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Alternating Direction Implicit Galerkin Finite Element Method for the Two-Dimensional Time Fractional Evolution Equation

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Abstract. New numerical techniques are presented for the solution of the two-dimensional time fractional evolution equation in the unit square. In these methods, Galerkin finite element is used for the spatial discretization, and, for the time stepping, new alternating direction implicit (ADI) method based on the backward Euler method combined with the first order convolution quadrature approximating the integral term are considered. The ADI Galerkin finite element method is proved to be convergent in time and in the L^2 norm in space. The convergence order is $\mathscr{O}(k|\ln k|+h^r)$, where k is the temporal grid size and h is spatial grid size in the x and y directions, respectively. Numerical results are presented to support our theoretical analysis.

AMS subject classifications: 65M06, 65M12, 65M15, 65M60

Key words: Fractional evolution equation, alternating direction implicit method, Galerkin finite element method, backward Euler.

1. Introduction

We study an alternating direction implicit (ADI) method for the numerical solution of the fractional evolution equation

$$u_t-\int_0^t \beta(t-s)\Delta u(x,y,s)ds=f(x,y,t),\quad (x,y)\in\Omega,\quad t\in J, \tag{1.1}$$

with boundary and initial conditions

$$u(x,y,t)=0,\qquad (x,y)\in\partial\Omega,\quad t\in J, \tag{1.2a}$$
$$u(x,y,0)=u^0(x,y),\qquad (x,y)\in\Omega\times\partial\Omega, \tag{1.2b}$$

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respectively, where $u_t = \partial u/\partial t$, Δ is the two-dimensional Laplacian, $J = (0, T]$, $\Omega = R \times R$, $R = (0, 1)$, $\partial\Omega$ is the boundary of the domain Ω, the kernel $\beta(t)$ is assumed to be $t^{-1/2}$, $u^0(x, y)$ and $f(x, y, t)$ are given functions.

This equation possesses the remarkable feature that it may be considered as an equation intermediate between the standard (parabolic) heat equation and the (hyperbolic) wave equation. In fact, the integral operator $I^{1/2}$ that maps each (locally integrable) function $\varphi(t)$, $t > 0$, into the function

$$(I^{1/2}\varphi)(t) = \int_0^t (t-s)^{-1/2}\varphi(s)ds$$

is such that

$$(I^{1/2}(I^{1/2}\varphi))(t) = \pi \int_0^t \varphi(s)ds =: \pi(I\varphi)(t).$$

In recent decade years, more and more attentions have been placed on the development and research of fractional differential and integral equations, because they can describe many phenomenons, physical and chemical process more accurately than classical integer-order differential equations and have been widely used in many other fields, such as viscoelasticity [1, 2], biological systems, finance [3], hydrology [4] and so on. Therefore, it is important to find some efficient methods to solve fractional differential and integral equations. Many considerable works on the theoretical analysis [5–8] have been carried on, but analytic solutions of most fractional differential equations cannot be obtained explicitly. Many authors have resorted to numerical solution strategies based on convergence and stability analysis [9–34]. For one-dimensional problems, Lopez-Marcos [10] and Tang [12] analyzed finite difference schemes for a partial integro-differential equation. Lin and Xu [16] considered numerical approximations based on a finite difference scheme in time and Legendre spectral methods in space. Li and Xu [17] extended their previous work and proposed a spectral method in both temporal and spatial discretizations. Deng [18] developed the finite element method for the numerical solution of the space-time fractional Fokker-Planck equation. Li and Xu [24] constructed and analyzed stable and high order scheme to solve the integro-differential equation which is discretized by the finite difference in time and by the finite element method in space.

For two-dimensional problems, Zhang et al. [31] presented the finite difference/element method for a two-dimensional modified fractional diffusion equation. They used the second-order backward differentiation formula in time and the finite element method in space. Ji and Tang [32] considered the Runge-Kutta and discontinuous Galerkin (DG) methods for the fractional diffusion equations. Chen and Liu [34] proposed a second-order accurate numerical method for the two-dimensional fractional advection-dispersion equation. This method combined the alternating directions implicit approach with the unshifted Grünwald formula for the advection term, the right-shifted Grünwald formula for the diffusion term, and a Richardson extrapolation to establish an unconditionally stable second order accurate difference scheme.

