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Traveling Waves and Capillarity Driven Spreading of Shear-Thinning Fluids

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Abstract. We study capillary spreadings of thin films of liquids of power-law rheology. These satisfy

$$u_t+(u^{\lambda+2}|u_{xxx}|^{\lambda-1}u_{xxx})_x=0,$$

where $u(x,t)$ represents the thickness of the one-dimensional liquid and $\lambda>1$. We look for traveling wave solutions so that $u(x,t)=g(x+ct)$ and thus g satisfies

$$g'''=\frac{|g-\epsilon|^{\frac{1}{\lambda}}}{g^{1+\frac{2}{\lambda}}}\operatorname{sgn}(g-\epsilon).$$

We show that for each $\epsilon>0$ there is an infinitely oscillating solution, g_ϵ, such that

$$\lim_{t\to\infty} g_\epsilon=\epsilon$$

and that $g_\epsilon\to g_0$ as $\epsilon\to 0$, where $g_0\equiv 0$ for $t\geq 0$ and

$$g_0=c_\lambda|t|^{\frac{3\lambda}{2\lambda+1}}\quad\text{for}\quad t<0$$

for some constant c_λ.

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1 Introduction

In this work, we study *capillary spreadings* of thin films of liquids of power-law rheology, also known as Ostwald-de Waele fluids. The following equation for one-dimensional

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motion was derived in [1, 2] and is

$$u_t + \left(u^{\lambda+2}|u_{xxx}|^{\lambda-1}u_{xxx}\right)_x = 0,$$

where λ is a real constant and $u(x,t)$ represents the thickness of the one-dimensional liquid film at position x and time t. See also [3, 4]. When $\lambda > 1$, the fluid is called *shear thinning* and the viscosity tends to zero at high strain rates [5]. Typical values for λ are between 1.7 and 6.7 [6].

For *gravity driven* spreadings studied in [7], $u(x,t)$ satisfies

$$u_t - \left(u^{\lambda+2}|u_x|^{\lambda-1}u_x\right)_x = 0.$$

If we look for traveling wave solutions of the above equation so that $u(x,t) = g(x+ct)$ for some nonzero $c \in \mathbf{R}$, we obtain

$$cg' = \left(g^{\lambda+2}|g'|^{\lambda-1}g'\right)'$$

and thus

$$c(g-K) = g^{\lambda+2}|g'|^{\lambda-1}g'$$

for some constant K. In the case $K=0$ we obtain

$$g(z) = d(z-z_0)^{\frac{\lambda}{2\lambda+1}}$$

for some constant d which represents a current advancing with constant speed, c, and front located at $x = -ct - z_0$. In particular, this differential equation has no oscillatory traveling wave solutions. Similarly, in the case $K \neq 0$ there are no oscillatory traveling wave solutions. If $g'(m_1) = g'(m_2) = 0$ with $m_1 < m_2$, then it follows from the differential equation that $g(m_1) = K = g(m_2)$. Now let M be the maximum (or minimum) of g on $[m_1, m_2]$. Then $g'(M) = 0$ and thus $g(M) = K$. Thus $g \equiv K$ on $[m_1, m_2]$.

In this paper, we will study traveling wave solutions for *capillarity-driven* spreadings in which case we obtain

$$cg' + \left(g^{\lambda+2}|g'''|^{\lambda-1}g'''\right)' = 0$$

and so

$$cg + g^{\lambda+2}|g'''|^{\lambda-1}g''' = K.$$

If we expect that g will be essentially constant as $t \to \infty$, say $\epsilon > 0$, then this gives the equation

$$c(g-\epsilon) + g^{\lambda+2}|g'''|^{\lambda-1}g''' = 0.$$

This reduces to

$$g''' = d\,\frac{|g-\epsilon|^{\frac{1}{\lambda}}}{g^{1+\frac{2}{\lambda}}}\operatorname{sgn}(g-\epsilon), \quad \text{where } d = -\frac{c}{|c|^{1-\frac{1}{\lambda}}}.$$

Letting $y(t) = g(\frac{t}{d^{1/3}})$ gives

$$y''' = \frac{|y-\epsilon|^{\frac{1}{\lambda}}}{y^{1+\frac{2}{\lambda}}} \mathrm{sgn}(y-\epsilon).$$

We now consider

$$y'''(t) = f_\epsilon(y(t)), \tag{1.1}$$

$$y(t_0) = y_0 > 0, \quad y'(t_0) = y'_0, \quad y''(t_0) = y''_0, \tag{1.2}$$

where

$$f_\epsilon(y) \equiv \frac{|y-\epsilon|^{\frac{1}{\lambda}}}{y^{1+\frac{2}{\lambda}}} \mathrm{sgn}(y-\epsilon), \quad y,\epsilon,\lambda \in \mathbf{R}, \ y>0, \ \epsilon>0, \ \lambda>1. \tag{1.3}$$

We note that f_ϵ is increasing for $0<y<(1+\frac{1}{\lambda+1})\epsilon$, decreasing for $(1+\frac{1}{\lambda+1})\epsilon<y<\infty$, and has an absolute maximum at $y=(1+\frac{1}{\lambda+1})\epsilon$. We also see that $f_\epsilon(y)$ is *not* integrable at $y=0$ and *is* integrable at $y=\infty$. Next we define

$$F_\epsilon(y) = \int_\epsilon^y f_\epsilon(t)\,\mathrm{d}t \quad \text{for } y>0.$$

We see that $F_\epsilon(y) \geq 0$, F_ϵ is decreasing on $(0,\epsilon)$, increasing on (ϵ,∞),

$$\lim_{y\to 0^+} F_\epsilon(y) = +\infty, \tag{1.4a}$$

and there exists $0 < F_{\epsilon,\infty} < \infty$ such that

$$\lim_{y\to\infty} F_\epsilon(y) = F_{\epsilon,\infty}. \tag{1.4b}$$

Also we see that there exists $0 < L_\epsilon < \epsilon$ such that

$$F_\epsilon(L_\epsilon) = F_{\epsilon,\infty}. \tag{1.5}$$

We now define the following "energy" type functions which will be useful in analyzing solutions of Eq. (1.1). Let

$$E_{1,y} = \frac{1}{2}(y')^2 - (y-\epsilon)y'', \tag{1.6a}$$

$$E_{2,y} = F_\epsilon(y) - y'y'',, \tag{1.6b}$$

$$E_{3,y} = \frac{1}{2}(y'')^2 - f_\epsilon(y)y'. \tag{1.6c}$$

Note that

$$E'_{1,y} = -(y-\epsilon)y''' = -(y-\epsilon)f_\epsilon(y) = -\frac{|y-\epsilon|^{1+\frac{1}{\lambda}}}{y^{1+\frac{2}{\lambda}}} \leq 0, \tag{1.7a}$$

$$E'_{2,y} = -(y'')^2 \leq 0, \tag{1.7b}$$

$$E'_{3,y} = -f'_\epsilon(y)(y')^2. \tag{1.7c}$$

It can be verified that

$$E_{3,y}'\leq 0\quad\text{for }\ 0<y\leq\Big(1+\frac{1}{\lambda+1}\Big)\epsilon$$

and

$$E_{3,y}'\geq 0\quad\text{for }\ y\geq\Big(1+\frac{1}{\lambda+1}\Big)\epsilon.$$

In this paper we prove the following:

Main Theorem. *Let $\epsilon>0$ and $\lambda>1$. There exists a solution of (1.1) with $y(0)=L_\epsilon$, $y'(0)=0$, and $y''(0)=b_\epsilon>0$ and y_{b_ϵ} is decreasing on $(-\infty,0)$, oscillates infinitely often on $[0,\infty)$ and*

$$\lim_{t\to\infty}y_{b_\epsilon}(t)=\epsilon. \tag{1.8}$$

In addition,

$$\lim_{\epsilon\to 0}y_{b_\epsilon}(t)=y_0(t), \tag{1.9}$$

where

$$y_0=\begin{cases}0, & \text{for }\ t\geq 0,\\ c_\lambda|t|^{\frac{3\lambda}{2\lambda+1}}, & \text{for }\ t<0,\end{cases} \tag{1.10a}$$

where

$$c_\lambda=\Big[\frac{(2\lambda+1)^3}{3\lambda(\lambda-1)(\lambda+2)}\Big]^{\frac{\lambda}{2\lambda+1}}. \tag{1.10b}$$

Note that y_0 satisfies the limiting differential equation

$$y'''=\frac{1}{y^{1+\frac{1}{\lambda}}}\quad\text{for }\ t<0.$$

Also, since $\lambda>1$ then $3\lambda/(2\lambda+1)>1$ so that y_0 has zero contact angle at $t=0$. According to [3], there are other solutions to

$$y'''=\frac{1}{y^{1+\frac{1}{\lambda}}}$$

with nonzero contact angle at $t=0$ which grow like $|t|^{3\lambda/(2\lambda+1)}$ at $-\infty$. However, zero contact angle is more physically reasonable.

2 Preliminaries

In this section, we fix $\epsilon>0$ and write f, F, E_1,E_2, and E_3 instead of f_ϵ, F_ϵ, $E_{1,y}$,$E_{2,y}$, and $E_{3,y}$.

Lemma 2.1. *Let $t_0\in\mathbf{R}$. There is a solution of (1.1)-(1.2) on $(t_0-\delta,t_0+\delta)$ for some $\delta>0$. Also, for*

$$y_0>0,\quad |y_0-\epsilon|+|y_0'|+|y_0''|>0,$$

the solution is unique and the solution varies continuously with respect to the parameters (y_0,y_0',y_0'').

Proof. The standard existence-uniqueness-continuous-dependence theorem applies for all $y_0 > 0$ with $y_0 \neq \epsilon$.

If $y_0 = \epsilon$ then we still have existence by the Peano existence theorem. Now suppose $y_0 = \epsilon$ but that $y_0' \neq 0$. Then near t_0 we have that

$$|(y-\epsilon) - y_0'(t-t_0)| \leq C|t-t_0|^2,$$

which implies

$$\frac{1}{2}|y_0'||t-t_0| \leq |y-\epsilon| \leq 2|y_0'||t-t_0| \quad \text{near } t_0.$$

Assuming without loss of generality that $y_0' > 0$ then we see that this means

$$\frac{1}{2}y_0'|t-t_0| \leq (y-\epsilon) \leq 2y_0'|t-t_0| \quad \text{for } t \text{ near } t_0 \text{ and } t > t_0. \tag{2.1}$$

Similarly, if z is another solution (1.1)-(1.2) with $z_0 = \epsilon_0$, $z_0' = y_0'$, and $z_0'' = y_0''$, then

$$\frac{1}{2}y_0'|t-t_0| \leq (z-\epsilon) \leq 2y_0'|t-t_0| \quad \text{for } t \text{ near } t_0 \text{ and } t > t_0. \tag{2.2}$$

Now

$$[y-z] = \int_{t_0}^{t}\int_{t_0}^{s}\int_{t_0}^{w} [f(y(x)) - f(z(x))]\,\mathrm{d}x\mathrm{d}w\mathrm{d}s,$$

so for any fixed x we have by the Mean-Value Theorem that

$$f(y(x)) - f(z(x)) = f'(\mu y(x) + (1-\mu)z(x))[y(x) - z(x)]$$

for some $0 < \mu < 1$. Using (2.1) and that $\lambda > 1$ gives for some constant $C > 0$

$$\begin{aligned}
&|f'(\mu y(x) + (1-\mu)z(x))| \\
&\leq C|\mu y + (1-\mu)z - \epsilon|^{\frac{1}{\lambda}-1} \\
&= C|\mu(y-\epsilon) + (1-\mu)(z-\epsilon)|^{\frac{1}{\lambda}-1} \\
&\leq C\left(\frac{1}{2}y_0'\right)^{\frac{1}{\lambda}-1}|x-t_0|^{\frac{1}{\lambda}-1}.
\end{aligned}$$

Therefore

$$\begin{aligned}
|y-z| &\leq \int_{t_0}^{t}\int_{t_0}^{s}\int_{t_0}^{w} |f(y)-f(z)|\,\mathrm{d}x\mathrm{d}w\mathrm{d}s \\
&\leq C\left(\frac{1}{2}y_0'\right)^{\frac{1}{\lambda}-1}\int_{t_0}^{t}\int_{t_0}^{s}\int_{t_0}^{w} |x-t_0|^{\frac{1}{\lambda}-1}|y-z|\,\mathrm{d}x\mathrm{d}w\mathrm{d}s \\
&\leq \left(\frac{1}{2}y_0'\right)^{\frac{1}{\lambda}-1}(t-t_0)^2\int_{t_0}^{t} |s-t_0|^{\frac{1}{\lambda}-1}|y-z|\,\mathrm{d}s.
\end{aligned}$$

It follows from (2.1) and (2.2) that the last integral on the right-hand side is defined. Thus for some constant $C>0$

$$|y-z|\leq C(t-t_0)^2\int_{t_0}^{t}|s-t_0|^{\frac{1}{\lambda}-1}|y-z|\,\mathrm{d}s. \tag{2.3}$$

Letting

$$w=\int_{t_0}^{t}|s-t_0|^{\frac{1}{\lambda}-1}|y-z|\,\mathrm{d}s\geq 0.$$

Then

$$w'=|t-t_0|^{\frac{1}{\lambda}-1}|y-z|.$$

Consequently, (2.3) becomes

$$w'|t-t_0|^{1-\frac{1}{\lambda}}\leq C(t-t_0)^2 w$$

so that

$$w'\leq C|t-t_0|^{1+\frac{1}{\lambda}}w\leq Cw \quad \text{for } t \text{ near } t_0.$$

Therefore,

$$\int_{t_0}^{t}(we^{-Ct})'\leq 0$$

which implies $w\equiv 0$ on (t_0,t). Hence $y\equiv z$ on (t_0,t). A similar argument shows $y\equiv z$ on (t,t_0).

