{aligned} f(x)&=W_-^1\circ W_+^1(f)(x)=W_-^1(F)\\ &=\frac{1}{\Delta(x)}\Big(\sum_{\gamma\in\Gamma_0}W_{-\gamma}^{\mathbb{R}}(F)(x)(\mathrm{th}x)^\gamma+\sum_{\gamma\in\Gamma_1}(\mathrm{th}x)^\gamma\int_x^\infty W_{-\gamma}^{\mathbb{R}}(F)(s)A_\gamma(x,s)ds\Big), \end{aligned} \tag{2.1a}$$

where $W_{-\gamma}^{\mathbb{R}}$ is the Weyl type fractional operator on $\mathbb{R}$, which is defined by replacing $\mathrm{ch}\sigma t$ and $\mathrm{ch}\sigma s$ in the above definition of $W_\gamma^\sigma(f)(s)$ with t and $s\in\mathbb{R}$ respectively. For some properties of $A_\gamma(x,s)$ see [7, Lemma 3.6]. In particular, if α and β both belong to $\mathbb{N}+\frac{1}{2}$, then the integral terms in (2.1) vanish; $\Gamma_1=\emptyset$ and $\Gamma_0=\{0,1,2,\cdots,\gamma_0\}$, $\gamma_0=\alpha+\frac{1}{2}$. Since $e^{-\rho x}F$ is an even function on $\mathbb{R}$, L^1 norm of $W_{-\gamma}^{\mathbb{R}}(F)(-x)$ on $\mathbb{R}_+$ is controlled by L^1 norms of $W_{-\gamma}^{\mathbb{R}}(F)(x)$ on $\mathbb{R}_+$ †. Hence it follows that

$$\|f\|_{L^1(\Delta)}\sim\sum_{\gamma\in\Gamma_0\cup\Gamma_1}\|W_{-\gamma}^{\mathbb{R}}(F)\|_{L^1_{w_\gamma}(\mathbb{R})},$$

where $L^1_{w_\gamma}(\mathbb{R})$ is the w_γ-weighted L^1 space on $\mathbb{R}$ and $w_\gamma(x)=(\mathrm{th}|x|)^\gamma$.

We now define the real Hardy space $H^1(\Delta)$ as the subspace of $L^1(\Delta)$ consisting of all functions with finite $H^1(\Delta)$-norm:

$$\|f\|_{H^1(\Delta)}=\sum_{\gamma\in\Gamma_0\cup\Gamma_1}\|W_{-\gamma}^{\mathbb{R}}(F)\|_{H^1_{w_\gamma}(\mathbb{R})}, \tag{2.2}$$

where $H^1_{w_\gamma}(\mathbb{R})$ is the w_γ-weighted H^1 Hardy space on $\mathbb{R}$ that coincides with the weighted homogeneous Triebel-Lizorkin space $\dot{F}_{1,2}^{\gamma,w_\gamma}$ (cf. [3]). Thereby the above $H^1(\Delta)$-norm is equivalent to

$$\|F\|_{H^1(\mathbb{R})}+\|W_{-\gamma_0}^{\mathbb{R}}(F)\|_{H^1_{w_{\gamma_0}}(\mathbb{R})}.$$

In [7], Section 4 we define a radial maximal operator M for the Jacobi hypergroup $(\mathbb{R}_+,*,\Delta)$ and deduce that $H^1(\Delta)$ coincides with the space consisting of all $f\in L^1_{\mathrm{loc}}(\mathbb{R}_+)$ whose radial maximal functions Mf belong to $L^1(\Delta)$ and $\|f\|_{H^1(\Delta)}\sim\|Mf\|_{L^1(\Delta)}$.

The letter c will be used to denote a positive constant which may assume different values at different places.

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†We also use the fact that $W_{-\gamma}^{\mathbb{R}}$, $0<\gamma<1$, corresponds to the Fourier multiplier of $-i|\lambda|^\gamma(\mathrm{sgn}(\lambda)\sin\frac{\gamma\pi}{2}-i\cos\frac{\gamma\pi}{2})$.