The numerical solutions of the high-dimensional fractional evolution equations are still a challenge. The purpose in this paper is to consider effective numerical methods for the high-dimensional fractional evolution equations. The ADI method can reduce a multidimensional problem to sets of independent one-dimensional problems. Thus the ADI method reduces computational complexities greatly. We propose and analyze an alternating direction implicit Galerkin finite element method for the two-dimensional time fractional evolution equation. The ADI Galerkin finite element method is proved to be convergent in time and in the L^2 norm in space.

Throughout, the functions $u^0(x,y)$ and $f(x,y,t)$ are assumed to be appropriately smooth such that the problem (1.1)-(1.2b) has sufficiently smooth solution in $\bar{\Omega}\times[0,T]$. In fact, we need that u_t, u_{tt}, u_{xyt}, u_{xxt} and u_{yyt} are continuous in $\bar{\Omega}\times[0,T]$ in the following analysis.

Remark 1.1. The solution of (1.1)-(1.2b) has the following regularities,

$$
\begin{aligned}
&u \in C([0,T];H^2\cap H_0^1),\\
&u_t \in C([0,T];L^2)\cap L^1([0,T];H^2\cap H_0^1),\\
&u_{tt} \in L^1([0,T];L^2),
\end{aligned}
$$

when $u^0(x,y)$ and $f(x,y,t)$ are sufficiently smooth (see [11]).

Remark 1.2. It is shown that, for the solution of (1.1)-(1.2b) with $f(x,y,t)=0$,

$$
\begin{aligned}
&|u(\cdot,t)|_{s+2\theta} \le c(\alpha)t^{-(\alpha+1)\theta}|u^0|_s, && t>0, \quad 0\le\theta\le 1, \quad 0<\alpha<1, && \text{(1.3a)}\\
&|D_t^m u(\cdot,t)|_{s+2\theta} \le c(m,\alpha)t^{-(\alpha+1)\theta-m}|u^0|_s, && t>0, \quad -1\le\theta\le 1, \quad 0<\alpha<1, && \text{(1.3b)}
\end{aligned}
$$

where the notations $|v|_s=\|A^{s/2}v\|$, $s\in\mathfrak{R}$, $A=-\Delta$, $\mathfrak{R}=(-\infty,+\infty)$, and $D_t^m u(\cdot,t)$ is the mth order derivatives in time (see [11]).

The remainder of the article is organized as follows. In Section 2, some notations and auxiliary lemmas are presented. The numerical scheme is given in Section 3. Section 4 studies the error estimate of the ADI scheme. The numerical example is given in Section 5 to support our analysis. The final section contains some concluding remarks.

2. Notation and preliminary lemmas

We denote the $L^2(\Omega)$ inner product and norm by

$$(f,g)=\int_\Omega f(x,y)g(x,y)dxdy, \quad \|f\|^2=(f,f),$$

respectively. With s a nonnegative integer, let $H^s(\Omega)$ denote the Sobolev space with norm

$$\|v\|_{H^s}=\Big(\sum_{0\le\alpha_1+\alpha_2\le s}\Big\|\frac{\partial^{\alpha_1+\alpha_2}v}{\partial x^{\alpha_1}\partial y^{\alpha_2}}\Big\|^2\Big)^{1/2}.$$

Further, for $v:[0,T]\rightarrow H^s(\Omega)$, define the norm $\|\cdot\|_{L^p(H^s)}$ by

$$\|v\|_{L^p(H^s)}=\left(\int_0^T\|v\|_s^p dt\right)^{1/p},\quad \|v\|_{L^\infty(H^s)}=\sup_{0<t\le T}\|v\|_s.$$

Throughout this paper, c will denote a generic positive constant which is independent of the mesh spacing. Let $S_{h,r}(\Omega)$, $r\geq 2$, denote a family of finite-dimensional subspaces of $H_0^1(\Omega)$ parameterized by h with the following properties:

$$S_{h,r}\in Z\cap H_0^1(\Omega), \tag{2.1a}$$

$$\left\|\frac{\partial^2 V}{\partial x\partial y}\right\|\le ch^{-2}\|V\|,\quad V\in S_{h,r}, \tag{2.1b}$$

$$\inf_{\chi\in S_{h,r}}\left[\sum_{m=0}^{2}h^m\sum_{i,j=0,1,i+j=m}\left\|\frac{\partial^m(u-\chi)}{\partial x^i\partial y^j}\right\|\right]\le ch^s\|u\|_{H^s}, \tag{2.1c}$$

for $u\in H^s(\Omega)\cap Z\cap H_0^1(\Omega)$, $2\le s\le r$, where

$$Z=\left\{u\,\middle|\, u,\ \frac{\partial u}{\partial x},\ \frac{\partial u}{\partial y},\ \frac{\partial^2 u}{\partial x\partial y}\in L^2(\Omega)\right\}.$$