Now suppose $y_0=\epsilon$ and $y_0'=0$ but $y_0''\neq 0$. Then a similar argument as above shows that

$$\frac{1}{4}|y_0''|(t-t_0)^2\leq |y-\epsilon|\leq |y_0''|(t-t_0)^2 \quad \text{for } t \text{ near } t_0.$$

Assuming without loss of generality that $y_0''>0$, we see that this means

$$\frac{1}{4}y_0''(t-t_0)^2\leq y-\epsilon\leq y_0''(t-t_0)^2 \quad \text{for } t \text{ near } t_0 \text{ and } t>t_0. \tag{2.4}$$

Similarly if z is another solution then

$$\frac{1}{4}y_0''(t-t_0)^2\leq z-\epsilon\leq y_0''(t-t_0)^2 \quad \text{for } t \text{ near } t_0 \text{ and } t>t_0. \tag{2.5}$$

Again by the Mean-Value Theorem we have for each fixed x

$$\begin{aligned}|f(y)-f(z)|&=|f'(\mu y(x)+(1-\mu)z(x))||y(x)-z(x)|\\&\leq C|\mu y+(1-\mu)z-\epsilon|^{\frac{1}{\lambda}-1}\\&=C|\mu(y-\epsilon)+(1-\mu)(z-\epsilon)|^{\frac{1}{\lambda}-1}\\&\leq C\Big(\frac{1}{4}y_0''\Big)^{\frac{1}{\lambda}-1}|x-t_0|^{\frac{2}{\lambda}-2}.\end{aligned}$$

Therefore

$$\begin{aligned}|y-z| &\le \int_{t_0}^{t}\int_{t_0}^{s}\int_{t_0}^{w}|f(y)-f(z)|\,\mathrm{d}x\,\mathrm{d}w\,\mathrm{d}s\\ &\le C\Big(\frac{1}{4}y_0''\Big)^{\frac{1}{\lambda}-1}\int_{t_0}^{t}\int_{t_0}^{s}\int_{t_0}^{w}|x-t_0|^{\frac{2}{\lambda}-2}|y-z|\,\mathrm{d}x\,\mathrm{d}w\,\mathrm{d}s\\ &\le C\Big(\frac{1}{2}y_0'\Big)^{\frac{1}{\lambda}-1}(t-t_0)^2\int_{t_0}^{t}|s-t_0|^{\frac{2}{\lambda}-2}|y-z|\,\mathrm{d}s.\end{aligned}$$

It follows from (2.4) and (2.5) that the last integral is defined. Therefore we have for some constant C

$$|y-z|\le C(t-t_0)^2\int_{t_0}^{t}|s-t_0|^{\frac{2}{\lambda}-2}|y-z|\,\mathrm{d}s. \tag{2.6}$$

Letting

$$w=\int_{t_0}^{t}|s-t_0|^{\frac{2}{\lambda}-2}|y-z|\,\mathrm{d}s\ge 0.$$

Then

$$w'=|t-t_0|^{\frac{2}{\lambda}-2}|y-z|$$

and thus (2.6) becomes

$$w'|t-t_0|^{2-\frac{2}{\lambda}}\le C(t-t_0)^2 w.$$

Consequently,

$$w'\le C|t-t_0|^{\frac{2}{\lambda}}w\le Cw \quad \text{for } t \text{ near } t_0.$$

Therefore,

$$\int_{t_0}^{t}(we^{-Ct})'\le 0,$$

which implies that $w\equiv 0$ on (t_0,t). Hence $y\equiv z$ on (t_0,t). A similar argument shows $y\equiv z$ on (t,t_0).

Thus we have shown that the solution is unique if $y_0=\epsilon$ and either $y_0'=0$ or $y_0''=0$ but not both.

Remark: If $y_0=\epsilon$ and $y_0'=y_0''=0$, then *there are nonlinearities* f for which there is *more than one solution of* (1.1)-(1.3). For example, if

$$f(y)=|y-\epsilon|^{\frac{1}{\lambda}}\operatorname{sgn}(y-\epsilon)$$

then $y=\epsilon$ is a solution and

$$y=\epsilon+a_\lambda t^{\frac{3\lambda}{\lambda-1}},$$

where

$$a_\lambda=\Big[\frac{3\lambda(2\lambda+1)(\lambda+2)}{(\lambda-1)^3}\Big]^{\frac{\lambda}{\lambda-1}},$$

is also a solution.

Suppose now that there is a triple (y_0,y_0',y_0'') with

$$y_0>0, \quad |y_0-\epsilon|+|y_0'|+|y_0''|>0 \tag{2.7}$$

and suppose $y_0(t)$ is the solution of (1.1) with

$$y_0(t_0)=y_0, \quad y_0'(t_0)=y_0', \quad y_0''(t_0)=y_0''. \tag{2.8}$$

Let $(y_{0,n},y_{0,n}',y_{0,n}'')$ be a sequence that converges to (y_0,y_0',y_0'') and let y_n be the solution of (1.1) with

$$y_n(t_0)=y_{0,n}, \quad y_n'(t_0)=y_{0,n}', \quad y_n''(t_0)=y_{0,n}''.$$

By the existence proof all of the y_n's are defined on $(t_0-\delta,t_0+\delta)$ for some $\delta>0$ which is independent of n. On this set we have that $|f(y_n(t))|$ is bounded by a constant M so that $|y_n'''|\le M$ and so $y_n,|y_n'|,|y_n''|,|y_n'''|$ are all bounded by a constant on $[t_0-\delta/2,t_0+\delta/2]$. By the Arzela-Ascoli theorem a subsequence (denoted by y_{n_k}) along with its first and second derivatives converges uniformly to a function y with initial condition (2.8). From Eq. (1.1) we see that y_{n_k}''' converges uniformly to y''' and y solves (1.1). With (2.7), by the uniqueness part of the proof established earlier we must have $y(t)\equiv y_0(t)$ and hence y_{n_k} converges uniformly to y_0. It then follows from this that y_n converges uniformly to y_0 for if not then there would be an $\eta>0$ and a sequence $t_{n_k}\in[t_0-\delta/2,t_0+\delta/2]$ with $t_{n_k}\to t^*$ such that

$$|y_{n_k}(t_{n_k})-y_0(t^*)|\ge\eta>0.$$

However, we could proceed through the same argument as above and find a subsequence $y_{n_{k_l}}$ of y_{n_k} such that $y_{n_{k_l}}$ converges uniformly to y_0 on $[t_0-\delta/2,t_0+\delta/2]$ contradicting the above inequality. This completes the proof of the lemma. $\square$

Lemma 2.2. *Let $y(t)$ be any solution of* (1.1)-(1.2). *Then there is a maximal open interval (T_1,T_2) with $T_1<t_0<T_2$ where $y(t)$ is defined. In addition, if $T_1>-\infty$ then y is increasing near T_1 and*

$$\lim_{t\to T_1^+} y(t)=0, \tag{2.9}$$

and if $T_2<\infty$ then y is decreasing near T_2 and

$$\lim_{t\to T_2^-} y(t)=0. \tag{2.10}$$

Proof. Let (T_1,T_2) with $T_1<t_0<T_2$ be the maximal open interval where $y(t)$ is defined (and $y(t)>0$). We now let

$$c_1\equiv\inf_{(T_1,t_0]} y(t) \quad \text{and} \quad c_2\equiv\inf_{[t_0,T_2)} y(t).$$

Clearly, $c_1\ge 0,c_2\ge 0$. If $c_2>0$ then from the definition of f we see that $y'''(t)$ is uniformly bounded on $[t_0,T_2)$. Thus if $T_2<\infty$ then y, y', and y'' are also uniformly bounded on

$[t_0,T_2)$ and so the solution y could be extended to $(T_1,T_2+\delta)$ for some $\delta>0$ contradicting the definition of T_2. Thus $T_2=\infty$ if $c_2>0$. A similar argument shows that $T_1=-\infty$ if $c_1>0$.

So now suppose that $c_2=0$. Then either there is a $T<T_2$ such that $y(t)$ is decreasing on (T,T_2) or there is an increasing sequence of local minimums, m_k, of y converging to T_2 such that $y(m_{k+1})<y(m_k)$ and $\lim_{k\to\infty}y(m_k)=0$. However, if the latter is true then by (1.7b) we would have

$$F(y(m_{k+1}))=E_2(m_{k+1})\leq E_2(m_k)=F(y(m_k)).$$

But also for large k, $y(m_k)<\epsilon$ and since F is decreasing for $0<y<\epsilon$ we would have

$$F(y(m_{k+1}))\geq F(y(m_k))$$

a contradiction. Thus there is a $T<T_2$ such that $y(t)$ is decreasing on (T,T_2). Thus (2.10) holds. Similarly, if $c_1=0$ then there is $T>T_1$ such that $y(t)$ is increasing on (T_1,T) and (2.9) holds. This completes the proof of the lemma. □

Lemma 2.3. *If there is an m such that $0<y(m)\leq L_\epsilon$, $y'(m)=0$, and $y''(m)\geq 0$, then $T_1=-\infty$, $y'<0$ and $y''>0$ for $t<m$, and*

$$\lim_{t\to-\infty}y(t)=\infty. \tag{2.11}$$

Proof. If $y''(m)>0$, then there exists $\delta>0$ such that $y'<0$ on $(m-\delta,m)$. If $y''(m)=0$, then since $y'''(m)=f(y(m))<0$, it follows that there exists $\delta>0$ such that $y''>0$ on $(m-\delta,m)$. Since $y'(m)=0$ it then follows that $y'<0$ on $(m-\delta,m)$. Thus we see that if $y''(m)\geq 0$ then there exists a $\delta>0$ such that $y'<0$ on $(m-\delta,m)$.

Now suppose there exists an $m^*<m$ such that $y'(m^*)=0$ and $y'<0$ on (m^*,m). Then $y(m^*)>y(m)$ and since E_2 is decreasing we see that

$$F(y(m^*))=E_2(m^*)\geq E_2(m)=F(y(m))\geq F_\infty. \tag{2.12}$$

Now if $y(m^*)\leq L_\epsilon$, then since F is strictly decreasing on $(0,L_\epsilon]$ we see that $F(y(m^*))<F(y(m))$ which contradicts (2.12). On the other hand, if $y(m^*)>L_\epsilon$, then we see that $F(y(m^*))<F_\infty$ which again contradicts (2.12). Thus, no such m^* can exist and therefore y is decreasing for $t<m$. Then from Lemma 2.2 it follows that $T_1=-\infty$.

Next, we show that y has no inflection points for $t<m$. First we show that if y has an inflection point, p, then $y(p)>\epsilon$. So suppose there is a $p<m$ with $y''(p)=0$ and $y''>0$ on (p,m) and $y(p)\leq\epsilon$. Then on $[p,m]$ we have by (1.7c)

$$E_3'=-f'(y)(y')^2\leq 0 \quad \text{since} \quad y<\Big(1+\frac{1}{\lambda+1}\Big)\epsilon \quad \text{on} \quad [p,m].$$

Also

$$E_3(m)=\frac{1}{2}(y''(m))^2\geq 0$$

so

$$\frac{1}{2}(y'')^2 - f(y)y' \geq 0 \quad \text{on} \ [p,m].$$

Evaluating at p we obtain $f(y(p))y'(p) \leq 0$ and since $y'(p) < 0$ it follows then that $f(y(p)) \geq 0$. Consequently, $y(p) \geq \epsilon$. Since we assumed $y(p) \leq \epsilon$ we see that the only possibility is $y(p) = \epsilon$. However, if $y(p) = \epsilon$ then $y''' < 0$ on (p,m) and since $y''(p) = 0$ this implies $y'' < 0$ on (p,m), which is a contradiction. Thus, $y(p) > \epsilon$. Since $y' < 0$ for $t < m$ it follows that $y''' > 0$ for $t < p$ so if $t < q < p$ then

$$y''(t) < y''(q) < 0.$$

Integrating on (t,q) gives

$$y'(q) - y'(t) < y''(q)(q-t).$$

Thus,

$$y'(q) - y''(q)(q-t) < y'(t)$$

and the left-hand side goes to $+\infty$ as $t \to -\infty$ contradicting with $y' < 0$ for $t < m$. Thus $y'' > 0$ for $t < m$. Since we also have that $y' < 0$ for $t < m$ we then see that (2.11) holds. This completes the proof of the lemma. □

3 Existence of a solution with $\lim\limits_{t\to\infty} y(t) = \epsilon$

We now fix $\epsilon > 0$ and $b \geq 0$. Let y_b be the solution of:

$$y'''(t) = f_\epsilon(y(t)), \tag{3.1}$$

$$y(0) = L_\epsilon, \quad y'(0) = 0, \quad y''(0) = b, \tag{3.2}$$

where L_ϵ is defined in the statement after (1.4b).

We denote the maximal open interval of existence of (3.1)-(3.2) as $(T_{1,b}, T_{2,b})$. From Lemma 2.3 it follows that $T_{1,b} = -\infty$.

Lemma 3.1. *If $b = 0$, then $T_{2,b} < \infty$.*

Proof. We see that $E_{1,y_b}(0) = 0$ and since $E'_{1,y_b}(t) \leq 0$ (by (1.7a)) and $E'_{1,y_b}(0) < 0$ it follows that

$$E_{1,y_b}(t) < 0 \quad \text{on} \ (0, T_{2,b}).$$

Hence

$$0 \leq \frac{1}{2}(y_b')^2 < (y_b - \epsilon)y_b'' \quad \text{on} \ (0, T_{2,b}).$$

Then since $y_b(0) = L_\epsilon < \epsilon$, we see that $y_b < \epsilon$ and $y_b'' < 0$ for $t > 0$. Since $y_b'(0) = 0$ it follows then that $y_b' < 0$ for $t > 0$ and therefore y_b is decreasing and concave down on $(0, T_{2,b})$. Hence y_b must become zero at some finite value of t. Thus, $T_{2,b} < \infty$. This completes the proof of the lemma. □

Lemma 3.2. *If $b > 0$ is sufficiently large, then $T_{2,b} = \infty$ and $y_b'(t) > 0$ for all $t > 0$ (and hence $y_b(t) > 0$ for all $t \in \mathbf{R}$ by Lemma 2.3).*

Proof. Since $y_b'(0)=0$ and $y_b''(0)=b>0$, we see that $y_b'>0$ on $(0,\delta)$ for some $\delta>0$. Suppose first that $T_{2,b}<\infty$. Then by Lemma 2.2, there is an $M>0$ such that $y_b'(M)=0$ and $y_b'>0$ on $(0,M)$. So we see that on $(0,M)$ we have

$$y_b(t) > y_b(0) = L_\epsilon$$

and therefore

$$y_b''' = f_\epsilon(y_b) > f_\epsilon(L_\epsilon).$$

Integrating on $(0,t)$ gives

$$y_b'' > b + f_\epsilon(L_\epsilon)t \quad \text{on } (0,M).$$

Integrating again on $(0,t)$ gives

$$y_b' > bt + \frac{f_\epsilon(L_\epsilon)}{2}t^2 \quad \text{on } (0,M).$$

Taking the limit as $t \to M^-$ we get $M \geq 2b/|f_\epsilon(L_\epsilon)|$. Therefore we see that

$$y_b' > 0 \quad \text{for } 0 < t < \frac{b}{|f_\epsilon(L_\epsilon)|}.$$

After another integration we see that

$$y_b > L_\epsilon + \frac{b}{2}t^2 + \frac{f_\epsilon(L_\epsilon)}{6}t^3 \quad \text{on } (0,M).$$

Evaluating this inequality and the y_b'' inequality at $t = b/|f_\epsilon(L_\epsilon)|$ we see that

$$y_b\Big(\frac{b}{|f_\epsilon(L_\epsilon)|}\Big) > L_\epsilon + \frac{b^3}{3|f_\epsilon(L_\epsilon)|^2}, \quad y_b''\Big(\frac{b}{|f_\epsilon(L_\epsilon)|}\Big) > 0.$$

Therefore, we see that

$$y_b\Big(\frac{b}{|f_\epsilon(L_\epsilon)|}\Big) > \epsilon \quad \text{if } b \text{ is chosen sufficiently large.}$$

Now since we already know that $y_b' > 0$ on $(0,M)$ so in particular this inequality is true on the interval $(b/|f_\epsilon(L_\epsilon)|, M)$, we see that

$$y_b''' = f_\epsilon(y_b) > 0 \quad \text{on } \Big(\frac{b}{|f_\epsilon(L_\epsilon)|}, M\Big)$$

so that y_b'' is increasing on this interval and since $y_b''(b/|f_\epsilon(L_\epsilon)|)>0$, this implies $y_b''(M)>0$. On the other hand, $y_b'(M) = 0$ and $y_b' > 0$ on $(0,M)$ which implies $y_b''(M) \leq 0$ and so we obtain a contradiction. Thus we see that $T_{2,b} = \infty$.