3 Key lemmas

The following lemmas on the generalized translation T_x will play a key role in the arguments in Section 4 and Section 5. The first one is obtained in [4, (5.2)], and the second one is essentially obtained in [6, Lemma 2.2], for group cases.

Lemma 3.1 (see [4]). *Let $f \in L^p(\Delta)$, $1 \le p \le \infty$, and $x \in \mathbb{R}_+$. Then*

$$\|T_x f\|_{L^p(\Delta)} \le \|f\|_{L^p(\Delta)}.$$

Moreover, if f is positive, then the equality holds.

Lemma 3.2. *Let $x,y \ge 0$. Then*

$$0 \le T_x e^{-2\rho(\cdot)}(y) \le c e^{-2\rho \max\{x,y\}},$$

where c is independent of x,y.

Proof. We may assume that $x \ge y$. It follows from [4, (4.19)], that

$$\begin{aligned} T_x e^{-2\rho(\cdot)}(y) &= \int_{x-y}^{x+y} e^{-2\rho z} K(x,y,z)\Delta(z)dz \\ &\le c(\mathrm{th}x)^{-2\alpha} e^{-\rho x} (\mathrm{th}y)^{-2\alpha} e^{-\rho y} \int_{x-y}^{x+y} \mathrm{th}z e^{-\rho z} dz \\ &\le c(\mathrm{th}x)^{-2\alpha} (\mathrm{th}y)^{-2\alpha} e^{-2\rho x} \mathrm{th}y \end{aligned}$$

and moreover, from [4, (4.20)], that

$$T_x e^{-2\rho(\cdot)}(y) \le \int_0^\infty K(x,y,z)\Delta(z)dz = 1. \tag{3.1}$$

Hence we can obtain the desired estimate. □

Lemma 3.3. *Let $x,t \ge 0$. Then*

$$\int_0^\infty T_x \chi_t(y) dy \le ct,$$

where c is independent of x, t.

Proof. Similarly as (3.1), $T_x\chi_t(y) \le 1$. Since $T_x\chi_t(y)$ is supported on $[|x-t|, x+t]$, the desired result is obvious. □

4 Littlewood-Paley g functions

As shown in [1, Corollary 6.2], g_σ is strongly bounded on $L^p(\Delta)$ provided $\sigma<2\rho/\sqrt{pp'}$ where $\frac{1}{p}+\frac{1}{p'}=1$ and g_0 satisfies a weak type L^1 estimate. We give a simple proof of the L^2 boundedness of g_σ for $\sigma<\rho$ and consider a modified operator $g_{\rho,\beta}$ when $\sigma=\rho$.

Theorem 4.1 (see [1])**.** *Let $\sigma<\rho$. Then g_σ is $(L^2(\Delta),L^2(\Delta))$ bounded.*

Proof. Since

$$2\sigma\frac{t}{\sqrt{\lambda^2+\rho^2}}\le\frac{2\sigma}{\rho}t<2t$$

for $\lambda\in\mathbb{R}$ and $t>0$, it follows that

$$\begin{aligned}
\|g_\sigma(f)\|_2^2=&\int_0^\infty\int_0^\infty e^{2\sigma t}\Big|f*t\frac{\partial}{\partial t}p_t\Big|^2(x)\frac{dt}{t}\Delta(x)dx\\
=&\int_0^\infty e^{2\sigma t}\Big\|f*t\frac{\partial}{\partial t}p_t\Big\|^2_{L^2(\Delta)}\frac{dt}{t}\\
=&\int_0^\infty e^{2\sigma t}\Big\|\hat{f}\cdot\Big(t\frac{\partial}{\partial t}p_t\Big)^\wedge\Big\|^2_{L^2(\mathbb{R}_+,|C(\lambda)|^{-2}d\lambda)}\frac{dt}{t}\\
=&\int_0^\infty e^{2\sigma t}\int_0^\infty\Big|\hat{f}(\lambda)t\sqrt{\lambda^2+\rho^2}e^{-t\sqrt{\lambda^2+\rho^2}}\Big|^2|C(\lambda)|^{-2}d\lambda\frac{dt}{t}\\
=&\int_0^\infty|\hat{f}(\lambda)|^2|C(\lambda)|^{-2}\Big(\int_0^\infty e^{2\sigma t}t^2(\lambda^2+\rho^2)e^{-2t\sqrt{\lambda^2+\rho^2}}\frac{dt}{t}\Big)d\lambda\\
=&\int_0^\infty|\hat{f}(\lambda)|^2|C(\lambda)|^{-2}\Big(\int_0^\infty e^{2\sigma\frac{t}{\sqrt{\lambda^2+\rho^2}}}t^2e^{-2t}\frac{dt}{t}\Big)d\lambda\\
=&\int_0^\infty|\hat{f}(\lambda)|^2|C(\lambda)|^{-2}\Big(\int_0^\infty e^{2\sigma t/\rho}te^{-2t}dt\Big)d\lambda\\
\le&c_\sigma\|f\|^2,
\end{aligned}$$