For a positive integer N, let $k=T/N$, $t_n=nk$ $(0<n<N)$. The time domain $[0,T]$ is covered by $\{t_n|0\le n\le N\}$. Given grid function $w=\{w^n|0\le n\le N\}$, denote

$$w^n(\cdot)=w(\cdot,t_n),\quad \partial_t w=\frac{w^n-w^{n-1}}{k}.$$

Following Sanz-Serna [9], we approximate the following integral operator $I^{1/2}$ by means of the convolution sum $I_k^{1/2}$ defined by

$$(I_k^{1/2}\varphi)(t_n)=(\pi k)^{1/2}\sum_{p=0}^{n-1}\epsilon_p\varphi(t_{n-p})\approx\int_0^{t_n}(t_n-s)^{-1/2}\varphi(s)ds, \tag{2.2}$$

where

$$\epsilon_p=(-1)^p\begin{pmatrix}-0.5\\ p\end{pmatrix},\quad p=0,1,2,\cdots. \tag{2.3}$$

To analyze the L^2 error estimate, we define the projection operator R_h as follows: for all $\phi\in H_0^1(\Omega)$, let $R_h\phi\in S_{h,r}(\Omega)$, such that

$$(\nabla(\phi-R_h\phi),\nabla v_h)=0,\quad \forall v_h\in S_{h,r}(\Omega). \tag{2.4}$$

Then the following projection estimates hold:

Lemma 2.1 (see [36]). *If $\partial^l\phi/\partial t^l \in L^p(H^r)$, $l=0,1,2$, $p=2,\infty$, then there exists a constant c, independent of h, such that*

$$\Big\|\frac{\partial^l(\phi-R_h\phi)}{\partial t^l}\Big\|_{L^p(H^j)} \le ch^{s-j}\Big\|\frac{\partial^l\phi}{\partial t^l}\Big\|_{L^p(H^s)}, \tag{2.5}$$

where $j=0,1$ and $1\le s\le r$.

For deriving the error estimate, we also need the following results.

Lemma 2.2. *Let $\varphi\in C(J;L^2)$, $\varphi_t\in L(J;L^2)$, then there exists a positive constant c that only depends on T such that*

$$\begin{aligned}&\|(I_k^{1/2}-I^{1/2})\varphi(t_n)\|^2\\ \le& c\Big[(k/n)\|\varphi(0)\|^2+k^2\int_{t_{n-1}}^{t_n}\|\varphi_t(s)\|^2ds+k^2|\ln k|\int_0^{t_{n-1}}\|\varphi_t(s)\|^2ds\Big],\end{aligned} \tag{2.6}$$

for $1\le n\le N$, $N=[T/k]$.

Proof. Noting that $R_k=I_k^{1/2}-I^{1/2}$, from [19], we obtain

$$|(R_k\varphi)(t_n)|\le c(k/n)^{1/2}|\varphi(0)|+ck^{1/2}\int_{t_{n-1}}^{t_n}|\varphi_t(s)|ds+ck\int_0^{t_{n-1}}(t_n-s)^{-1/2}|\varphi_t(s)|ds.$$

Using Hölder inequality, these follow that

$$\Big\|k^{1/2}\int_{t_{n-1}}^{t_n}|\varphi_t(s)|ds\Big\|^2=k\int_\Omega\Big(\int_{t_{n-1}}^{t_n}|\varphi_t(s)|ds\Big)^2dxdy\le k^2\int_{t_{n-1}}^{t_n}\|\varphi_t(s)\|^2ds,$$

and

$$\begin{aligned}&\Big\|k\int_0^{t_{n-1}}(t_n-s)^{-1/2}|\varphi_t(s)|ds\Big\|^2\\ =&\int_\Omega\Big(k\int_0^{t_{n-1}}(t_n-s)^{-1/2}|\varphi_t(s)|ds\Big)^2dxdy\\ \le& k^2\int_0^{t_{n-1}}(t_n-s)^{-1}ds\int_0^{t_{n-1}}\|\varphi_t(s)\|^2ds\\ \le& k^2|\ln t_n-\ln(t_n-t_{n-1})|\int_0^{t_{n-1}}\|\varphi_t(s)\|^2ds\\ \le& k^2|\ln T-\ln k|\int_0^{t_{n-1}}\|\varphi_t(s)\|^2ds\\ \le& ck^2|\ln k|\int_0^{t_{n-1}}\|\varphi_t(s)\|^2ds.\end{aligned}$$