So we now assume that $T_{2,b}=\infty$ but that y_b is not increasing for all $t>0$. So suppose there is an M so that $y_b'>0$ on $(0,M)$ and $y_b'(M)=0$. Then repeating the same argument as at the beginning of the proof of this lemma, we will obtain again a contradiction. Thus this completes the proof of the lemma. $\square$

Now we define

$$S=\{b\geq 0 \mid T_{2,b}<\infty\}. \tag{3.3}$$

It follows that S is nonempty (since $0\in S$ by Lemma 3.1) and bounded above (by Lemma 3.2). Thus we define

$$b_\epsilon=\sup S \tag{3.4}$$

and note that $b_\epsilon\geq 0$.

Lemma 3.3. *$y_{b_\epsilon}(t)>0$ for all t. (That is, $T_{2,b_\epsilon}=\infty$ and hence $b_\epsilon>0$ by Lemma 3.1).*

Proof. Suppose not. Then $T_{2,b_\epsilon}<\infty$ and so by Lemma 2.2 it follows that y_{b_ϵ} is decreasing on $(T_{2,b_\epsilon}-\delta,T_{2,b_\epsilon})$ for some $\delta>0$ and

$$\lim_{t\to T_{2,b_\epsilon}^-} y_{b_\epsilon}(t)=0. \tag{3.5}$$

Since $E_{2,y_{b_\epsilon}}$ is decreasing (by (1.7b)) we have

$$F_\epsilon(y_{b_\epsilon})-y_{b_\epsilon}'y_{b_\epsilon}''=E_{2,y_{b_\epsilon}}(t)\leq E_{2,y_{b_\epsilon}}(0)=F_\epsilon(L_\epsilon) \quad \text{for } 0\leq t\leq T_{2,b_\epsilon}. \tag{3.6}$$

Now it follows from (1.4a) and Lemma 2.2 that

$$\lim_{t\to T_{2,b_\epsilon}^-} F_\epsilon(y_{b_\epsilon}(t))=+\infty. \tag{3.7}$$

Therefore since the right hand side of (3.6) is bounded (since ϵ is fixed), it follows that

$$\lim_{t\to T_{2,b_\epsilon}^-} y_{b_\epsilon}'(t)y_{b_\epsilon}''(t)=+\infty.$$

From this and Lemma 2.2 it follows that there exists a neighborhood of T_{2,b_ϵ}, $(T_{2,b_\epsilon}-\delta,T_{2,b_\epsilon})$ (where we decrease the size of the δ chosen at the beginning of the proof if necessary), such that

$$0<y_{b_\epsilon}(t)<\epsilon, \quad y_{b_\epsilon}'(t)<0, \quad y_{b_\epsilon}''(t)<0 \quad \text{for all } t\in(T_{2,b_\epsilon}-\delta,T_{2,b_\epsilon}).$$

Now by Lemma 2.1, it follows that

$$0<y_b<\epsilon, \quad y_b'<0, \quad y_b''<0 \quad \text{on } \left(T_{2,b_\epsilon}-\frac{2}{3}\delta,T_{2,b_\epsilon}-\frac{1}{3}\delta\right)$$

if b is sufficiently close to b_ϵ. If we also require $b > b_\epsilon$, then $T_{2,b} = \infty$ (by definition of b_ϵ) and so $y_b(t) > 0$ for all t. Let us now denote $(T_{2,b_\epsilon} - \frac{2}{3}\delta, A_b)$ as the maximal interval for which

$$0 < y_b < \epsilon, \quad y_b' < 0, \quad y_b'' < 0. \tag{3.8}$$

From (1.1) we see that $y_b''' < 0$ on $(T_{2,b_\epsilon} - \frac{2}{3}\delta, A_b)$. Thus, $0 < y_b < \epsilon$, y_b is decreasing, concave down, and y_b'' is decreasing on $(T_{2,b_\epsilon} - \frac{2}{3}\delta, A_b)$. Now A_b must be finite for if A_b were infinite then y_b would be decreasing and concave down for t large forcing y_b to become zero in a finite value of t contradicting the fact that $y_b > 0$ for all t (since $b > b_\epsilon$). Thus, A_b is finite. Thus, either

$$y_b(A_b) = 0 \quad \text{or} \quad y_b'(A_b) = 0 \quad \text{or} \quad y_b''(A_b) = 0. \tag{3.9}$$

However, since $b > b_\epsilon$, $y_b > 0$ for all t, the first condition is impossible. Also

$$y_b\left(T_{2,b_\epsilon} - \frac{2}{3}\delta\right) < \epsilon, \quad y_b'\left(T_{2,b_\epsilon} - \frac{2}{3}\delta\right) < 0, \quad y_b''\left(T_{2,b_\epsilon} - \frac{2}{3}\delta\right) < 0,$$

and so from (3.8) we see that y_b is decreasing, concave down, and y_b'' is decreasing on $(T_{2,b_\epsilon} - \frac{2}{3}\delta, A_b)$. Thus

$$y_b'(A_b) < y_b'\left(T_{2,b_\epsilon} - \frac{2}{3}\delta\right) < 0,$$

and

$$y_b''(A_b) < y_b''\left(T_{2,b_\epsilon} - \frac{2}{3}\delta\right) < 0$$

which contradict (3.9). Thus the assumption that $T_{2,b_\epsilon} < \infty$ must be false and so $T_{2,b_\epsilon} = \infty$. This completes the proof of the lemma. □

Lemma 3.4. *$y_{b_\epsilon}(t)$ has a first critical point, $m_{1,\epsilon} > 0$, which is a local maximum, and $y_{b_\epsilon}' > 0$ on $(0, m_{1,\epsilon})$. Also,*

$$y_{b_\epsilon}(m_{1,\epsilon}) > \epsilon, \quad y_{b_\epsilon}''(m_{1,\epsilon}) < 0, \tag{3.10}$$

and

$$F_\epsilon(y_{b_\epsilon}(m_{1,\epsilon})) < F_\epsilon(L_\epsilon). \tag{3.11}$$

Proof. If not then $y_{b_\epsilon}'(t) > 0$ for all $t > 0$. We will now show that this implies y_{b_ϵ} increases without bound. If not then

$$\lim_{t\to\infty} y_{b_\epsilon}(t) = B_\epsilon < \infty.$$

In this case, we see that

$$\lim_{t\to\infty} y_{b_\epsilon}'''(t) = \frac{|B_\epsilon - \epsilon|^{\frac{1}{\lambda}}}{B_\epsilon^{1+\frac{2}{\lambda}}} \operatorname{sgn}(B_\epsilon - \epsilon) \equiv C_\epsilon. \tag{3.12}$$

If $B_\epsilon > \epsilon$ then $y_{b_\epsilon}''' \geq C_\epsilon > 0$ for large t and integrating three times we see that this would imply that y_{b_ϵ} would be increasing without bound contradicting the fact that

$$\lim_{t\to\infty} y_{b_\epsilon}(t) = B_\epsilon. \tag{3.13}$$

On the other hand if $0\leq B_\epsilon<\epsilon$ then $y'''_{b_\epsilon}\leq C_\epsilon<0$ for large t and integrating twice we see that this would imply that y_{b_ϵ} is decreasing for large t contradicting the fact that we are assuming that $y'_{b_\epsilon}(t)>0$ for all $t>0$. Thus it must be $B_\epsilon=\epsilon$ so that $y'_{b_\epsilon}>0$ and $y_{b_\epsilon}<\epsilon$ for all $t>0$.

Next since $y''_{b_\epsilon}(0)=b_\epsilon>0$, we see that y_{b_ϵ} must have a first inflection point $p_\epsilon>0$ and $y''_{b_\epsilon}>0$ on $(0,p_\epsilon)$. Then from (1.1) we see that y''_{b_ϵ} is decreasing for $t>0$ so it follows that $y''_{b_\epsilon}<0$ for $t>p_\epsilon$, and it also follows that there is a $q_\epsilon>p_\epsilon$ such that

$$y''_{b_\epsilon}<y''_{b_\epsilon}(q_\epsilon)<0 \quad \text{for} \;\; t>q_\epsilon.$$

Integrating on (q_ϵ,t) gives

$$y'_{b_\epsilon}<y'_{b_\epsilon}(q_\epsilon)+y''_{b_\epsilon}(q_\epsilon)(t-q_\epsilon)$$

which implies that $y'_{b_\epsilon}<0$ for large enough t which contradicts that $y'_{b_\epsilon}>0$ for $t>0$. Thus, we see that if $y'_{b_\epsilon}>0$ for all $t>0$ then it must be the case that y_{b_ϵ} does not stay bounded on $[0,\infty)$.

In particular, then there is a $z_\epsilon>0$ with $y_{b_\epsilon}(z_\epsilon)=\epsilon$ and y_{b_ϵ} is increasing for all $t>0$. Thus from (1.1), $y'''_{b_\epsilon}>0$ for $t>z_\epsilon$. So there is a $q_\epsilon>z_\epsilon$ and a $c_\epsilon>0$ such that $y'''_{b_\epsilon}>c_\epsilon$ for $t>q_\epsilon$ hence

$$y''_{b_\epsilon}(t)>y''_{b_\epsilon}(q_\epsilon)+c_\epsilon(t-q_\epsilon) \quad \text{for} \;\; t>q_\epsilon$$

and so we see that there is an r_ϵ such that $y''_{b_\epsilon}(t)>0$ for $t>r_\epsilon$. Integrating again we see that $y'_{b_\epsilon}(t)>0$ for $t>r_\epsilon$ and another integration gives that $y_{b_\epsilon}(t)>\epsilon$ for $t>r_\epsilon$.

Now if $b<b_\epsilon$ and b is sufficiently close to b_ϵ then by Lemma 2.1 $y_b>\epsilon$, $y'_b>0$ and $y''_b>0$ for $r_\epsilon<t<r_\epsilon+1$. Then from (1.1) $y'''_b>0$ for $r_\epsilon<t<r_\epsilon+1$. Therefore, y_b, y'_b, and y''_b are increasing and $y_b>\epsilon$ for $r_\epsilon<t<r_\epsilon+1$ and so we see that these conditions continue to hold for $r_\epsilon<t<\infty$, but this contradicts the fact that for $b<b_\epsilon$, y_b must have a zero. Thus we finally see that y_{b_ϵ} cannot be increasing for all $t>0$ and so we see that there exists $m_{1,\epsilon}>0$ such that

$$y'_{b_\epsilon}>0 \quad \text{on} \;\; (0,m_{1,\epsilon}) \quad \text{and} \quad y'_{b_\epsilon}(m_{1,\epsilon})=0.$$

From calculus, it also follows that $y''_{b_\epsilon}(m_{1,\epsilon})\leq 0$.

We next claim that $y_{b_\epsilon}(m_{1,\epsilon})>\epsilon$. First we suppose that $y_{b_\epsilon}(m_{1,\epsilon})<\epsilon$. Then

$$E_{1,y_{b_\epsilon}}(m_{1,\epsilon})\leq 0 \quad \text{and} \quad E'_{1,y_{b_\epsilon}}(m_{1,\epsilon})<0$$

so that since $E_{1,y_{b_\epsilon}}$ is decreasing (by (1.7a)), we see that $E_{1,y_{b_\epsilon}}<0$ for $t>m_{1,\epsilon}$. Thus

$$0\leq\frac{1}{2}(y'_{b_\epsilon})^2<(y_{b_\epsilon}-\epsilon)y''_{b_\epsilon} \quad \text{for} \;\; t>m_{1,\epsilon}$$

and since $y_{b_\epsilon}(m_{1,\epsilon})<\epsilon$ we see that

$$y_{b_\epsilon}(t)<\epsilon \quad \text{for} \;\; t>m_{1,\epsilon} \quad \text{and} \quad y''_{b_\epsilon}(t)<0 \quad \text{for} \;\; t>m_{1,\epsilon}.$$

Since $y'_{b_\epsilon}(m_{1,\epsilon})=0$, this implies $y_{b_\epsilon}(t)$ will become 0 at some finite value of t contradicting Lemma 3.3. Thus we see that $y_{b_\epsilon}(m_{1,\epsilon})\geq\epsilon$.

Next we suppose that $y_{b_\epsilon}(m_{1,\epsilon})=\epsilon$. In this case either

$$y''_{b_\epsilon}(m_{1,\epsilon})=0 \quad \text{or} \quad y''_{b_\epsilon}(m_{1,\epsilon})<0.$$

If $y''_{b_\epsilon}(m_{1,\epsilon})<0$ then $y_{b_\epsilon}<\epsilon$ on $(m_{1,\epsilon},m_{1,\epsilon}+\delta)$ for some $\delta>0$. Hence $E'_{1,y_{b_\epsilon}}<0$ on $(m_{1,\epsilon},m_{1,\epsilon}+\delta)$ and by (1.7a) since $E_{1,y_{b_\epsilon}}(m_{1,\epsilon})=0$ we see that $E_{1,y_{b_\epsilon}}(t)<0$ for $t>m_{1,\epsilon}$. Then as in the previous paragraph this implies $y_{b_\epsilon}(t)$ will become 0 at some finite value of t again contradicting Lemma 3.3.

Finally, we suppose that $y_{b_\epsilon}(m_{1,\epsilon})=\epsilon$ and $y''_{b_\epsilon}(m_{1,\epsilon})=0$. Since $y_{b_\epsilon}(t)<\epsilon$ for $0<t<m_{1,\epsilon}$, we have $y'''_{b_\epsilon}(t)<0$ for $0<t<m_{1,\epsilon}$. Thus, $y''_{b_\epsilon}(t)$ is decreasing for $0<t<m_{1,\epsilon}$. Since $y''_{b_\epsilon}(m_{1,\epsilon})=0$ this implies $y''_{b_\epsilon}>0$ for $0<t<m_{1,\epsilon}$. However, the mean value theorem implies that there exists a c with $0<c<m_{1,\epsilon}$ such that

$$0=y'_{b_\epsilon}(m_{1,\epsilon})-y'_{b_\epsilon}(0)=y''_{b_\epsilon}(c)m_{1,\epsilon}$$

which contradicts with $y''_{b_\epsilon}>0$ for $0<t<m_{1,\epsilon}$.

Thus we demonstrate that $y_{b_\epsilon}(m_{1,\epsilon})>\epsilon$.