where

$$c_\sigma=\int_0^\infty e^{-2(1-\sigma/\rho)t}tdt.$$

Thus, we complete the proof. □

Theorem 4.2. *Let $g_{\rho,\beta}$ be the operator defined by replacing $e^{2\rho t}$ in the definition* (1.2) *of g_ρ by*

$$e^{2\rho t}\frac{1}{(1+t)^\beta},\quad\beta>2.$$

Then $g_{\rho,\beta}$ is $(L^2(\Delta),L^2(\Delta))$ bounded.

Proof. We note that

$$\int_0^\infty e^{2\rho\frac{t}{\sqrt{\lambda^2+\rho^2}}}\frac{1}{\Big(1+\frac{t}{\sqrt{\lambda^2+\rho^2}}\Big)^\beta}t^2e^{-2t}\frac{dt}{t}$$

is dominated by

$$\begin{cases} \displaystyle\int_0^\infty \frac{t}{\big(1+\frac{t}{\sqrt{2}\rho}\big)^\beta}dt, & \text{if } 0\le\lambda<\rho,\\ \displaystyle\int_0^\infty e^{-(2-\sqrt{2})t}tdt, & \text{if } \lambda\ge\rho. \end{cases}$$

Hence the desired result follows similarly as in Theorem 4.1. □

As shown in [7], Section 6, g_0 is bounded from $H^1(\Delta)$ to $L^1(\Delta)$. In order to understand the usage of the formula (2.1) we give a sketch of the proof.

Theorem 4.3 (see [7])**.** *g_0 is $(H^1(\Delta),L^1(\Delta))$ bounded.*

Proof. We recall (2.1) and, for simplicity, we suppose that the integral terms vanish, that corresponds to the case of $\alpha,\beta\in\mathbb{N}+\frac{1}{2}$. For general α, β, we refer to the arguments in [7], Section 6. Hence, we see that

$$f*t\frac{\partial}{\partial t}p_t(x)=\frac{1}{\Delta(x)}\sum_{\gamma\in\Gamma_0}W_{-\gamma}^{\mathbb{R}}(F)\otimes P_t(x)(\mathrm{th}x)^\gamma,$$

where $F=W_+^1(f)$ and $P_t=W_+^1(t\frac{\partial}{\partial t}p_t)$. Since P_t behaves similarly as the Euclidean Poisson kernel (see [7, Lemma 6.3]), it follows that

$$\begin{aligned} g_0(f)(x)&\le\frac{1}{\Delta(x)}\sum_{\gamma\in\Gamma_0}\Big(\int_0^\infty|W_{-\gamma}^{\mathbb{R}}(F)\otimes P_t(x)|^2\frac{dt}{t}\Big)^{1/2}(\mathrm{th}x)^\gamma\\ &\le\frac{c}{\Delta(x)}\sum_{\gamma\in\Gamma_0}g^{\mathbb{R}}(W_{-\gamma}^{\mathbb{R}}(F))(\mathrm{th}x)^\gamma, \end{aligned}$$