So, with the above inequalities, we obtain the desired result (2.6). $\square$

Lemma 2.3 (see [10]). *Let $\{a_0, a_1, \cdots a_n, \cdots\}$ be a sequence of real numbers with the properties*

$$a_n \geq 0, \quad a_n - a_{n-1} \leq 0, \quad a_{n+1} - 2a_n + a_{n-1} \geq 0. \tag{2.7}$$

Then for any positive integer M, and for each vector $(V^1, V^2, \cdots, V^M)$ with M real entries,

$$\sum_{n=1}^{M} \Big(\sum_{p=0}^{n-1} a_p V^{n-p} \Big) V^n \geq 0. \tag{2.8}$$

Note that for the numbers ϵ_p, $p = 0, 1, \cdots$, introduced in (2.3), properties (2.7) hold, and hence the associated quadratic form is nonnegative.

Lemma 2.4 (see [35]). *Let D denote the operators ∂_t, $\partial/\partial t$ or $\partial^2/\partial t^2$. Using the triangle inequality and the inequality* (2.1b) *and* (2.1c)*, it holds that*

$$\Big\| \frac{\partial^2 (D\eta^n)}{\partial x \partial y} \Big\| \leq ch^{r-2} \|Du\|_{H^r} + ch^{-2} \|D\eta^n\|. \tag{2.9}$$

3. Construction of the ADI scheme

We consider first discretisation in space by the Galerkin finite-element method to (1.1) as follows: find $U \in S_{h,r}$, such that for all $v_h \in S_{h,r}$:

$$\Big(\frac{\partial U}{\partial t}, v_h \Big) + \int_0^t (t-s)^{-1/2} (\nabla U(s), \nabla v_h) ds = (f, v_h), \quad \forall v_h \in S_{h,r}, \quad t \in (0, T], \tag{3.1}$$

with $U(0) = R_h u^0$. Then, using (2.2), we define the backward Euler method in time by

$$(\partial_t U^n, v_h) + (\pi k)^{1/2} \sum_{p=0}^{n-1} \epsilon_p (\nabla U^{n-p}, \nabla v_h) = (f^n, v_h), \quad \forall v_h \in S_{h,r}, \quad n = 1, 2, \cdots, N, \tag{3.2}$$

with the initial values $U^0 = R_h u^0$, and where $f^n = f(x, y, t_n)$. With $E^n = U^n - U^{n-1}$, then (3.2) may be written as

$$(E^n, v_h) + \sqrt{\pi} k^{3/2} \epsilon_0 (\nabla E^n, \nabla v_h) = F^n, \quad v_h \in S_{h,r}, \quad n = 1, 2, \cdots, N, \tag{3.3}$$

where

$$F^n = k(f^n, v_h) - \sqrt{\pi} k^{3/2} \sum_{p=1}^{n-1} \epsilon_p (\nabla U^{n-p}, \nabla v_h) - \sqrt{\pi} k^{3/2} \epsilon_0 (\nabla U^{n-1}, \nabla v_h). \tag{3.4}$$

If we add the term

$$\pi k^3 \epsilon_0^2 \Big(\frac{\partial^2 E^n}{\partial x \partial y}, \frac{\partial^2 v_h}{\partial x \partial y} \Big)$$

to the left-hand side of (3.3), we obtain the backward Euler ADI Galerkin finite method

$$\begin{aligned}&(E^n,v_h)+\sqrt{\pi}k^{3/2}\epsilon_0(\nabla E^n,\nabla v_h)+\pi k^3\epsilon_0^2\Big(\frac{\partial^2 E^n}{\partial x\partial y},\frac{\partial^2 v_h}{\partial x\partial y}\Big)\\ =&F^n,\quad v_h\in S_{h,r},\quad n=1,2,\cdots,N,\end{aligned} \tag{3.5}$$

with the initial values $U^0=R_hu^0$.