Next we show that $y''_{b_\epsilon}(m_{1,\epsilon})<0$. From calculus it follows that $y''_{b_\epsilon}(m_{1,\epsilon})\leq 0$. so we assume now by way of contradiction that $y''_{b_\epsilon}(m_{1,\epsilon})=0$. This implies that $E_{1,y_{b_\epsilon}}(m_{1,\epsilon})=0$. Also, since $y_{b_\epsilon}(m_{1,\epsilon})>\epsilon$ we see that $E'_{1,y_{b_\epsilon}}(m_{1,\epsilon})<0$ and since $E_{1,y_{b_\epsilon}}$ is decreasing (by (1.7a)) we see that

$$\frac{1}{2}(y'_{b_\epsilon})^2-(y_{b_\epsilon}-\epsilon)y''_{b_\epsilon}=E_{1,y_{b_\epsilon}}<0 \quad \text{for} \ \ t>m_{1,\epsilon}.$$

Thus there is a $\delta>0$ such that $E_{1,y_{b_\epsilon}}<0$ for $t\geq m_{1,\epsilon}+\delta$. Thus for $b<b_\epsilon$ and b sufficiently close to b_ϵ we also have $E_{1,y_b}<0$ for $t\geq m_{1,\epsilon}+\delta$.

Also, perhaps by choosing a smaller δ if necessary, we see that

$$y'_{b_\epsilon}>0 \quad \text{on} \ \ (0,m_{1,\epsilon}-\delta] \qquad \text{and} \qquad y_{b_\epsilon}>\epsilon \quad \text{on} \ \ [m_{1,\epsilon}-\delta,m_{1,\epsilon}+\delta].$$

So by Lemma 2.1 and since $b_\epsilon>0$, if b is sufficiently close to b_ϵ then $y'_b>0$ on $(0,m_{1,\epsilon}-\delta]$ and $y_b>\epsilon$ on $[m_{1,\epsilon}-\delta,m_{1,\epsilon}+\delta]$. Now if we choose $b>b_\epsilon$, then by definition of b_ϵ we see there exists an $r_b>m_{1,\epsilon}+\delta$ such that $y_b(r_b)=0$. Therefore by the intermediate value theorem there is a z_b with $m_{1,\epsilon}+\delta<z_b<r_b$ such that $y_b(z_b)=\epsilon$. Hence

$$E_{1,y_b}(z_b)=\frac{1}{2}[y'_b(z_b)]^2\geq 0.$$

On the other hand, we know from earlier that since $z_b>m_{1,\epsilon}+\delta$ then $E_{1,y_b}(z_b)<0$. Thus we obtain a contradiction. Therefore it must be that $y''_{b_\epsilon}(m_{1,\epsilon})<0$.

Finally, since $E_{2,y_{b_\epsilon}}$ is decreasing (by (1.7b)) and $E'_{2,y_{b_\epsilon}}(0)<0$ we have

$$E_{2,y_{b_\epsilon}}(m_{1,\epsilon})<E_{2,y_{b_\epsilon}}(0)$$

and hence (3.11) holds. This completes the proof of the lemma. $\square$

Lemma 3.5. *$y_{b_\epsilon}(t)$ has a second critical point at $m_{2,\epsilon}>0$ which is a local minimum, and $y'_{b_\epsilon}<0$ on $(m_{1,\epsilon},m_{2,\epsilon})$. Also,*

$$y_{b_\epsilon}(m_{2,\epsilon})<\epsilon \quad \textit{and} \quad y''_{b_\epsilon}(m_{2,\epsilon})>0 \tag{3.14}$$

and

$$F_\epsilon(y_{b_\epsilon}(m_{2,\epsilon}))<F_\epsilon(y_{b_\epsilon}(m_{1,\epsilon})). \tag{3.15}$$

Proof. The proof of this lemma is nearly identical to the proof of Lemma 3.4 and we omit it here. □

In order to simplify notation a bit we now write $E_{1,\epsilon},E_{2,\epsilon}$, and $E_{3,\epsilon}$ instead of $E_{1,y_{b_\epsilon}}$, $E_{2,y_{b_\epsilon}}$, and $E_{3,y_{b_\epsilon}}$, respectively.

Continuing in this way we see that there is a sequence of extrema with

$$m_{1,\epsilon}<m_{2,\epsilon}<m_{3,\epsilon}<m_{4,\epsilon}<\cdots$$

such that the $m_{2k,\epsilon}$ are local minima, the $m_{2k-1,\epsilon}$ are local maxima, y is monotone of $(m_{n,\epsilon},m_{n+1,\epsilon})$, and since $E_{2,\epsilon}$ is decreasing, we have

$$F_\epsilon(y_{b_\epsilon}(m_{k+1,\epsilon}))<F_\epsilon(y_{b_\epsilon}(m_{k,\epsilon})).$$

Note that this implies

$$y_{b_\epsilon}(m_{2k,\epsilon})<y_{b_\epsilon}(m_{2k+2,\epsilon})<\epsilon \quad \text{and} \quad \epsilon<y_{b_\epsilon}(m_{2k+1,\epsilon})<y_{b_\epsilon}(m_{2k-1,\epsilon}). \tag{3.16}$$

We now let

$$M_\epsilon=\lim_{n\to\infty} m_{n,\epsilon} \tag{3.17}$$

and note that $M_\epsilon\le\infty$.

Lemma 3.6. *$y_{b_\epsilon}(t)$ oscillates infinitely often, and*

$$\lim_{t\to M_\epsilon^-} y_{b_\epsilon}(t)=\epsilon, \quad \lim_{t\to M_\epsilon^-} y'_{b_\epsilon}(t)=0, \quad \lim_{t\to M_\epsilon^-} y''_{b_\epsilon}(t)=0.$$

Proof. We have $0\equiv m_{0,\epsilon}<m_{1,\epsilon}<m_{2,\epsilon}<m_{3,\epsilon}<\cdots$ and

$$F_\epsilon(L_\epsilon)>F_\epsilon(y_{b_\epsilon}(m_{1,\epsilon}))>F_\epsilon(y_{b_\epsilon}(m_{2,\epsilon}))>F_\epsilon(y_{b_\epsilon}(m_3,\epsilon))>\cdots.$$

Also, there exists $z_{k,\epsilon}$ such that

$$0<z_{1,\epsilon}<m_{1,\epsilon}<z_{2,\epsilon}<m_{2,\epsilon}<z_{3,\epsilon}<\cdots, \quad y_{b_\epsilon}(z_{n,\epsilon})=\epsilon, \quad \lim_{n\to\infty} z_{n,\epsilon}=M_\epsilon.$$

Next we observe that since $y'_{b_\epsilon}(m_k)=y'_{b_\epsilon}(m_{k+1})=0$ the extrema of y'_{b_ϵ} on $(m_{k,\epsilon},m_{k+1,\epsilon})$ must occur at points p where $y''_{b_\epsilon}(p)=0$ so

$$\frac{1}{2}[y'_{b_\epsilon}(p)]^2=E_{1,\epsilon}(p)\le E_{1,\epsilon}(0)=(\epsilon-L_\epsilon)b_\epsilon.$$

Thus for every $k \geq 0$

$$|y'_{b_\epsilon}(t)| \leq \sqrt{2(\epsilon - L_\epsilon)b_\epsilon} \equiv K_\epsilon \quad \text{on } [m_{k,\epsilon}, m_{k+1,\epsilon}].$$

Then since $m_{k,\epsilon} \to M_\epsilon$ as $k \to \infty$ we obtain

$$|y'_{b_\epsilon}(t)| \leq \sqrt{2(\epsilon - L_\epsilon)b_\epsilon} \equiv K_\epsilon \quad \text{on } [0, M_\epsilon]. \tag{3.18}$$

Next, since $E_{1,\epsilon}$ is decreasing, $E_{1,\epsilon}(z_{k,\epsilon}) = \frac{1}{2}[y'_{b_\epsilon}(z_{k,\epsilon})]^2 \geq 0$, and $z_{k,\epsilon} \to M_\epsilon$ we see that

$$\lim_{t \to M_\epsilon^-} E_{1,\epsilon}(t) = e_{1,\epsilon} \geq 0. \tag{3.19}$$

Integrating (1.7a) on $(0,t)$ we obtain

$$E_{1,\epsilon}(t) = (\epsilon - L_\epsilon)b_\epsilon - \int_0^t (y_{b_\epsilon} - \epsilon) f_\epsilon(y_{b_\epsilon}).$$

Using (3.19) and taking limits as $t \to M_\epsilon^-$ give

$$(\epsilon - L_\epsilon)b_\epsilon = e_{1,\epsilon} + \int_0^{M_\epsilon} (y_{b_\epsilon} - \epsilon) f_\epsilon(y_{b_\epsilon}).$$

Thus we see that

$$\int_0^{M_\epsilon} (y_{b_\epsilon} - \epsilon) f_\epsilon(y_{b_\epsilon}) \text{ is finite.} \tag{3.20}$$

We have $y'''_{b_\epsilon} > 0$ on $(z_{1,\epsilon}, m_{1,\epsilon})$ so that y''_{b_ϵ} is increasing on $(z_{1,\epsilon}, m_{1,\epsilon})$. Also from Lemma 3.4 we know that $y''_{b_\epsilon}(m_{1,\epsilon}) < 0$ therefore it follows that $y''_{b_\epsilon} < 0$ on $(z_{1,\epsilon}, m_{1,\epsilon})$. Therefore, y_{b_ϵ} is concave down on $(z_{1,\epsilon}, m_{1,\epsilon})$ and so it follows that

$$y_{b_\epsilon} - \epsilon \geq \frac{y_{b_\epsilon}(m_{1,\epsilon}) - \epsilon}{m_{1,\epsilon} - z_{1,\epsilon}}(t - z_{1,\epsilon}) \quad \text{on } (z_{1,\epsilon}, m_{1,\epsilon}). \tag{3.21}$$

Similarly, since $y''_{b_\epsilon} > 0$ on $(z_{2,\epsilon}, m_{2,\epsilon})$ we see that

$$y_{b_\epsilon} - \epsilon \leq \frac{y_{b_\epsilon}(m_{2,\epsilon}) - \epsilon}{m_{2,\epsilon} - z_{2,\epsilon}}(t - z_{2,\epsilon}) \quad \text{on } (z_{2,\epsilon}, m_{2,\epsilon}). \tag{3.22}$$

Thus, it follows from (3.21) that

$$\begin{aligned} &\int_{z_{1,\epsilon}}^{m_{1,\epsilon}} (y_{b_\epsilon} - \epsilon) f(y_{b_\epsilon}) \, \mathrm{d}t = \int_{z_{1,\epsilon}}^{m_{1,\epsilon}} \frac{|y_{b_\epsilon} - \epsilon|^{1+\frac{1}{\lambda}}}{y_{b_\epsilon}^{1+\frac{2}{\lambda}}} \mathrm{d}t \\ \geq &\frac{1}{y_{b_\epsilon}(m_{1,\epsilon})^{1+\frac{2}{\lambda}}} \left| \frac{y_{b_\epsilon}(m_{1,\epsilon}) - \epsilon}{m_{1,\epsilon} - z_{1,\epsilon}} \right|^{1+\frac{1}{\lambda}} \int_{z_{1,\epsilon}}^{m_{1,\epsilon}} (t - z_{1,\epsilon})^{1+\frac{1}{\lambda}} \mathrm{d}t \\ = &\frac{\lambda}{2\lambda + 1} \frac{|y_{b_\epsilon}(m_{1,\epsilon}) - \epsilon|^{1+\frac{1}{\lambda}}}{y_{b_\epsilon}(m_{1,\epsilon})^{1+\frac{2}{\lambda}}} (m_{1,\epsilon} - z_{1,\epsilon}). \end{aligned}$$

Also, by the mean value theorem and (3.18) we have

$$\begin{aligned}|y_{b_\epsilon}(m_{1,\epsilon})-\epsilon| &= |y_{b_\epsilon}(m_{1,\epsilon})-y_{b_\epsilon}(z_{1,\epsilon})|\\ &= |y'_{b_\epsilon}(c_{1,\epsilon})||(m_{1,\epsilon}-z_{1,\epsilon})| \le K_\epsilon |m_{1,\epsilon}-z_{1,\epsilon}|.\end{aligned}$$

Thus

$$\int_{z_{1,\epsilon}}^{m_{1,\epsilon}} (y_{b_\epsilon}-\epsilon) f_\epsilon(y_{b_\epsilon}) \ge \frac{\lambda |y_{b_\epsilon}(m_{1,\epsilon})-\epsilon|^{2+\frac{1}{\lambda}}}{(2\lambda+1)K_\epsilon y_{b_\epsilon}(m_{1,\epsilon})^{1+\frac{2}{\lambda}}}. \tag{3.23}$$

A similar inequality holds over $(z_{2,\epsilon}, m_{2,\epsilon})$ and thus

$$\int_{z_{2,\epsilon}}^{m_{2,\epsilon}} (y_{b_\epsilon}-\epsilon) f_\epsilon(y_{b_\epsilon}) \ge \frac{\lambda |y_{b_\epsilon}(m_{2,\epsilon})-\epsilon|^{2+\frac{1}{\lambda}}}{(2\lambda+1)K_\epsilon y_{b_\epsilon}(m_{2,\epsilon})^{1+\frac{2}{\lambda}}}.$$

Now using (3.16) we see that

$$\int_{z_{2,\epsilon}}^{m_{2,\epsilon}} (y_{b_\epsilon}-\epsilon) f_\epsilon(y_{b_\epsilon}) \ge \frac{\lambda |y_{b_\epsilon}(m_{2,\epsilon})-\epsilon|^{2+\frac{1}{\lambda}}}{(2\lambda+1)K_\epsilon y_{b_\epsilon}(m_{1,\epsilon})^{1+\frac{2}{\lambda}}}.$$

Similarly we can show

$$\int_{z_{k,\epsilon}}^{m_{k,\epsilon}} (y_{b_\epsilon}-\epsilon) f_\epsilon(y_{b_\epsilon}) \ge \frac{\lambda |y_{b_\epsilon}(m_{k,\epsilon})-\epsilon|^{2+\frac{1}{\lambda}}}{(2\lambda+1)K_\epsilon y_{b_\epsilon}(m_{1,\epsilon})^{1+\frac{2}{\lambda}}}. \tag{3.24}$$

Next using (3.20) and the fact that $(y_{b_\epsilon}-\epsilon) f_\epsilon(y_{b_\epsilon}) \ge 0$ for all t we obtain

$$\begin{aligned}\infty &> \int_0^{M,\epsilon} (y_{b_\epsilon}-\epsilon) f_\epsilon(y_{b_\epsilon})\,\mathrm{d}t\\ &\ge \sum_{k=1}^{\infty} \int_{z_{k,\epsilon}}^{m_{k,\epsilon}} (y_{b_\epsilon}-\epsilon) f_\epsilon(y_{b_\epsilon})\,\mathrm{d}t\\ &\ge \frac{\lambda}{(2\lambda+1)K_\epsilon y_{b_\epsilon}(m_{1,\epsilon})^{1+\frac{2}{\lambda}}} \sum_{k=1}^{\infty} |y_{b_\epsilon}(m_{k,\epsilon})-\epsilon|^{2+\frac{1}{\lambda}}.\end{aligned}$$