where $g^{\mathbb{R}}$ is the Euclidean g-function on $\mathbb{R}$ (see (1.1)). Since $g^{\mathbb{R}}$ is bounded form $H^1_{w_\gamma}(\mathbb{R})$ to $L^1_{w_\gamma}(\mathbb{R})$ (see [12, XII, Section 3], with a slight modification by a weight function), it follows from (2.2) that

$$\begin{aligned} \|g_0(f)\|_{L^1(\Delta)}&\le c\sum_{\gamma\in\Gamma_0}\|g^{\mathbb{R}}(W_{-\gamma}^{\mathbb{R}}(F))\|_{L^1_{w_\gamma}(\mathbb{R})}\\ &\le c\sum_{\gamma\in\Gamma_0}\|W_{-\gamma}^{\mathbb{R}}(F))\|_{H^1_{w_\gamma}(\mathbb{R})}=c\|f\|_{H^1(\Delta)}. \end{aligned}$$

Thus, we complete the proof. □

5 Lusin area functions

We shall consider strong type estimates of the modified area function $S_{a,h}$. Similarly as in the Euclidean case, the L^2 boundedness of $S_{a,h}$ is reduced to the one of g_σ.

Theorem 5.1. *$S_{a,h}$ is $(L^2(\Delta),L^2(\Delta))$ bounded provided that $a<2$ and h is the following:*
(a) $h=1$,
(b) $h=\sqrt{\Delta}$,
(c) $h=(\mathrm{th})^{\gamma_0}$.

Proof. We note that $\|S_{a,h}(f)\|_2^2$ is given by

$$\begin{aligned}&\int_0^\infty \frac{1}{h(x)^2}\int_0^\infty\Big(\int_0^\infty T_x\tilde{\chi}_{at}(y)\Big|h(y)f*t\frac{\partial}{\partial t}p_t(y)\Big|^2\Delta(y)dy\Big)\frac{dt}{t}\Delta(x)dx\\ =&\int_0^\infty\int_0^\infty\Big(\int_0^\infty\frac{h^2(y)}{h^2(x)}T_x\tilde{\chi}_{at}(y)\Delta(x)dx\Big)\Big|f*t\frac{\partial}{\partial t}p_t(y)\Big|^2\Delta(y)dy\Big)\frac{dt}{t}.\end{aligned}$$

Therefore, if we can deduce that

$$\int_0^\infty\frac{h^2(y)}{h^2(x)}T_x\tilde{\chi}_{at}(y)\Delta(x)dx\le ce^{2\sigma t}, \tag{5.1}$$

where c is independent of y,t, then we see that $\|S_{a,h}(f)\|_2^2\le c\|g_\sigma(f)\|_2^2$ and thus, $\|S_{a,h}(f)\|_2\le c\|f\|_2$ provided $\sigma<\rho$ by Theorem 4.1.

(a) $h=1$: It follows from Lemma 3.1 that

$$\int_0^\infty T_x\tilde{\chi}_{at}(y)\Delta(x)dx=\|\tilde{\chi}_{at}\|_{L^1(\Delta)}\le\frac{|B(at)|}{|B(t)|}\le ce^{2(a-1)\rho t}.$$

Therefore, (5.1) holds for $\sigma=(a-1)\rho$. Hence, if $a<2$, then $\sigma<\rho$.

(b) $h=\sqrt{\Delta}$: We divide the integral (5.1) over $[0,\infty)$ into several segments.

Let $x\ge y$. Since

$$\frac{\Delta(y)}{\Delta(x)}\le 1,$$

it follows that

$$\int_y^\infty\frac{\Delta(y)}{\Delta(x)}T_x\tilde{\chi}_{at}(y)\Delta(x)dx\le\|\tilde{\chi}_{at}\|_{L^1(\Delta)}\le ce^{2(a-1)\rho t}.$$

Let $x<y$ and $x\ge 1$. Since

$$\frac{\Delta(y)}{\Delta(x)}\le ce^{2\rho(y-x)},$$

it follows from Lemma 3.2 that

$$\int_1^y\frac{\Delta(y)}{\Delta(x)}T_x\tilde{\chi}_{at}(y)\Delta(x)dx\le e^{2\rho y}\tilde{\chi}_{at}*e^{-2\rho(\cdot)}(y)\le\|\tilde{\chi}_{at}\|_{L^1(\Delta)}\le ce^{2(a-1)\rho t}.$$