We now rewrite (3.5) in a more familiar ADI form. To this end, suppose $S_{h,r}=S^x_{h,r}\otimes S^y_{h,r}$, where $S^x_{h,r}$ and $S^y_{h,r}$ are finite-dimensional subspaces of $H^1_0(R)$, and let $\{\varphi_i\vartheta_j\}_{i=1,j=1}^{M_x-1,M_y-1}$ be a tensor product basis for $S_{h,r}$, where $\{\varphi_i\}_{i=1}^{M_x-1}$ and $\{\vartheta_j\}_{j=1}^{M_y-1}$ are bases for the subspaces $S^x_{h,r}$ and $S^y_{h,r}$, respectively. Set

$$U^n(x,y)=\sum_{i=1}^{M_x-1}\sum_{j=1}^{M_y-1}\alpha_{ij}^{(n)}\varphi_i(x)\vartheta_j(y),$$

then

$$E^n(x,y)=\sum_{i=1}^{M_x-1}\sum_{j=1}^{M_y-1}\beta_{ij}^{(n)}\varphi_i(x)\vartheta_j(y),\quad \beta_{ij}^{(n)}=\alpha_{ij}^{(n)}-\alpha_{ij}^{(n-1)}.$$

Choosing $v_h=\varphi_l\vartheta_m$, $l=1,\cdots,M_x-1$; $m=1,\cdots,M_y-1$, then (3.5) becomes

$$\begin{aligned}&\sum_{i=1}^{M_x-1}\sum_{j=1}^{M_y-1}\Big\{(\varphi_i\vartheta_j,\varphi_l\vartheta_m)+\sqrt{\pi}k^{3/2}\epsilon_0\Big[\Big(\frac{\partial\varphi_i}{\partial x}\vartheta_j,\frac{\partial\varphi_l}{\partial x}\vartheta_m\Big)\\ &\quad+\Big(\varphi_i\frac{\partial\vartheta_j}{\partial y},\varphi_l\frac{\partial\vartheta_m}{\partial y}\Big)\Big]+\pi k^3\epsilon_0^2\Big(\frac{\partial\varphi_i}{\partial x}\frac{\partial\vartheta_j}{\partial y},\frac{\partial\varphi_l}{\partial x}\frac{\partial\vartheta_m}{\partial y}\Big)\Big\}\beta_{ij}^{(n)}=F^n,\end{aligned} \tag{3.6}$$

where

$$\begin{aligned}F^n=&k(f^n,\varphi_l\vartheta_m)-\sum_{i=1}^{M_x-1}\sum_{j=1}^{M_y-1}\Big\{\sqrt{\pi}k^{3/2}\sum_{p=1}^{n-1}\epsilon_p\Big[\Big(\frac{\partial\varphi_i}{\partial x}\vartheta_j,\frac{\partial\varphi_l}{\partial x}\vartheta_m\Big)\\ &+\Big(\varphi_i\frac{\partial\vartheta_j}{\partial y},\varphi_l\frac{\partial\vartheta_m}{\partial y}\Big)\Big]\alpha_{ij}^{(n-p)}+\sqrt{\pi}k^{3/2}\epsilon_0\Big[\Big(\frac{\partial\varphi_i}{\partial x}\vartheta_j,\frac{\partial\varphi_l}{\partial x}\vartheta_m\Big)\\ &+\Big(\varphi_i\frac{\partial\vartheta_j}{\partial y},\varphi_l\frac{\partial\vartheta_m}{\partial y}\Big)\Big]\alpha_{ij}^{(n-1)}\Big\}.\end{aligned} \tag{3.7}$$

We define the matrices

$$\begin{aligned}A_x&=((\varphi_i,\varphi_j)_x)_{i,j}^{M_x-1}, & A_y&=((\vartheta_i,\vartheta_j)_y)_{i,j=1}^{M_y-1},\\ B_x&=\Big(\Big(\frac{\partial\varphi_i}{\partial x},\frac{\partial\varphi_j}{\partial x}\Big)_x\Big)_{i,j=1}^{M_x-1}, & B_y&=\Big(\Big(\frac{\partial\vartheta_i}{\partial y},\frac{\partial\vartheta_j}{\partial y}\Big)_y\Big)_{i,j=1}^{M_y-1},\end{aligned}$$

and

$$F^{(n)} = [F^n(\varphi_1,\vartheta_1), F^n(\varphi_1,\vartheta_2), \cdots, F^n(\varphi_1,\vartheta_{M_y-1}), F^n(\varphi_2,\vartheta_1), \cdots, F^n(\varphi_{M_x-1},\vartheta_{M_y-1})]^T,$$