Thus

$$\sum_{k=1}^{\infty} |y_{b_\epsilon}(m_{k,\epsilon})-\epsilon|^{2+\frac{1}{\lambda}} < \infty.$$

Consequently,

$$\lim_{k\to\infty} |y_{b_\epsilon}(m_{k,\epsilon})-\epsilon| = 0$$

and since $m_{k,\epsilon} \to M_\epsilon^-$ and the $m_{k,\epsilon}$ are extrema of y_{b_ϵ} we see that

$$\lim_{t\to M_\epsilon^-} |y_{b_\epsilon}(t)-\epsilon| = 0. \tag{3.25}$$

Then by (1.1) we obtain

$$\lim_{t\to M_\epsilon^-} y_{b_\epsilon}'''(t) = 0. \tag{3.26}$$

We also know that $E_{2,\epsilon}' \leq 0$ (by (1.7b)) and by (1.7c) and (3.25) we know that $E_{3,\epsilon}' \leq 0$ for t close to M_ϵ so that

$$\lim_{t\to M_\epsilon^-} E_{2,\epsilon}(t) = e_{2,\epsilon}, \quad \lim_{t\to M_\epsilon^-} E_{3,\epsilon}(t) = e_{3,\epsilon}. \tag{3.27}$$

Also since $E_{2,\epsilon}(m_{k,\epsilon}) \geq 0$ and $E_{3,\epsilon}(m_{k,\epsilon}) \geq 0$ and since $m_{k,\epsilon} \to M_\epsilon$ we see that

$$e_{2,\epsilon} \geq 0 \quad \text{and} \quad e_{3,\epsilon} \geq 0. \tag{3.28}$$

From (3.18) and (3.25) it follows that

$$f_\epsilon(y_{b_\epsilon}) y_{b_\epsilon} \to 0 \quad \text{as } t \to M_\epsilon^-.$$

Combining this with (3.27) we see that

$$\lim_{t\to M_\epsilon^-} \frac{1}{2}(y_{b_\epsilon}'')^2 = e_{3,\epsilon}.$$

Since y_{b_ϵ}' is bounded (by (3.18)) we see that the only possibility is that $e_{3,\epsilon} = 0$ thus

$$\lim_{t\to M_\epsilon^-} y_{b_\epsilon}'' = 0. \tag{3.29}$$

Now using (3.19), (3.25), and (3.29) we see that

$$\lim_{t\to M_\epsilon^-} \frac{1}{2}(y_{b_\epsilon}')^2 = \lim_{t\to M_\epsilon^-} E_{1,\epsilon} = e_{1,\epsilon}. \tag{3.30}$$

Since y_{b_ϵ} is bounded (by (3.25)) we see that the only possibility is that $e_{1,\epsilon} = 0$ and so

$$\lim_{t\to M_\epsilon^-} y_{b_\epsilon}'(t) = 0. \tag{3.31}$$

Using (3.25), (3.29), and (3.31) completes the proof of the lemma. □

One final note, if $M_\epsilon < \infty$ then since

$$\lim_{t\to M_\epsilon^-} y_{b_\epsilon}(t) = \epsilon, \quad \lim_{t\to M_\epsilon^-} y_{b_\epsilon}'(t) = 0, \quad \lim_{t\to M_\epsilon^-} y_{b_\epsilon}''(t) = 0,$$

we see that we may extend $y_{b_\epsilon}(t)$ for $t \geq M_\epsilon$ by simply defining

$$y_{b_\epsilon}(t) \equiv \epsilon \quad \text{for } t \geq M_\epsilon.$$

Then whether $M_\epsilon < \infty$ or $M_\epsilon = \infty$ we see that

$$\lim_{t\to\infty} y_{b_\epsilon}(t) = \epsilon.$$

4 Determination of $\lim_{\epsilon\to 0} y_{b_\epsilon}(t)$

Lemma 4.1. *Let* L_ϵ *be defined by* (1.5). *Then*

$$L_\epsilon = L_1\epsilon \quad \textit{where} \;\; 0<L_1<1. \tag{4.1}$$

Proof. First we denote

$$I=\int_1^\infty \frac{(t-1)^{\frac{1}{\lambda}}}{t^{1+\frac{2}{\lambda}}}\mathrm{d}t. \tag{4.2}$$

Next, by definition we have

$$F_\epsilon(y)=\int_\epsilon^y \frac{|s-\epsilon|^{\frac{1}{\lambda}}\mathrm{sgn}(s-\epsilon)}{s^{1+\frac{2}{\lambda}}}\mathrm{d}s.$$

Making the change of variables $s=\epsilon t$ we obtain

$$F_\epsilon(y)=\epsilon^{-\frac{1}{\lambda}}F_1(y/\epsilon). \tag{4.3}$$

Hence, by (1.4b), (4.2), and (4.3) we see that

$$F_{\epsilon,\infty}=\lim_{y\to\infty}F_\epsilon(y)=\epsilon^{-\frac{1}{\lambda}}\int_1^\infty \frac{(t-1)^{\frac{1}{\lambda}}}{t^{1+\frac{2}{\lambda}}}\mathrm{d}t=\epsilon^{-\frac{1}{\lambda}}I.$$

Also, by the statement after (1.4b) and (4.3) we see that

$$\epsilon^{-\frac{1}{\lambda}}\int_{\frac{L_\epsilon}{\epsilon}}^1 \frac{(1-t)^{\frac{1}{\lambda}}}{t^{1+\frac{2}{\lambda}}}\mathrm{d}t=F_\epsilon(L_\epsilon)=F_{\epsilon,\infty}=\epsilon^{-\frac{1}{\lambda}}I.$$

So we see from (4.2) and the above line that

$$\int_1^\infty \frac{(t-1)^{\frac{1}{\lambda}}}{t^{1+\frac{2}{\lambda}}}\mathrm{d}t=I=\int_{\frac{L_\epsilon}{\epsilon}}^1 \frac{(1-t)^{\frac{1}{\lambda}}}{t^{1+\frac{2}{\lambda}}}\mathrm{d}t,$$

which implies that L_ϵ/ϵ is independent of ϵ since I does not depend on ϵ (by (4.2)). Thus $L_\epsilon/\epsilon=L_1$. Also, from the statement after (1.4b) we see that $0<L_\epsilon<\epsilon$ and thus $0<L_1<1$. This completes the proof of the lemma. □

Lemma 4.2. *If*

$$b>[3f_\epsilon^2(L_\epsilon)(\epsilon-L_\epsilon)]^{\frac{1}{3}}, \tag{4.4}$$

then $y_b(t)>0$ *for all* $t\geq 0$ (*and thus* $b\notin S$ (*see* (3.3))). *Hence,*

$$b_\epsilon\leq[3f_\epsilon^2(L_\epsilon)(\epsilon-L_\epsilon)]^{\frac{1}{3}}. \tag{4.5}$$

Proof. Since

$$y_b(0)=L_\epsilon,\quad y_b'(0)=0,\quad y_b''(0)=b>0,$$

it follows that $y_b(t)$ is initially increasing and so $y_b(t)>L_\epsilon$ on $(0,\delta)$ for some $\delta>0$. So on this interval we have

$$y_b'''>f_\epsilon(L_\epsilon).$$

Successively integrating on $(0,t]$ we get

$$y_b''>b+tf_\epsilon(L_\epsilon),\quad y_b'>bt+\frac{t^2f_\epsilon(L_\epsilon)}{2},\quad y_b>L_\epsilon+\frac{bt^2}{2}+\frac{t^3f_\epsilon(L_\epsilon)}{6}.$$

Next, we observe that

$$y_b'>0,\quad y_b''>0\quad \text{for } 0<t\le\frac{b}{|f_\epsilon(L_\epsilon)|}.$$

From the inequality for y_b and (4.4) we see that

$$y_b\Big(\frac{b}{|f_\epsilon(L_\epsilon)|}\Big)>L_\epsilon+\frac{b^3}{3|f_\epsilon(L_\epsilon)|^2}>L_\epsilon+\epsilon-L_\epsilon=\epsilon.$$

Then since

$$y_b'\Big(\frac{b}{|f_\epsilon(L_\epsilon)|}\Big)>0,\quad y_b''\Big(\frac{b}{|f_\epsilon(L_\epsilon)|}\Big)>0,$$

it follows from (1.1) that

$$y_b'''\Big(\frac{b}{|f_\epsilon(L_\epsilon)|}\Big)>0.$$

This in fact implies hence $y_b'>0$ and $y_b''>0$ for all $t>b/|f_\epsilon(L_\epsilon)|$ so that in fact $y_b(t)>0$ for all $t\ge 0$. This completes the proof of the lemma. □

Lemma 4.3.

$$b_\epsilon\le\frac{Q}{\epsilon^{\frac{1}{3}+\frac{2}{3\lambda}}}\quad \textit{where}\ \ Q=\left(\frac{3(1-L_1)^{1+\frac{2}{\lambda}}}{L_1^{2+\frac{4}{\lambda}}}\right)^{\frac{1}{3}}.$$

Proof. We know that $L_\epsilon=L_1\epsilon$ by Lemma 4.1 so that

$$|f_\epsilon(L_\epsilon)|=|f_\epsilon(L_1\epsilon)|=\frac{(1-L_1)^{\frac{1}{\lambda}}}{L_1^{1+\frac{2}{\lambda}}}\frac{1}{\epsilon^{1+\frac{1}{\lambda}}}.$$

Substituting this equation and that $L_\epsilon=L_1\epsilon$ into the consequence of Lemma 4.2 we see that

$$b_\epsilon^3\le 3f_\epsilon^2(L_\epsilon)(\epsilon-L_\epsilon)=\frac{3(1-L_1)^{\frac{2}{\lambda}}}{L_1^{2+\frac{4}{\lambda}}}\frac{1}{\epsilon^{2+\frac{2}{\lambda}}}(1-L_1)\epsilon=\frac{Q^3}{\epsilon^{1+\frac{2}{\lambda}}}.$$

Taking cube roots we see that this completes the proof of the lemma. □

Lemma 4.4. *$y_{b_\epsilon}\to 0$ and $y'_{b_\epsilon}\to 0$ uniformly on compact subsets of $[0,\infty)$.*

Proof. Since $E_{1,\epsilon}$ is decreasing by (1.7a), for $t\geq 0$ we have by Lemma 4.3 that

$$\begin{aligned}&\frac{1}{2}(y'_{b_\epsilon})^2-(y_{b_\epsilon}-\epsilon)y''_{b_\epsilon}\\ &=E_{1,\epsilon}\leq E_{1,\epsilon}(0)=(\epsilon-L_\epsilon)b_\epsilon\leq\epsilon b_\epsilon\leq Q\epsilon^{\frac{2}{3}(1-\frac{1}{\lambda})}.\end{aligned}\tag{4.6}$$

Also, since

$$y'_{b_\epsilon}(0)=0 \quad\text{and}\quad \lim_{t\to M_\epsilon^-}y'_{b_\epsilon}(t)=0 \quad\text{(by Lemma 3.6)}$$

we see that the maximum of $|y'_{b_\epsilon}|$ occurs at some point p where $y''_{b_\epsilon}(p)=0$. Evaluating (4.6) at p gives

$$\frac{1}{2}(y'_{b_\epsilon}(p))^2\leq Q\epsilon^{\frac{2}{3}(1-\frac{1}{\lambda})}.$$

Thus

$$|y'_{b_\epsilon}(t)|\leq\sqrt{2Q}\epsilon^{\frac{1}{3}(1-\frac{1}{\lambda})}\quad\text{for all } t\geq 0.$$

Consequently,

$$|y'_{b_\epsilon}(t)|\to 0\quad\text{uniformly on } [0,\infty).$$

Now letting $P>0$ and integrating on $[0,P]$ we see that

$$|y_{b_\epsilon}(t)-L_\epsilon|\leq P\sqrt{2Q}\epsilon^{\frac{1}{3}(1-\frac{1}{\lambda})}$$

and since $L_\epsilon\to 0$ as $\epsilon\to 0$ (by Lemma 4.1) we see that $y_{b_\epsilon}(t)\to 0$ uniformly on compact subsets of $[0,\infty)$. This completes the proof of the lemma. □

We now investigate the behavior of $y_{b_\epsilon}(t)$ as $t\to-\infty$. From Lemma 2.3 we know that

$$y'_{b_\epsilon}(t)<0,\quad y''_{b_\epsilon}(t)>0\quad\text{for } t<0\quad\text{and}\quad\lim_{t\to-\infty}y_{b_\epsilon}(t)=\infty.$$

Thus, for t sufficiently negative we have that

$$y_{b_\epsilon}(t)>\Big(1+\frac{1}{\lambda+1}\Big)\epsilon$$

and thus by (1.7c) $E'_{3,\epsilon}\geq 0$ if t is sufficiently negative. Thus, there exists $t_{0,\epsilon}<0$ such that $E_{3,\epsilon}(t)\leq E_{3,\epsilon}(t_{0,\epsilon})$ for $t<t_{0,\epsilon}$. Thus,

$$\frac{1}{2}(y''_{b_\epsilon})^2-f_\epsilon(y_{b_\epsilon})y'_{b_\epsilon}\leq E_{3,\epsilon}(t_{0,\epsilon})\quad\text{for } t<t_{0,\epsilon}.$$

Since $y'_{b_\epsilon}<0$ for $t<0$ and $y_{b_\epsilon}>(1+\frac{1}{\lambda+1})\epsilon>\epsilon$ for $t<t_{0,\epsilon}$ we see that

$$0\leq\frac{1}{2}(y''_{b_\epsilon})^2\leq E_{3,\epsilon}(t_{0,\epsilon}),\quad 0\leq -f_\epsilon(y_{b_\epsilon})y'_{b_\epsilon}\leq E_{3,\epsilon}(t_{0,\epsilon})\quad\text{for } t<t_{0,\epsilon}.$$

Thus $E_{3,\epsilon}(t) \geq 0$ for $t < t_{0,\epsilon}$ and since $E_{3,\epsilon}(t)$ is increasing for $t < t_{0,\epsilon}$ it follows that

$$\lim_{t \to -\infty} E_{3,\epsilon}(t) = e_{3,\epsilon} \geq 0.$$

Since $y'''_{b_\epsilon} = f_\epsilon(y_{b_\epsilon}) > 0$ for $t < t_{0,\epsilon}$, we see that y''_{b_ϵ} is increasing for $t < t_{0,\epsilon}$ and since we also have $y''_{b_\epsilon} > 0$ for $t < 0$, it follows that

$$\lim_{t \to -\infty} y''_{b_\epsilon}(t) = A_\epsilon \geq 0.$$

Combining this with the fact that $E_{3,\epsilon}$ has a limit as $t \to -\infty$ it follows that

$$\lim_{t \to -\infty} -f_\epsilon(y_{b_\epsilon}) y'_{b_\epsilon} = G_\epsilon \geq 0.$$

Lemma 4.5.