Let $x<y$ and $\alpha t<x<1$ for sufficiently small $\alpha>0$. Since $y<x+at<(1+\frac{a}{\alpha})x$ and $x<1$,

$$\frac{\Delta(y)}{\Delta(x)}\le\frac{\Delta((1+\frac{a}{\alpha})x)}{\Delta(x)}\le c\Big(1+\frac{a}{\alpha}\Big)^{\gamma_0}=c_{a,\alpha}$$

and thus,

$$\int_{\alpha t}^{1}\frac{\Delta(y)}{\Delta(x)}T_x\tilde{\chi}_{at}(y)\Delta(x)dx\le c_{a,\alpha}\|\tilde{\chi}_{at}\|_{L^1(\Delta)}\le c_{a,\alpha}e^{2(a-1)\rho t}.$$

Let $x<y,x<1$ and $x<\alpha t$. Since $y<x+at<(\alpha+a)t$, it follows from Lemma 3.3 that

$$\int_0^{\alpha t}\frac{\Delta(y)}{\Delta(x)}T_x\tilde{\chi}_{at}(y)\Delta(x)dx\le c\Delta((\alpha+a)t)\int_0^{\infty}T_y\tilde{\chi}_{at}(x)dx\le c\frac{\Delta((\alpha+a)t)t}{|B(t)|}=J(t).$$

We note that, if $t\le1$, then $J(t)\le c(\alpha+a)^{\gamma_0}$ and if $t>1$, then $J(t)\le ce^{2\rho(\alpha+a-1)t}t$. Therefore, for $a<2$, we can take a sufficiently small $\alpha>0$ for which $\alpha+a-1<1$.

Therefore, in each case, if $a<2$, then there exists $0<\sigma<\rho$ for which (5.1) holds.

(c) $h=(\mathrm{th})^{\gamma_0}$: Similarly as in (b), we divide the integral (5.1) over $[0,\infty)$.

Let $x\ge y$. Since

$$\frac{\mathrm{th}y}{\mathrm{th}x}\le c, \tag{5.2}$$

it follows that

$$\int_y^{\infty}\frac{(\mathrm{th}y)^{2\gamma_0}}{(\mathrm{th}x)^{2\gamma_0}}T_x\chi_{at}(y)\Delta(x)dx\le ce^{2(a-1)\rho t}. \tag{5.3}$$

Let $x<y$ and $x\ge1$. Clearly (5.2) and thus, (5.3) hold.

Let $x<y$ and $\alpha t<x<1$ for $\alpha>0$. Since $y<x+at<(1+\frac{a}{\alpha})x$, (5.2) and thus, (5.3) hold.

Let $x<y$, $x<1$ and $x<\alpha t$. Since $y<x+at<(\alpha+a)t$ and $(\mathrm{th}x)^{-2\gamma_0}\Delta(x)\le c$ for $x<1$, $J(t)$ in the case of (b) is replaced by

$$\frac{(\mathrm{th}(\alpha+a)t)^{2\gamma_0}t}{|B(t)|}.$$

Hence, if $t\le1$, then $J(t)\le c(\alpha+a)^{\gamma_0}$ and if $t>1$, then $J(t)\le ce^{-2\rho t}at\le c$. Therefore, in each case, if $a<2$, then there exists $0<\sigma<2$ for which (5.1) holds. ☐

Theorem 5.2. *$S_{a,h}$ is $(H^1(\Delta),L^1(\Delta))$ bounded provided that a and h are the following:*

(a) $h=\sqrt{\Delta}$ and $a\le1$,

(b) $h=(\mathrm{th})^{\gamma_0}$ and $a\le\frac{1}{2}$.