where

$$(\chi,\psi)_x = \int_R \chi(x)\psi(x)dx, \quad (\chi,\psi)_y = \int_R \chi(y)\psi(y)dy,$$

and let

$$\alpha^{(p)} = \left[\alpha_{11}^{(p)}, \alpha_{12}^{(p)}, \cdots, \alpha_{1M_y-1}^{(p)}, \alpha_{21}^{(p)}, \cdots, \alpha_{M_x-1,M_y-1}^{(p)}\right]^T,$$

with $\beta^{(p)}$ defined similarly. So the matrix form of (3.6) is

$$\left[A_x \otimes A_y + \sqrt{\pi}k^{3/2}\epsilon_0\{B_x \otimes A_y + A_x \otimes B_y\} + \pi k^3\epsilon_0^2 B_x \otimes B_y\right]\beta^{(n)} = F^{(n)},$$

or

$$[(A_x + \sqrt{\pi}k^{3/2}\epsilon_0 B_x) \otimes I_{M_y-1}][I_{M_x-1} \otimes (A_y + \sqrt{\pi}k^{3/2}\epsilon_0 B_y)]\beta^{(n)} = F^{(n)}, \tag{3.8}$$

where $\otimes$ denotes the matrix tensor product, and I_{M_x-1} and I_{M_y-1} denote the identity matrices of order $M_x - 1$ and $M_y - 1$, respectively. Introducing the auxiliary vector $\hat{\beta}^{(n)}$, (3.8) is equivalent to

$$\begin{cases} \left[(A_x + \sqrt{\pi}k^{3/2}\epsilon_0 B_x) \otimes I_{M_y-1}\right]\hat{\beta}^{(n)} = F^{(n)}, \\ \left[I_{M_x-1} \otimes (A_y + \sqrt{\pi}k^{3/2}\epsilon_0 B_y)\right]\beta^{(n)} = \hat{\beta}^{(n)}. \end{cases} \tag{3.9}$$

Thus we determine $\beta^{(n)}$ by solving two sets of independent one-dimensional problems, first

$$(A_x + \sqrt{\pi}k^{3/2}\epsilon_0 B_x)\hat{\beta}_m^{(n)} = F_m^{(n)}, \quad m = 1, 2, \cdots, M_y - 1, \tag{3.10}$$

in the x-direction, where

$$v_m = [v_{1m}, v_{2m}, \cdots, v_{M_x-1,m}]^T$$

followed by

$$(A_y + \sqrt{\pi}k^{3/2}\epsilon_0 B_y)\beta_l^{(n)} = \hat{\beta}_l^{(n)}, \quad l = 1, 2, \cdots, M_x - 1, \tag{3.11}$$

in the y-direction, where

$$v_l = [v_{l1}, v_{l2}, \cdots, v_{l,M_y-1}]^T.$$

Since $\epsilon_0 \geq 0$, the matrices $A_x + \sqrt{\pi}k^{3/2}\epsilon_0 B_x$ and $A_y + \sqrt{\pi}k^{3/2}\epsilon_0 B_y$ are nonsingular. Hence, there exists a unique backward Euler ADI Galerkin finite element approximation. With standard choices of bases for $S_{h,r}^x$ and $S_{h,r}^y$, the linear systems (3.10) and (3.11) are almost block diagonal and can be solved efficiently using Gauss elimination-based algorithms.

4. Convergence and stability of the ADI scheme

In this section we present the L^2 error estimate for the backward Euler ADI Galerkin finite element approximation.

Theorem 4.1. *Let u and $\{U^n\}_{n=0}^N$ denote the solutions of* (1.1)-(1.2b) *and* (3.5)*, respectively. Assume that $u \in L^\infty(H^r)$, $\partial u/\partial t$, $\partial^3 u/\partial x\partial y\partial t \in L^2((0,T];H^r)$, $u_{tt} \in L^2((0,T];L^2)$ and $u^0 \in H^r \cap H_0^1$, where $r \geq 2$. Then, for k sufficiently small,*