$$\lim_{t \to -\infty} f_\epsilon(y_{b_\epsilon}) y'_{b_\epsilon} = 0.$$

Proof. Suppose that $G_\epsilon > 0$. Then there exists a sufficiently negative $t_{1,\epsilon}$ such that

$$-f_\epsilon(y_{b_\epsilon}) y'_{b_\epsilon} \geq \frac{G_\epsilon}{2} \quad \text{for} \;\; t < t_{1,\epsilon}.$$

Therefore

$$\int_t^{t_{1,\epsilon}} -f_\epsilon(y_{b_\epsilon}) y'_{b_\epsilon} \, \mathrm{d}s \geq \int_t^{t_{1,\epsilon}} \frac{G_\epsilon}{2} \mathrm{d}s$$

so that

$$\infty > F_{\epsilon,\infty} \geq F_\epsilon(y_{b_\epsilon}(t)) \geq -F_\epsilon(y_{b_\epsilon}(t_{1,\epsilon})) + F_\epsilon(y_{b_\epsilon}(t)) \geq \frac{G_\epsilon}{2}(t_{1,\epsilon} - t) \quad \text{for} \;\; t < t_{1,\epsilon}.$$

However, as $t \to -\infty$ the right hand side goes to ∞ as $t \to -\infty$ which is a contradiction to the above inequality. Hence it must be that $G_\epsilon = 0$. This completes the proof of the lemma. □

Lemma 4.6.

$$\lim_{t \to -\infty} \frac{-y'_{b_\epsilon}}{\sqrt{y_{b_\epsilon} - \epsilon}} = \sqrt{2A_\epsilon}.$$

Proof. Since $E'_{1,\epsilon} \leq 0$ and $E_{1,\epsilon}(0) = (\epsilon - L_\epsilon) b_\epsilon \geq 0$, it follows that $E_{1,\epsilon} \geq 0$ for $t \leq 0$. Since $y'_{b_\epsilon}(t) < 0$ for $t < 0$ and $y_{b_\epsilon}(t) > \epsilon$ for t sufficiently negative we see that

$$\left(\frac{-y'_{b_\epsilon}}{\sqrt{y_{b_\epsilon} - \epsilon}} \right)' = \frac{E_{1,\epsilon}}{(y_{b_\epsilon} - \epsilon)^{\frac{3}{2}}} > 0$$

for t sufficiently negative. Thus the function within the bracket above is positive and increasing for t sufficiently negative. Consequently,

$$\lim_{t \to -\infty} \frac{-y'_{b_\epsilon}}{\sqrt{y_{b_\epsilon} - \epsilon}} = V_\epsilon \geq 0.$$

Also, since

$$0 \leq E_{1,\epsilon} = \frac{1}{2}(y'_{b_\epsilon})^2 - (y_{b_\epsilon} - \epsilon)y''_{b_\epsilon} \quad \text{for } t < 0$$

and $y_{b_\epsilon}(t) > \epsilon$, for t sufficiently negative we have

$$\frac{(y'_{b_\epsilon})^2}{y_{b_\epsilon} - \epsilon} \geq 2y''_{b_\epsilon}.$$

Taking limits as $t \to -\infty$ we obtain $V_\epsilon^2 \geq 2A_\epsilon$. Thus, if $V_\epsilon = 0$ then $A_\epsilon = 0$. If $V_\epsilon > 0$, then since $y_{b_\epsilon}(t) \to \infty$ as $t \to -\infty$ then also $-y'_{b_\epsilon} \to \infty$ as $t \to -\infty$. Thus we may apply L'Hopital's rule and obtain

$$V_\epsilon^2 = \lim_{t \to -\infty} \frac{(y'_{b_\epsilon})^2}{y_{b_\epsilon} - \epsilon} = \lim_{t \to -\infty} \frac{2y'_{b_\epsilon} y''_{b_\epsilon}}{y'_{b_\epsilon}} = 2A_\epsilon.$$

Thus in all cases we obtain $V_\epsilon = \sqrt{2A_\epsilon}$. This completes the proof of the lemma. □

We now define

$$w_\epsilon(t) = \frac{1}{\epsilon} y_{b_\epsilon}\left(\epsilon^{\frac{2\lambda+1}{3\lambda}} t\right) \tag{4.7}$$

and observe that w_ϵ satisfies

$$\frac{w_\epsilon(t)}{|t|^{\frac{3\lambda}{2\lambda+1}}} = \frac{y_{b_\epsilon}(s)}{|s|^{\frac{3\lambda}{2\lambda+1}}}, \quad \frac{w'_\epsilon(t)}{|t|^{\frac{\lambda-1}{2\lambda+1}}} = \frac{y'_{b_\epsilon}(s)}{|s|^{\frac{\lambda-1}{2\lambda+1}}}, \quad |t|^{\frac{\lambda+2}{2\lambda+1}} w''_\epsilon(t) = |s|^{\frac{\lambda+2}{2\lambda+1}} y''_{b_\epsilon}(s), \tag{4.8}$$

where $s = \epsilon^{\frac{2\lambda+1}{3\lambda}} t$. Also, we see that w_ϵ satisfies

$$\begin{aligned} w'''_\epsilon &= \frac{|w_\epsilon - 1|^{\frac{1}{\lambda}}}{w_\epsilon^{1+\frac{2}{\lambda}}} \operatorname{sgn}(w_\epsilon - 1) = f_1(w_\epsilon), \qquad (4.9)\\ w_\epsilon(0) &= \frac{L_\epsilon}{\epsilon} = L_1 \text{ by Lemma 4.1,} \\ w'_\epsilon(0) &= 0, \quad w''_\epsilon(0) = \epsilon^{\frac{1}{3} + \frac{2}{3\lambda}} b_\epsilon. \end{aligned}$$

We also define

$$\tilde{E}_{1,\epsilon} = \frac{1}{2}(w'_\epsilon)^2 - (w_\epsilon - 1)w''_\epsilon, \quad \tilde{E}_{2,\epsilon} = F_1(w_\epsilon) - w'_\epsilon w''_\epsilon, \tag{4.10}$$

$$\tilde{E}_{3,\epsilon} = \frac{1}{2}(w''_\epsilon)^2 - f_1(w_\epsilon) w'_\epsilon. \tag{4.11}$$

Note that

$$\tilde{E}'_{1,\epsilon} = -(w_\epsilon - 1)w'''_\epsilon = -(w_\epsilon - 1) f_1(w_\epsilon) = -\frac{|w_\epsilon - 1|^{1+\frac{1}{\lambda}}}{w_\epsilon^{1+\frac{2}{\lambda}}} \leq 0, \tag{4.12}$$

$$\tilde{E}'_{2,\epsilon} = -(w''_\epsilon)^2 \leq 0, \tag{4.13}$$

$$\tilde{E}'_{3,\epsilon} = -f'_1(w_\epsilon)(w'_\epsilon)^2 \tag{4.14}$$

so that

$$\tilde{E}'_{3,\epsilon} \le 0 \quad \text{for } 0 < w_\epsilon \le 1 + \frac{1}{\lambda+1} \qquad \text{and} \qquad \tilde{E}'_{3,\epsilon} \ge 0 \quad \text{for } w_\epsilon \ge 1 + \frac{1}{\lambda+1}.$$

In Lemma 4.3 we showed that $\epsilon^{\frac{1}{3}+\frac{2}{3\lambda}} b_\epsilon \le Q$, where Q is independent of ϵ. Thus there is a subsequence of the ϵ (still denoted ϵ) such that

$$\lim_{\epsilon\to 0} \epsilon^{\frac{1}{3}+\frac{2}{3\lambda}} b_\epsilon = c_0 \ge 0$$

and for which w_ϵ converges uniformly on compact sets to w_0 and w_0 satisfies

$$w_0''' = \frac{|w_0 - 1|^{\frac{1}{\lambda}}}{w_0^{1+\frac{2}{\lambda}}} \operatorname{sgn}(w_0 - 1) = f_1(w_0), \tag{4.15a}$$

$$w_0(0) = L_1, \quad w_0'(0) = 0, \quad w_0''(0) = c_0 \ge 0. \tag{4.15b}$$

We note in fact that $c_0 > 0$ for if $c_0 = 0$ then since $w_0'''(0) < 0$ we see that w_0'' is decreasing near $t=0$ so that $w_0'' < 0$ for $t > 0$ and t small. From (4.10) it follows that w_0 continues to be concave down and decreasing so that w_0 becomes 0 at some finite value of t, say t_0. Since $w_\epsilon \to w_0$ uniformly on compact sets and since $w_\epsilon > 0$ (since $y_{b_\epsilon} > 0$ by Lemma 3.3) then w_ϵ must have a local minimum, t_ϵ, near t_0 and $w_\epsilon(t_\epsilon) < L_1$. However, this implies from (4.13)

$$F_1(w_\epsilon(t_\epsilon)) = \tilde{E}_{2,\epsilon}(t_\epsilon) \le \tilde{E}_2(0) = F_1(L_1).$$

On the other hand, since $0 < w_\epsilon(t_\epsilon) < L_1$ and F_1 is decreasing on $(0, L_1)$ we have $F_1(w_\epsilon(t_\epsilon)) > F_1(L_1)$ which is a contradiction. Thus $c_0 > 0$.

Lemma 4.7.

$$\lim_{t\to-\infty} w_\epsilon''(t) = 0 \quad \textit{for } \epsilon > 0.$$

Proof. From Lemma 2.3 it follows that $y'_{b_\epsilon} < 0$ and $y''_{b_\epsilon} > 0$ for $t < 0$ and also that $y_{b_\epsilon} \to \infty$ as $t \to -\infty$. Hence from (4.7) we see that $w_\epsilon' < 0$ and $w_\epsilon'' > 0$ for $t < 0$ and also that $w_\epsilon \to \infty$ as $t \to -\infty$. Thus, $w_0' \le 0$, $w_0'' \ge 0$, and $w_0 \to \infty$ as $t \to -\infty$.

Thus from (4.14) we see that $\tilde{E}'_{3,\epsilon} \ge 0$ for t sufficiently negative. Thus $\tilde{E}_{3,\epsilon}$ defined by (4.11) is increasing for t sufficiently negative and since $-f_1(w_\epsilon)w_\epsilon' \ge 0$ for t sufficiently negative we see that $0 \le \frac{1}{2}(w_\epsilon'')^2$ and $0 \le -f_1(w_\epsilon)w_\epsilon'$ are both bounded above for t sufficiently negative. Also, $w_\epsilon''' > 0$ for t sufficiently negative and since $w_\epsilon'' > 0$ for t sufficiently negative, it follows that

$$\lim_{t\to-\infty} w_\epsilon''(t) = H_\epsilon \quad \text{for some } H_\epsilon \ge 0.$$

Assume now by the way of contradiction that $H_\epsilon > 0$. Then it follows that

$$\lim_{t\to-\infty} w_\epsilon' = -\infty$$

and it follows then from L'Hopital's rule that

$$\lim_{t\to-\infty}\frac{w'_\epsilon(t)}{t}=H_\epsilon,\quad \lim_{t\to-\infty}\frac{w_\epsilon(t)}{t^2}=\frac{H_\epsilon}{2},\quad \lim_{t\to-\infty}\frac{(w'_\epsilon)^2}{w_\epsilon-1}=2H_\epsilon. \tag{4.16}$$

Integrating (4.9) for t sufficiently negative when $w_\epsilon\geq 1$ we obtain

$$w''_\epsilon-H_\epsilon=\int_{-\infty}^{t}\frac{[w_\epsilon-1]^{\frac{1}{\lambda}}}{w_\epsilon^{1+\frac{2}{\lambda}}}\mathrm{d}t=\int_{-\infty}^{t}\frac{1}{w_\epsilon^{1+\frac{1}{\lambda}}}\Big(1-\frac{1}{w_\epsilon}\Big)^{\frac{1}{\lambda}}\mathrm{d}t.$$

Using L'Hopital's rule and (4.16) it follows that

$$\lim_{t\to-\infty}|t|^{1+\frac{2}{\lambda}}[w''_\epsilon-H_\epsilon]=\frac{\lambda}{\lambda+2}\Big(\frac{2}{H_\epsilon}\Big)^{1+\frac{1}{\lambda}}. \tag{4.17}$$

Also, we know from (4.12) that $\tilde{E}_{1,\epsilon}$ defined by (4.10) satisfies

$$\tilde{E}'_{1,\epsilon}=-\frac{|w_\epsilon-1|^{1+\frac{1}{\lambda}}}{w_\epsilon^{1+\frac{2}{\lambda}}}=-\frac{1}{w_\epsilon^{\frac{1}{\lambda}}}\Big|1-\frac{1}{w_\epsilon}\Big|^{1+\frac{1}{\lambda}}$$

and so integrating on $(t,0)$ gives:

$$\tilde{E}_{1,\epsilon}=\frac{1}{2}(w'_\epsilon)^2-(w_\epsilon-1)w''_\epsilon=\tilde{E}_{1,\epsilon}(0)+\int_t^0\frac{1}{w_\epsilon^{\frac{1}{\lambda}}}\Big|1-\frac{1}{w_\epsilon}\Big|^{\frac{1}{\lambda}+1}\mathrm{d}t.$$

We now first consider the case where $1<\lambda<2$. The integral on the right converges as $t\to-\infty$ since $\lim_{t\to-\infty}w_\epsilon/t^2=H_\epsilon/2$ and $\lambda<2$ (by (1.3)). Thus, $\tilde{E}_{1,\epsilon}(t)\to J_\epsilon$ for some J_ϵ as $t\to-\infty$ and thus for t sufficiently negative

$$\frac{1}{2}(w'_\epsilon)^2-(w_\epsilon-1)w''_\epsilon-J_\epsilon=-\int_{-\infty}^{t}\frac{1}{w_\epsilon^{\frac{1}{\lambda}}}\Big(1-\frac{1}{w_\epsilon}\Big)^{\frac{1}{\lambda}+1}\mathrm{d}t.$$

Also, since $w_\epsilon(0)=L_1<1$ and $w_\epsilon\to\infty$ as $t\to-\infty$ it follows then that there exists a $t_{1,\epsilon}<0$ such that $w_\epsilon(t_{1,\epsilon})=1$. Then we see since $\tilde{E}'_{1,\epsilon}\leq 0$ (by (4.12)) that

$$J_\epsilon\geq\tilde{E}_{1,\epsilon}(t_{1,\epsilon})=\frac{1}{2}(w'_\epsilon(t_{1,\epsilon}))^2\geq 0.$$

Thus

$$J_\epsilon\geq 0. \tag{4.18}$$

Moreover, by L'Hopital's rule it follows that

$$\lim_{t\to-\infty}|t|^{\frac{2}{\lambda}-1}\Big(\frac{1}{2}(w'_\epsilon)^2-(w_\epsilon-1)w''_\epsilon-J_\epsilon\Big)=-\frac{\lambda}{2-\lambda}\Big(\frac{2}{H_\epsilon}\Big)^{\frac{1}{\lambda}}. \tag{4.19}$$