Proof. Similarly as in the proof of Theorem 4.3, for simplicity, we may suppose that the integral terms in (2.1) vanish (see [7, Section 6], for general case). Then we see that $S_{a,h}(f)(x)$ is dominated as

$$\begin{aligned}&\frac{1}{h(x)}\Big(\int_0^\infty \tilde{\chi}_{at}*\Big|\frac{h}{\Delta}\sum_{\gamma\in\Gamma_0}W^{\mathbb{R}}_{-\gamma}(F)\otimes P_t(\mathrm{th})^\gamma\Big|^2(x)\frac{dt}{t}\Big)^{1/2}\\ \le & c\sum_{\gamma\in\Gamma_0}\frac{1}{h(x)}\Big(\int_0^\infty \tilde{\chi}_{at}*\Big|\frac{h}{\Delta}W^{\mathbb{R}}_{-\gamma}(F)\otimes P_t(\mathrm{th})^\gamma\Big|^2(x)\frac{dt}{t}\Big)^{1/2}.\end{aligned}\tag{5.4}$$

Hence $\|S_{a,h}(f)\|_{L^1(\Delta)}$ is dominated by the sum of the L^1-norm of each term in (5.4) with respect to $\Delta(x)dx$:

$$\begin{aligned}&\int_0^\infty\frac{1}{h(x)}\Big(\int_0^\infty \tilde{\chi}_{at}*\Big|\frac{h}{\Delta}W^{\mathbb{R}}_{-\gamma}(F)\otimes P_t(\mathrm{th})^\gamma\Big|^2(x)\frac{dt}{t}\Big)^{1/2}\Delta(x)dx\\ =&\int_0^\infty\frac{\Delta(x)}{h(x)(\mathrm{th}x)^\gamma}\Big(\int_0^\infty \tilde{\chi}_{at}*\Big|\frac{h}{\Delta}W^{\mathbb{R}}_{-\gamma}(F)\otimes P_t(\mathrm{th})^\gamma\Big|^2(x)\frac{dt}{t}\Big)^{1/2}(\mathrm{th}x)^\gamma dx\\ =&\int_0^\infty\Big(\int_0^\infty\int_0^\infty T_x\tilde{\chi}_{at}(y)\frac{h(y)^2\Delta(x)^2(\mathrm{th}y)^{2\gamma}}{h(x)^2\Delta(y)(\mathrm{th}x)^{2\gamma}}\times|W^{\mathbb{R}}_{-\gamma}(F)\otimes P_t(y)|^2dy\frac{dt}{t}\Big)^{1/2}(\mathrm{th}x)^\gamma dx.\end{aligned}\tag{5.5}$$

We note that, for $f\in H^1(\Delta)$, each $W^{\mathbb{R}}_{-\gamma}(F)$ belongs to $H^1_{w_\gamma}(\mathbb{R})$ and P_t behaves similarly as the Euclidean Poisson kernel. Therefore, if we can deduce that

$$\int_0^\infty T_x\tilde{\chi}_{at}(y)\frac{h(y)^2}{h(x)^2}\frac{\Delta(x)^2}{\Delta(y)}\frac{(\mathrm{th}y)^{2\gamma}}{(\mathrm{th}x)^{2\gamma}}dx\le c \tag{5.6}$$

and

$$\int_0^\infty T_x\tilde{\chi}_{at}(y)\frac{h(y)^2}{h(x)^2}\frac{\Delta(x)^2}{\Delta(y)}\frac{(\mathrm{th}y)^{2\gamma}}{(\mathrm{th}x)^{2\gamma}}dy\le c, \tag{5.7}$$

where c is independent of x,y,t, then we can apply the arguments used in the Euclidean case (see [12, Proposition 1.2]). Then (5.5) is dominated by $\|W^{\mathbb{R}}_{-\gamma}(F)\|_{H^1_{w_\gamma}(\mathbb{R})}$ and thus,

$$\|S_{a,h}(f)\|_{L^1(\Delta)}\le c\sum_\gamma\|W_{-\gamma}(F)\|_{H^1_{w_\gamma}(\mathbb{R})}=c\|f\|_{H^1(\Delta)}.$$

(a): $h=\sqrt{\Delta}$ and $a\le 1$. The integrand of (5.6) and (5.7) is the following:

$$T_x\tilde{\chi}_{at}(y)\Delta(x)\frac{(\mathrm{th}y)^{2\gamma}}{(\mathrm{th}x)^{2\gamma}}.$$