$$\begin{aligned}\|U^N - u^N\| \leq c\Bigg\{ & h^r\left[\|u\|_{L^\infty(H^r)} + \left\|\frac{\partial u}{\partial t}\right\|_{L^2(H^r)}\right] + k^{3/2}h^{r-2}\left\|\frac{\partial u}{\partial t}\right\|_{L^2(H^r)} \\ & + k^{3/2}\left[\|\Delta u_t\|_{L^2(L^2)} + \left\|\frac{\partial^3 u}{\partial x\partial y\partial t}\right\|_{L^2(L^2)}\right] + k\big[(|\ln k|)^{1/2}\|\Delta u(0)\| \\ & + \|u_{tt}\|_{L^2(L^2)} + (|\ln k|)^{1/2}\|\Delta u_t\|_{L^2(L^2)}\big]\Bigg\},\end{aligned}$$

provided the initial value U^0 satisfies $U^0 = R_h u^0$.

Proof. Let $\xi^n = U^n - R_h u^n$ and $\eta^n = u^n - R_h u^n$. Then

$$U^n - u^n = \xi^n - \eta^n.$$

Since bounds on η^n are known from Lemma 2.1, it is sufficient to bound ξ^n and then use the triangle inequality to bound $U^n - u^n$.

First note that (3.5) can be written as

$$\begin{aligned}&(\partial_t U^n, v_h) + (\pi k)^{1/2}\sum_{p=0}^{n-1}\epsilon_p(\nabla U^{n-p}, \nabla v_h) + \pi k^3\epsilon_0^2\Big(\frac{\partial^2[\partial_t U^n]}{\partial x\partial y}, \frac{\partial^2 v_h}{\partial x\partial y}\Big) \\ =&(f^n, v_h), \quad \forall v_h \in S_{h,r}, \quad 1 \leq n \leq N.\end{aligned} \tag{4.1}$$

Also,

$$\begin{aligned}&(\partial_t u^n, v_h) + (\pi k)^{1/2}\sum_{p=0}^{n-1}\epsilon_p(\nabla u^{n-p}, \nabla v_h) \\ =&(f^n, v_h) + (\partial_t u^n - u_t^n, v_h) - (\varepsilon_n(\Delta u), v_h), \quad \forall v_h \in S_{h,r},\end{aligned} \tag{4.2}$$

where ε_n is the quadrature error associated with the quadrature rule (2.2), that is,

$$\varepsilon_n(\varphi) = I_k^{1/2}(\varphi)(t_n) - \int_0^{t_n}(t_n - s)^{-1/2}\varphi(s)ds. \tag{4.3}$$

Then, using (4.1), (4.2), and (2.4) and choosing $v_h = \xi^n$, we obtain

$$\begin{aligned}&(\partial_t\xi^n, \xi^n) + \pi k^3\epsilon_0^2\Big(\frac{\partial^2[\partial_t\xi^n]}{\partial x\partial y}, \frac{\partial^2\xi^n}{\partial x\partial y}\Big) + (\pi k)^{1/2}\sum_{p=0}^{n-1}\epsilon_p(\nabla\xi^{n-p}, \nabla\xi^n) \\ =&(\sigma^n, \xi^n) + \pi k^3\epsilon_0^2\Big(\frac{\partial^2}{\partial x\partial y}[\partial_t\eta^n - \partial_t u^n], \frac{\partial^2\xi^n}{\partial x\partial y}\Big),\end{aligned} \tag{4.4}$$

where $\sigma^n = \sigma_1^n + \sigma_2^n + \sigma_3^n$ with

$$\sigma_1^n = u_t^n - \partial_t u^n, \quad \sigma_2^n = \partial_t \eta^n, \quad \sigma_3^n = \varepsilon_n(\Delta u).$$

Then

$$(\partial_t \xi^n, \xi^n) \geq \frac{1}{2}\partial_t \|\xi^n\|^2, \tag{4.5}$$

and

$$\Big(\frac{\partial^2[\partial_t \xi^n]}{\partial x \partial y}, \frac{\partial^2 \xi^n}{\partial x \partial y}\Big) \geq \frac{1}{2}\partial_t \Big\|\frac{\partial^2 \xi^n}{\partial x \partial y}\Big\|^2. \tag{4.6}$$

For the right-hand-side terms in (4.4), we have, respectively,

$$|(\sigma^n, \xi^n)| \leq \frac{1}{2}(\|\sigma\|^2 + \|\xi^n\|^2), \tag{4.7a}$$

$$\Big|\Big(\frac{\partial^2}{\partial x \partial y}[\partial_t \eta^n - \partial_t u^n], \frac{\partial^2 \xi^n}{\partial x \partial y}\Big)\Big| \leq \frac{1}{2}\Big(\Big\|\frac{\partial^2}{\partial x \partial y}[\partial_t \eta^n - \partial_t u^n]\Big\|^2 + \Big\|\frac{\partial^2 \xi^n}{\partial x \partial y}\Big\|^2\Big). \tag{4.7b}$$