Combining (4.17) and (4.19) we obtain

$$\lim_{t\to-\infty}|t|^{\frac{2}{\lambda}-1}\Big(\frac{1}{2}(w_\epsilon')^2-H_\epsilon w_\epsilon-(J_\epsilon-H_\epsilon)\Big)=-\frac{2\lambda^2}{4-\lambda^2}\Big(\frac{2}{H_\epsilon}\Big)^{\frac{1}{\lambda}}. \tag{4.20}$$

It follows from (4.20) that

$$\lim_{t\to-\infty}\Big(\frac{1}{2}(w_\epsilon')^2-H_\epsilon w_\epsilon-(J_\epsilon-H_\epsilon)\Big)=0. \tag{4.21}$$

We also know that when $w_\epsilon>1$

$$\Big(-\frac{w_\epsilon'}{\sqrt{w_\epsilon-1}}\Big)'=\frac{\tilde{E}_{1,\epsilon}}{(w_\epsilon-1)^{\frac{3}{2}}}$$

and since $\tilde{E}_{1,\epsilon}\to J_\epsilon$ as $t\to-\infty$ we see that

$$\lim_{t\to-\infty}\Big[(w_\epsilon-1)^{\frac{3}{2}}\Big(-\frac{w_\epsilon'}{\sqrt{w_\epsilon-1}}\Big)'\Big]=J_\epsilon$$

and from the second result of (4.16) it follows that

$$\lim_{t\to-\infty}\Big[t^3\Big(-\frac{w_\epsilon'}{\sqrt{w_\epsilon-1}}\Big)'\Big]=\frac{2\sqrt{2}J_\epsilon}{H_\epsilon^{\frac{3}{2}}}.$$

Using (4.16) again and applying L'Hopital's rule we see that

$$\lim_{t\to-\infty}\Big[t^2\Big(\frac{w_\epsilon'}{\sqrt{w_\epsilon-1}}+\sqrt{2H_\epsilon}\Big)\Big]=\frac{\sqrt{2}J_\epsilon}{H_\epsilon^{\frac{3}{2}}}. \tag{4.22}$$

Now let $\delta>0$. Then for t sufficiently negative we have by (4.22)

$$0\le -w_\epsilon'\le\Big[\sqrt{2H_\epsilon}+\Big(\frac{-\sqrt{2}J_\epsilon}{H_\epsilon^{\frac{3}{2}}}+\delta\Big)\frac{1}{t^2}\Big]\sqrt{w_\epsilon-1}.$$

Squaring both sides and simplifying we obtain

$$\frac{1}{2}(w_\epsilon')^2\le H_\epsilon(w_\epsilon-1)+\frac{\sqrt{2H_\epsilon}(w_\epsilon-1)}{t^2}\Big(\frac{-\sqrt{2}J_\epsilon}{H_\epsilon^{\frac{3}{2}}}+\delta\Big)+\frac{1}{2}\Big(\frac{-\sqrt{2}J_\epsilon}{H_\epsilon^{\frac{3}{2}}}+\delta\Big)^2\frac{(w_\epsilon-1)}{t^4}$$

and then

$$\begin{aligned}&\frac{1}{2}(w_\epsilon')^2-H_\epsilon w_\epsilon-(J_\epsilon-H_\epsilon)\\ &\le\frac{\sqrt{2H_\epsilon}(w_\epsilon-1)}{t^2}\Big(\frac{-\sqrt{2}J_\epsilon}{H_\epsilon^{\frac{3}{2}}}+\delta\Big)+\frac{1}{2}\Big(\frac{-\sqrt{2}J_\epsilon}{H_\epsilon^{\frac{3}{2}}}+\delta\Big)^2\frac{w_\epsilon-1}{t^4}-J_\epsilon.\end{aligned} \tag{4.23}$$

Taking limits in (4.23) using (4.16) and (4.22) yields

$$0 \le -2J_\epsilon + \frac{H_\epsilon^{\frac{3}{2}}}{\sqrt{2}}\delta.$$

This along with (4.18) gives

$$0 \le J_\epsilon \le \frac{H_\epsilon^{\frac{3}{2}}}{2\sqrt{2}}\delta.$$

Finally, since $\delta > 0$ is arbitrary we see therefore that $J_\epsilon = 0$.

Therefore $\lim_{t\to-\infty} \tilde{E}_{1,\epsilon} = 0$ but since $\tilde{E}'_{1,\epsilon} \le 0$ and $\tilde{E}_{1,\epsilon}(t_{1,\epsilon}) \ge 0$ it follows that $\tilde{E}_{1,\epsilon} \equiv 0$ on $(-\infty, t_{1,\epsilon})$. Thus

$$-\frac{|w_\epsilon - 1|^{1+\frac{1}{\lambda}}}{w_\epsilon^{1+\frac{2}{\lambda}}} = \tilde{E}'_{1,\epsilon} \equiv 0 \quad \text{on} \ \ (-\infty, t_{1,\epsilon})$$

and thus $w_\epsilon \equiv 1$ on $(-\infty, t_{1,\epsilon})$ contradicting that

$$\lim_{t\to-\infty} \frac{w_\epsilon}{t^2} = \frac{H_\epsilon}{2} > 0.$$

Hence it must be the case that $H_\epsilon = 0$ completing the proof of the lemma in the case where $1 < \lambda < 2$.

We now consider the case where $\lambda \ge 2$. We see from (4.16) and the equation after (4.17) that if $\lambda \ge 2$ then

$$\lim_{t\to-\infty} \tilde{E}_{1,\epsilon} = \infty. \tag{4.24}$$

Next, we see that

$$\frac{1}{2}(w'_\epsilon)^2 - H_\epsilon(w_\epsilon - 1) = \tilde{E}_{1,\epsilon} + (w_\epsilon - 1)(w''_\epsilon - H_\epsilon).$$

Using (4.17) $w''_\epsilon - H_\epsilon \ge 0$ for sufficiently negative t and (4.24), we obtain

$$\lim_{t\to-\infty} \frac{1}{2}(w'_\epsilon)^2 - H_\epsilon(w_\epsilon - 1) = \infty. \tag{4.25}$$

Also from the equation after (4.21) we see that

$$\Big(-\frac{w'_\epsilon}{\sqrt{w_\epsilon - 1}}\Big)' = \frac{\tilde{E}_{1,\epsilon}}{(w_\epsilon - 1)^{\frac{3}{2}}},$$

which gives

$$\lim_{t\to-\infty} \left[(w_\epsilon - 1)^{\frac{3}{2}}\Big(-\frac{w'_\epsilon}{\sqrt{w_\epsilon - 1}}\Big)'\right] = \infty.$$

Also it follows from the second result of (4.16) that

$$\lim_{t\to-\infty}\left[t^3\Big(-\frac{w_\epsilon'}{\sqrt{w_\epsilon-1}}\Big)'\right]=\infty.$$

Then by L'Hopital's rule we see that

$$\lim_{t\to-\infty}\left[t^2\Big(\frac{w_\epsilon'}{\sqrt{w_\epsilon-1}}+\sqrt{2H_\epsilon}\Big)\right]=\infty. \tag{4.26}$$

For $M>0$ large and t sufficiently negative we see from (4.26) that

$$0\le -w_\epsilon'\le\Big(\sqrt{2H_\epsilon}-\frac{M}{t^2}\Big)\sqrt{w_\epsilon-1}.$$

Squaring both sides and rewriting gives

$$\frac{1}{2}(w_\epsilon')^2-H_\epsilon(w_\epsilon-1)\le -M\sqrt{2H_\epsilon}\Big(\frac{w_\epsilon-1}{t^2}\Big)+\frac{M^2}{2t^2}\Big(\frac{w_\epsilon-1}{t^2}\Big).$$

However, as $t\to-\infty$ the left hand side goes to ∞ by (4.25) and by (4.16) the right hand side goes to $-MH_\epsilon^{3/2}/\sqrt{2}\le 0$. This is a contradiction. As a result, if $\lambda\ge 2$, then it also must have $H_\epsilon=0$. This completes the proof of the lemma. ☐

Lemma 4.8. *There are constants $c_1>0$ and $c_2>0$ with c_1,c_2 independent of ϵ and $c_{1,\epsilon}>0$, $c_{2,\epsilon}>0$ with*

$$\lim_{\epsilon\to 0}c_{1,\epsilon}=\lim_{\epsilon\to 0}c_{2,\epsilon}=0$$

such that

$$\frac{y_{b_\epsilon}(s)}{|s|^{\frac{3\lambda}{2\lambda+1}}}\ge c_1\quad on\ \ (-\infty,-c_{1,\epsilon});\qquad \frac{-y_{b_\epsilon}'(s)}{|s|^{\frac{\lambda-1}{2\lambda+1}}}\ge c_2\quad on\ \ (-\infty,-c_{2,\epsilon}).$$

Proof. Recall that

$$\tilde{E}_{2,\epsilon}'=(F_1(w_\epsilon)-w_\epsilon'w_\epsilon'')'=-(w_\epsilon'')^2\le 0.$$

Integrating on $(t,0)$ and using (4.3) gives for $t<0$

$$\int_1^\infty f_1(s)\,\mathrm{d}s=F_{1,\infty}=F_1(L_1)\le F_1(w_\epsilon)-w_\epsilon'w_\epsilon''=\int_1^{w_\epsilon}f_1(s)\,\mathrm{d}s-w_\epsilon'w_\epsilon''.$$

Thus

$$\int_{w_\epsilon}^\infty f_1(s)\,\mathrm{d}s\le -w_\epsilon'w_\epsilon''. \tag{4.27}$$

Recall from the remark at the beginning of Lemma 4.7 that $\lim_{t\to-\infty}w_\epsilon=\infty$ and along with the fact that $w_\epsilon(0)=L_1<1$ we see that there exists $t_{2,\epsilon}<0$ such that $w_\epsilon(t_{2,\epsilon})=2$. Thus for $t<t_{2,\epsilon}$ we have

$$\int_{w_\epsilon}^\infty f_1(s)\,\mathrm{d}s=\int_{w_\epsilon}^\infty\frac{|s-1|^{\frac{1}{\lambda}}}{s^{1+\frac{2}{\lambda}}}\mathrm{d}s\ge\frac{1}{2^{\frac{1}{\lambda}}}\int_{w_\epsilon}^\infty\frac{1}{s^{1+\frac{1}{\lambda}}}\mathrm{d}s=\frac{\lambda}{2^{\frac{1}{\lambda}}}w_\epsilon^{-\frac{1}{\lambda}}. \tag{4.28}$$

Thus from (4.27)-(4.28) we see that

$$-w_\epsilon' w_\epsilon'' \geq \frac{\lambda}{2^{\frac{1}{\lambda}}} w_\epsilon^{-\frac{1}{\lambda}} \quad \text{when } \ t < t_{2,\epsilon}.$$

Multiplying this by $-w_\epsilon' > 0$ gives

$$(w_\epsilon')^2 w_\epsilon'' \geq \frac{\lambda}{2^{\frac{1}{\lambda}}} w_\epsilon^{-\frac{1}{\lambda}} (-w_\epsilon')$$

and integrating on $(t, t_{2,\epsilon})$ and using that $w_\epsilon' < 0$ gives

$$-(w_\epsilon')^3 \geq \frac{3\lambda^2}{2^{\frac{1}{\lambda}}(\lambda-1)} \Big(w_\epsilon^{1-\frac{1}{\lambda}} - 2^{1-\frac{1}{\lambda}} \Big). \tag{4.29}$$

Now let $t_{3,\epsilon} < 0$ be such that $w_\epsilon(t_{3,\epsilon}) = 3$. Then for $t < t_{3,\epsilon}$we have

$$w_\epsilon^{1-\frac{1}{\lambda}} - 2^{1-\frac{1}{\lambda}} \geq \Big(1 - (\frac{2}{3})^{1-\frac{1}{\lambda}} \Big) w_\epsilon^{1-\frac{1}{\lambda}}.$$

Thus, using this in (4.29) we obtain

$$\frac{1}{\Big(1-(\frac{2}{3})^{1-\frac{1}{\lambda}}\Big)^{\frac{1}{3}}} \int_t^{t_{3,\epsilon}} \frac{-w_\epsilon'}{w_\epsilon^{\frac{1}{3}(1-\frac{1}{\lambda})}} \mathrm{d}s \geq \int_t^{t_{3,\epsilon}} \frac{-w_\epsilon'}{\Big(w_\epsilon^{1-\frac{1}{\lambda}} - 2^{1-\frac{1}{\lambda}} \Big)^{\frac{1}{3}}} \mathrm{d}s \geq \int_t^{t_{3,\epsilon}} \Big(\frac{3\lambda^2}{2^{\frac{1}{\lambda}}(\lambda-1)} \Big)^{\frac{1}{3}} \mathrm{d}s.$$

Therefore, we have

$$w_\epsilon^{\frac{2\lambda+1}{3\lambda}} \geq \Big(w_\epsilon^{\frac{2\lambda+1}{3\lambda}} - 3^{\frac{2\lambda+1}{3\lambda}} \Big) \geq C_1 (t_{3,\epsilon} - t),$$

where

$$C_1 = \Big(1 - (\frac{2}{3})^{1-\frac{1}{\lambda}} \Big)^{\frac{1}{3}} \Big(\frac{3\lambda^2}{2^{\frac{1}{\lambda}}(\lambda-1)} \Big)^{\frac{1}{3}} \Big(\frac{2\lambda+1}{3\lambda} \Big).$$

Thus for $t < 2t_{3,\epsilon}$,

$$\frac{w_\epsilon}{|t|^{\frac{3\lambda}{2\lambda+1}}} \geq C_1^{\frac{3\lambda}{2\lambda+1}} \Big(1 - \Big| \frac{t_{3,\epsilon}}{t} \Big| \Big)^{\frac{3\lambda}{2\lambda+1}} \geq \Big(\frac{C_1}{2} \Big)^{\frac{3\lambda}{2\lambda+1}} \equiv c_1. \tag{4.30}$$

Letting $c_{1,\epsilon} = \epsilon^{\frac{2\lambda+1}{3\lambda}} (2|t_{3,\epsilon}|)$ and using the rescaling mentioned in (4.7)-(4.8) we see that

$$\frac{y_{b_\epsilon}(s)}{|s|^{\frac{3\lambda}{2\lambda+1}}} \geq c_1 \quad \text{on } \ (-\infty, -c_{1,\epsilon}). \tag{4.31}$$

Also, since $w_\epsilon \to w_0$ uniformly on compact sets and $w_0 \to \infty$ as $t \to -\infty$ then $t_{3,\epsilon} \to t_{3,0}$ where $t_{3,0}$ is finite and $t_{3,0} < 0$. Thus, $\lim_{\epsilon \to 0} c_{1,\epsilon} = 0$. Substituting (4.30) into (4.29) gives for $t < 2t_{3,\epsilon}$

$$-(w_\epsilon')^3 \geq \frac{3\lambda^2}{2^{\frac{1}{\lambda}}(\lambda-1)} \Big(w_\epsilon^{1-\frac{1}{\lambda}} - 2^{1-\frac{1}{\lambda}} \Big) \geq \frac{3\lambda^2}{2^{\frac{1}{\lambda}}(\lambda-1)} \Big([c_1 |t|^{\frac{3\lambda}{2\lambda+1}}]^{\frac{\lambda-1}{\lambda}} - 2^{1-\frac{1}{\lambda}} \Big).$$