The proof of (5.6): We divide the integral (5.6) over $[0,\infty)$. Let $x>y$ or $x\le y$ and $x\ge 1$ or $x\le y$ and $\frac{at}{2}<x<1$. In these cases, similarly as in the proof of (b) in Theorem 5.1, it follows that

$$\frac{(\mathrm{th}y)^{2\gamma}}{(\mathrm{th}x)^{2\gamma}}\le c$$

and thus, (5.6) is dominated by $e^{2(a-1)\rho t}$. Let $x\le y$, $x<1$ and $x<\frac{at}{2}$. Since $y\le x+at<\frac{3}{2}at$, it follows that

$$\Delta(x)\frac{(\mathrm{th}y)^{2\gamma}}{(\mathrm{th}x)^{2\gamma}}\le\Delta(x)\frac{(\mathrm{th}y)^{2\gamma_0}}{(\mathrm{th}x)^{2\gamma_0}}\le c(\mathrm{th}\tfrac{3}{2}at)^{2\gamma_0}.$$

Hence we see from Lemma 3.3 that

$$\int_0^1 T_x\tilde{\chi}_{at}(y)\Delta(x)\frac{(\mathrm{th}y)^{2\gamma}}{(\mathrm{th}x)^{2\gamma}}dx\le c\frac{(\mathrm{th}t)^{2\gamma_0}t}{|B(t)|}\le c.$$

Therefore, in each case, if $a\le 1$, then (5.6) holds.

The proof of (5.7): We divide the integral (5.7) over $[0,\infty)$.

Let $x>y$, $t>1$ and $y>1$. Since

$$\Delta(x)\le\frac{\Delta(x)}{\Delta(y)}\Delta(y)\le e^{2\rho(x-y)}\Delta(y),$$

it follows from Lemma 3.2 that

$$\int_1^x T_x\tilde{\chi}_{at}(y)\Delta(x)\frac{(\mathrm{th}y)^{2\gamma}}{(\mathrm{th}x)^{2\gamma}}dy\le ce^{2\rho x}\tilde{\chi}_{at}*e^{-2\rho(\cdot)}(x)\le c\|\tilde{\chi}_{at}\|_{L^1(\Delta)}\le ce^{2\rho(a-1)t}.$$

Let $x>y$, $t>1$ and $y\le 1$. Since $x\le y+at\le 1+at$, it follows that

$$\int_0^1 T_x\tilde{\chi}_{at}(y)\Delta(x)\frac{(\mathrm{th}y)^{2\gamma}}{(\mathrm{th}x)^{2\gamma}}dy\le c\Delta(1+at)\int_0^1 T_x\tilde{\chi}_{at}(y)dy\le c\frac{\Delta(1+at)}{|B(t)|}\le ce^{2\rho(a-1)t}.$$

Let $x>y$, $t\le 1$ and $at>\frac{x}{2}$. Since $x\le 2at$, it follows Lemma 3.3 that

$$\int_0^x T_x\tilde{\chi}_{at}(y)\Delta(x)\frac{(\mathrm{th}y)^{2\gamma}}{(\mathrm{th}x)^{2\gamma}}dy\le c\Delta(2at)\int_0^\infty T_x\tilde{\chi}_{at}(y)dy\le c\frac{\Delta(2at)t}{|B(t)|}\le ca^{2\gamma_0}.$$

Let $x>y$, $t\le 1$ and $at\le\frac{x}{2}$. Since $x\le y+at\le y+a$ and $y>x-at>\frac{x}{2}$, it follows that

$$\begin{aligned}\frac{\Delta(x)}{\Delta(y)}\le c&\begin{cases}\dfrac{\Delta(x)}{\Delta(x-a)}, & \text{if } x>2a,\\ \dfrac{\Delta(x)}{\Delta(\frac{x}{2})}, & \text{if } x\le 2a,\end{cases}\\ \le c_a&.\end{aligned}$$