Thus, for $t<2t_{3,\epsilon}$

$$-\frac{w_\epsilon'}{|t|^{\frac{\lambda-1}{2\lambda+1}}}\geq\left(\frac{3\lambda^2}{2^{\frac{1}{\lambda}}(\lambda-1)}\right)^{\frac{1}{3}}\left(c_1^{1-\frac{1}{\lambda}}-\frac{2^{1-\frac{1}{\lambda}}}{|t|^{\frac{3(\lambda-1)}{2\lambda+1}}}\right)^{\frac{1}{3}}.$$

The right-hand side of the above is larger than

$$\frac{1}{2}\left(\frac{3\lambda^2}{2^{\frac{1}{\lambda}}(\lambda-1)}\right)^{\frac{1}{3}}c_1^{\frac{1}{3}(1-\frac{1}{\lambda})}\equiv c_2$$

when

$$|t|\geq t^*\equiv 2^{\frac{(2\lambda-1)(2\lambda+1)}{3\lambda(\lambda-1)}}/c_1^{2+\frac{1}{\lambda}}.$$

So letting $c_{2,\epsilon}=\epsilon^{\frac{2\lambda+1}{3\lambda}}\cdot t^*$, we see that $c_{2,\epsilon}\to 0$ as $\epsilon\to 0$ and using the rescaling from (4.7)-(4.8) we see that

$$\frac{-y_{b_\epsilon}'(s)}{|s|^{\frac{\lambda-1}{2\lambda+1}}}\geq c_2\quad\text{on }\ (-\infty,-c_{2,\epsilon}).$$

This completes the proof of the lemma. □

Lemma 4.9. *There are constants $c_3>0$, $c_4>0$, and $c_5>0$ with c_3,c_4,c_5 independent of ϵ and $c_{3,\epsilon}>0$, $c_{4\epsilon}>0$, $c_{5,\epsilon}>0$ with*

$$\lim_{\epsilon\to 0}c_{3,\epsilon}=\lim_{\epsilon\to 0}c_{4,\epsilon}=\lim_{\epsilon\to 0}c_{5,\epsilon}=0$$

such that

$$\frac{y_{b_\epsilon}(s)}{|s|^{\frac{3\lambda}{2\lambda+1}}}\leq c_3\quad\text{on }\ (-\infty,-c_{3,\epsilon}),\qquad\frac{-y_{b_\epsilon}'(s)}{|s|^{\frac{\lambda-1}{2\lambda+1}}}\leq c_4\quad\text{on }\ (-\infty,-c_{4,\epsilon}),$$

and

$$0\leq|s|^{\frac{\lambda+2}{2\lambda+1}}y_{b_\epsilon}''(s)\leq c_5\quad\text{on }\ (-\infty,-c_{5,\epsilon}).$$

Proof. From Lemma 4.7 we know that $\lim_{t\to-\infty}w_\epsilon''=0$ and from Lemma 2.3 we know that $w_\epsilon''\geq 0$ when $t<0$. Thus, when $t<t_{2,\epsilon}$ (defined in Lemma 4.8) we have

$$0\leq w_\epsilon''(t)=\int_{-\infty}^t w_\epsilon''' \text{ and } \mathrm{d}s=\int_{-\infty}^t\frac{|w_\epsilon-1|^{\frac{1}{\lambda}}}{w_\epsilon^{1+\frac{2}{\lambda}}}\mathrm{sgn}(w_\epsilon-1)\,\mathrm{d}s\leq\int_{-\infty}^t\frac{1}{w_\epsilon^{1+\frac{1}{\lambda}}}\mathrm{d}s.$$

Then using (4.30) gives

$$0\leq w_\epsilon''(t)\leq\frac{1}{c_1^{1+\frac{1}{\lambda}}}\int_{-\infty}^t|s|^{\frac{-3\lambda-3}{2\lambda+1}}\mathrm{d}s=\frac{1}{c_1^{1+\frac{1}{\lambda}}}|t|^{\frac{-\lambda-2}{2\lambda+1}}\quad\text{for }\ t<2t_{3,\epsilon}.$$

Letting $c_5=1/c_1^{1+1/\lambda}$ we have

$$0\leq|t|^{\frac{\lambda+2}{2\lambda+1}}w_\epsilon''(t)\leq c_5\quad\text{for }\ t<2t_{3,\epsilon}.\tag{4.32}$$

Letting $c_{5,\epsilon}=\epsilon^{\frac{2\lambda+1}{3\lambda}}(2|t_{3,\epsilon}|)$ and using the rescaling (4.7)-(4.8) gives

$$0\le |s|^{\frac{\lambda+2}{2\lambda+1}}y''_{b_\epsilon}(s)\le c_5 \quad \text{on } (-\infty,c_{5,\epsilon}).$$

Also, as mentioned after Eq. (4.31), $t_{3,\epsilon}\to t_{3,0}$ and $t_{3,0}$ is finite so that $c_{5,\epsilon}\to 0$ as $\epsilon\to 0$. Dividing (4.32) by $|t|^{\frac{\lambda+2}{2\lambda+1}}$ and integrating the resulting inequality on $(t,2t_{3,\epsilon})$ gives

$$w'_\epsilon(2t_{3,\epsilon})-w'_\epsilon(t)\le c_5\Big(\frac{2\lambda+1}{\lambda-1}\Big)|t|^{\frac{\lambda-1}{2\lambda+1}}.$$

Therefore

$$0\le -\frac{w'_\epsilon(t)}{|t|^{\frac{\lambda-1}{2\lambda+1}}}\le -\frac{w'_\epsilon(2t_{3,\epsilon})}{|t|^{\frac{\lambda-1}{2\lambda+1}}}+c_5\Big(\frac{2\lambda+1}{\lambda-1}\Big) \quad \text{for } t<2t_{3,\epsilon}.$$

Since $w'_\epsilon\to w'_0$ uniformly on compact sets and $t_{3,\epsilon}\to t_{3,0}$, where $t_{3,0}$ is finite and $t_{3,0}<0$ as mentioned after (4.31), we have $w'_\epsilon(t_{3,\epsilon})\to w'_0(t_{3,0})$ which is finite so we see for ϵ small enough

$$0\le -\frac{w'_\epsilon(t)}{|t|^{\frac{\lambda-1}{2\lambda+1}}}\le -\frac{2w'_0(2t_{3,0})}{|t_{3,0}|^{\frac{\lambda-1}{2\lambda+1}}}+c_5\Big(\frac{2\lambda+1}{\lambda-1}\Big)\equiv c_4 \tag{4.33}$$

for $t<3t_{3,\epsilon_0}$. Then by the rescaling mentioned in (4.7) we see that

$$0\le \frac{-y'_{b_\epsilon}(s)}{|s|^{\frac{\lambda-1}{2\lambda+1}}}\le c_4 \quad \text{on } (-\infty,-c_{4,\epsilon}), \tag{4.34}$$

where $c_{4,\epsilon}=\epsilon^{\frac{2\lambda+1}{3\lambda}}(3t_{3,\epsilon_0})\to 0$ as $\epsilon\to 0$. Multiplying (4.33) by $|t|^{\frac{\lambda-1}{2\lambda+1}}$ and integrating on $(s,0)$ gives

$$w_\epsilon(t)\le w_\epsilon(3t_{3,0})+\Big(\frac{2\lambda+1}{3\lambda}\Big)c_4|t|^{\frac{3\lambda}{2\lambda+1}}.$$

Consequently,

$$\frac{w_\epsilon}{|t|^{\frac{3\lambda}{2\lambda+1}}}\le\frac{w_\epsilon(3t_{3,0})}{|t|^{\frac{3\lambda}{2\lambda+1}}}+\Big(\frac{2\lambda+1}{3\lambda}\Big)c_4\le\frac{w_\epsilon(3t_{3,0})}{|3t_{3,0}|^{\frac{3\lambda}{2\lambda+1}}}+\Big(\frac{2\lambda+1}{3\lambda}\Big)c_4\equiv c_3.$$

Then by the rescaling mentioned in (4.7) we see that

$$\frac{y_{b_\epsilon}}{|s|^{\frac{3\lambda}{2\lambda+1}}}\le c_3 \quad \text{on } (-\infty,-c_{3,\epsilon}),$$

where $c_{3,\epsilon}=\epsilon^{\frac{2\lambda+1}{3\lambda}}(3t_{3,\epsilon_0})\to 0$ as $\epsilon\to 0$. This completes the proof of the lemma. □

It follows from Lemmas 4.8 and 4.9 that $|y_{b_\epsilon}|,|y'_{b_\epsilon}|,|y''_{b_\epsilon}|$ are uniformly bounded on compact subsets of $(-\infty,0)$ and from (3.1) we see that $|y'''_{b_\epsilon}|$ is also uniformly bounded on compact subsets of $(-\infty,0)$. Consequently, $y_{b_\epsilon},y'_{b_\epsilon}$, and y''_{b_ϵ} converge uniformly on compact subsets of $(-\infty,0)$ to a function y_0 and from (3.1) we see that y_{b_ϵ}''' converges uniformly on compact sets and that y_0 satisfies:

$$y_0''' = \frac{1}{y_0^{1+\frac{1}{\lambda}}}, \tag{4.35}$$

$$\lim_{t\to 0^-} y_0(t) = 0, \quad \lim_{t\to 0^-} y_0'(t) = 0, \tag{4.36}$$

$$0 \le |t|^{\frac{\lambda+2}{2\lambda+1}} y_0''(t) \le c_5 \quad \text{for } t<0. \tag{4.37}$$

Finally, we have the following result.

Lemma 4.10.

$$y_0 = c_\lambda |t|^{\frac{3\lambda}{2\lambda+1}}, \quad \textit{where} \quad c_\lambda = \left(\frac{(2\lambda+1)^3}{3\lambda(\lambda-1)(\lambda+2)}\right)^{\frac{\lambda}{2\lambda+1}}.$$

Proof. It is straightforward to show that y given above is a solution of

$$y''' = \frac{1}{y^{1+\frac{1}{\lambda}}}, \tag{4.38}$$

$$\lim_{t\to 0^-} y(t) = 0, \quad \lim_{t\to 0^-} y'(t) = 0, \tag{4.39}$$

and

$$0 \le |t|^{\frac{\lambda+2}{2\lambda+1}} y''(t) \le C < \infty \quad \text{for } t<0. \tag{4.40}$$

Now we let $v = y_0 - y$. From the Mean-Value Theorem we see that for any fixed $t<0$ there is an $0<\mu<1$ such that

$$\begin{aligned} v''' = y_0''' - y''' &= \frac{1}{y_0^{1+\frac{1}{\lambda}}} - \frac{1}{y^{1+\frac{1}{\lambda}}} \\ &= -\frac{(1+\frac{1}{\lambda})}{\big(\mu y + (1-\mu)y_0\big)^{2+\frac{1}{\lambda}}}[y_0 - y] = -p(t)v, \end{aligned}$$

where $p(t)>0$. Now we observe that

$$\left(\frac{1}{2}(v')^2 - vv''\right)' = -vv''' = p(t)v^2 \ge 0.$$

It follows from Lemmas 4.8 and 4.9, and (4.36)-(4.37) and (4.39)-(4.41) that

$$\lim_{t\to 0^-} \frac{1}{2}(v')^2 - vv'' = 0,$$

so we see that

$$\frac{1}{2}(v')^2 - vv'' \le 0 \quad \text{for } t<0.$$

Thus it follows that $vv'' \geq 0$ for $t<0$. Then $(vv')' = vv'' + (v')^2 \geq 0$. Integrating on $(t,0)$ and using Lemmas 4.8 and 4.9, (4.36) and (4.39) give $vv' \leq 0$ for $t<0$. Suppose now that there is a $t_0<0$ for which $v(t_0)=0$. Integrating on (t_0,t) gives $v^2(t) \leq 0$ and so we see that $v \equiv 0$ on $(t_0,0)$. Therefore either $v \geq 0$ for $t<0$ or $v \leq 0$ for $t<0$.

Suppose first that $v \geq 0$ for $t<0$. Then we have

$$y_0 \geq y \equiv c_\lambda |t|^{\frac{3\lambda}{2\lambda+1}} \quad \text{for } t<0. \tag{4.41}$$

Then by (4.37) and (4.39)

$$y_0'' = \int_{-\infty}^{t} \frac{1}{y_0^{1+\frac{1}{\lambda}}} \mathrm{d}s \leq \frac{1}{c_\lambda^{1+\frac{1}{\lambda}}} |s|^{\frac{-3\lambda-3}{2\lambda+1}} = \frac{1}{c_\lambda^{1+\frac{1}{\lambda}}} \Big(\frac{2\lambda+1}{\lambda+2}\Big) |t|^{\frac{-\lambda-2}{2\lambda+1}}.$$

Integrating on $(t,0)$ gives

$$-y_0' \leq \int_t^0 \frac{1}{c_\lambda^{1+\frac{1}{\lambda}}} \Big(\frac{2\lambda+1}{\lambda+2}\Big) |s|^{\frac{-\lambda-2}{2\lambda+1}} \mathrm{d}s = \frac{1}{c_\lambda^{1+\frac{1}{\lambda}}} \Big(\frac{2\lambda+1}{\lambda+2}\Big) \Big(\frac{2\lambda+1}{\lambda-1}\Big) |t|^{\frac{\lambda-1}{2\lambda+1}}$$

and integrating again on $(t,0)$ and using the definition of c_λ given in Lemma 4.10 we see that

$$y_0 \leq \frac{1}{c_\lambda^{1+\frac{1}{\lambda}}} \Big(\frac{2\lambda+1}{\lambda+2}\Big) \Big(\frac{2\lambda+1}{\lambda-1}\Big) \Big(\frac{2\lambda+1}{3\lambda}\Big) |t|^{\frac{3\lambda}{2\lambda+1}} = c_\lambda |t|^{\frac{3\lambda}{2\lambda+1}}. \tag{4.42}$$

Thus combining (4.41)-(4.42) we see that

$$y_0 \equiv c_\lambda |t|^{\frac{3\lambda}{2\lambda+1}} \quad \text{for } t<0.$$

Similarly if $v \leq 0$ for $t<0$ then we have

$$y_0 \leq c_\lambda |t|^{\frac{3\lambda}{2\lambda+1}} \quad \text{for } t<0.$$

Then as earlier we may go through a similar computation and show that

$$y_0 \geq c_\lambda |t|^{\frac{3\lambda}{2\lambda+1}} \quad \text{for } t<0$$

and finally obtain

$$y_0 \equiv c_\lambda |t|^{\frac{3\lambda}{2\lambda+1}} \quad \text{for } t<0.$$

This completes the proof of the lemma and the proof of the Main Theorem. $\square$

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