Hence, replacing $\Delta(x)$ by $c_a\Delta(y)$, we can deduce that

$$\int_0^x T_x\tilde{\chi}_{at}(y)\Delta(x)\frac{(\mathrm{th}y)^{2\gamma}}{(\mathrm{th}x)^{2\gamma}}dy\le c_a\int_0^\infty T_x\tilde{\chi}_{at}(y)\Delta(y)dy=c\|\tilde{\chi}_{at}\|_{L^1(\Delta)}\le ce^{2\rho(a-1)t}.$$

Let $x<y$ and $1<x$. Since

$$\Delta(x)\frac{(\mathrm{th}y)^{2\gamma}}{(\mathrm{th}x)^{2\gamma}}\le\Delta(x)\le\Delta(y),$$

it follows that

$$\int_x^\infty T_x\tilde{\chi}_{at}(y)\Delta(x)\frac{(\mathrm{th}y)^{2\gamma}}{(\mathrm{th}x)^{2\gamma}}dy\le c\int_0^\infty T_x\tilde{\chi}_{at}(y)\Delta(y)dy\le ce^{2\rho(a-1)t}.$$

Let $x<y$ and $2at<x<1$. Since $y\le x+at\le\frac{3}{2}x$, we see that

$$\Delta(x)\frac{(\mathrm{th}y)^{2\gamma}}{(\mathrm{th}x)^{2\gamma}}\le\Delta(x)\frac{(\mathrm{th}\frac{3}{2}x)^{2\gamma}}{(\mathrm{th}x)^{2\gamma}}\le c\Delta(x)\le c\Delta(y)$$

and thus, we can obtain the above estimate.

Let $x<y$, $x<1$ and $x<2at$. Since $y\le x+at\le 3at$, it follows that

$$\Delta(x)\frac{(\mathrm{th}y)^{2\gamma}}{(\mathrm{th}x)^{2\gamma}}\le\Delta(x)\frac{(\mathrm{th}y)^{2\gamma_0}}{(\mathrm{th}x)^{2\gamma_0}}\le c(\mathrm{th}3at)^{2\gamma_0}.$$

Therefore, we see from Lemma 3.3 that

$$\int_x^\infty T_x\tilde{\chi}_{at}(y)\Delta(x)\frac{(\mathrm{th}y)^{2\gamma}}{(\mathrm{th}x)^{2\gamma}}dy\le c(\mathrm{th}3at)^{2\gamma_0}\int_0^\infty T_x\tilde{\chi}_{at}(y)dy\le c\frac{(\mathrm{th}3at)^{2\gamma_0}t}{|B(t)|}\le c.$$

Therefore, in each case, (5.7) holds if $a\le 1$.

(b): $h=(\mathrm{th})^{\gamma_0}$ and $a\le\frac{1}{2}$. The integrand of (5.6) and (5.7) is the following.

$$cT_x\tilde{\chi}_{at}(y)e^{2\rho(x-y)}\Delta(x)\frac{(\mathrm{th}y)^{2\gamma}}{(\mathrm{th}x)^{2\gamma}}.$$

Since $x-y\le at$, this is dominated by

$$cT_x\tilde{\chi}_{at}(y)e^{2\rho at}\Delta(x)\frac{(\mathrm{th}y)^{2\gamma}}{(\mathrm{th}x)^{2\gamma}}.$$

Hence it follows from the previous arguments in (a) that the integrals in (5.6) and (5.7) are dominated by

$$e^{2\rho at}e^{2\rho(a-1)t}=e^{2\rho(2a-1)t}.$$

Therefore, if $a\le\frac{1}{2}$, then (5.6) and (5.7) hold. □

Remark 5.1. In the definition of $S_{a,h}$ in (1.2) we can insert the term $e^{2\sigma t}$ as in the one of g_σ. Then it is easy to see that the condition $a<2$ in Theorem 5.1 is replaced by

$$\sigma+(a-1)\rho<\rho$$

and the conditions $a\le 1$ and $a\le\frac{1}{2}$ in Theorem 5.2(a), (b) are respectively replaced by

$$\begin{aligned}&\sigma+(a-1)\rho\le 0,\\&\sigma+(2a-1)\rho\le 0.\end{aligned}$$

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