If we substitute (4.5)-(4.7b) into (4.4), multiply both sides by $2k$, and then sum the resulting expression from $n = 1$ to $n = N$, we obtain

$$\begin{aligned}
&\|\xi^N\|^2 + \pi k^3 \epsilon_0^2 \Big\|\frac{\partial^2 \xi^N}{\partial x \partial y}\Big\|^2 + 2\sqrt{\pi} k^{3/2} \sum_{n=1}^{N} \sum_{p=0}^{n-1} \epsilon_p (\nabla \xi^{n-p}, \nabla \xi^n) \\
\leq& \|\xi^0\|^2 + \pi k^3 \epsilon_0^2 \Big\|\frac{\partial^2 \xi^0}{\partial x \partial y}\Big\|^2 + k\sum_{n=1}^{N}\|\sigma^n\|^2 + k\sum_{n=1}^{N}\|\xi^n\|^2 \\
&+ \pi k^4 \epsilon_0^2 \sum_{n=1}^{N} \Big\|\frac{\partial^2}{\partial x \partial y}[\partial_t \eta^n - \partial_t u^n]\Big\|^2 + \pi k^4 \epsilon_0^2 \sum_{n=1}^{N} \Big\|\frac{\partial^2 \xi^n}{\partial x \partial y}\Big\|^2.
\end{aligned} \tag{4.8}$$

From Lemma 2.3, it holds that

$$2\sqrt{\pi} k^{3/2} \sum_{n=1}^{N} \sum_{p=0}^{n-1} \epsilon_p (\nabla \xi^{n-p}, \nabla \xi^n) \geq 0. \tag{4.9}$$

So we have

$$\begin{aligned}
&\|\xi^N\|^2 + \pi k^3 \epsilon_0^2 \Big\|\frac{\partial^2 \xi^N}{\partial x \partial y}\Big\|^2 \\
\leq& k\sum_{n=1}^{N}\Big[\|\xi^n\|^2 + \pi k^3 \epsilon_0^2 \Big\|\frac{\partial^2 \xi^n}{\partial x \partial y}\Big\|^2\Big] + \Big[\|\xi^0\|^2 + \pi k^3 \epsilon_0^2 \Big\|\frac{\partial^2 \xi^0}{\partial x \partial y}\Big\|^2 \\
&+ k\sum_{n=1}^{N}\|\sigma^n\|^2 + \pi k^4 \epsilon_0^2 \sum_{n=1}^{N} \Big\|\frac{\partial^2}{\partial x \partial y}[\partial_t \eta^n - \partial_t u^n]\Big\|^2\Big].
\end{aligned}$$

From the Gronwall inequality, for k sufficiently small, it follows that

$$\begin{aligned}\|\xi^N\|^2+\pi k^3\epsilon_0^2\Big\|\frac{\partial^2\xi^N}{\partial x\partial y}\Big\|^2 \le& c\bigg[\|\xi^0\|^2+\pi k^3\epsilon_0^2\Big\|\frac{\partial^2\xi^0}{\partial x\partial y}\Big\|^2+k\sum_{n=1}^N\|\sigma^n\|^2\\ &+\pi k^4\epsilon_0^2\sum_{n=1}^N\Big\|\frac{\partial^2}{\partial x\partial y}[\partial_t\eta^n-\partial_t u^n]\Big\|^2\bigg].\end{aligned} \tag{4.10}$$

Taking $U^0=R_hu^0$, we can obtain

$$\|\xi^N\|^2\le c\bigg[k\sum_{n=1}^N\|\sigma^n\|^2+\pi k^4\epsilon_0^2\sum_{n=1}^N\Big\|\frac{\partial^2}{\partial x\partial y}[\partial_t\eta^n-\partial_t u^n]\Big\|^2\bigg]. \tag{4.11}$$

Using the triangle inequality and (4.11), we obtain

$$\begin{aligned}\|U^N-u^N\|^2\le& c[\|\xi^N\|^2+\|\eta^N\|^2]\\ \le& c\bigg[\|\eta^N\|^2+k\sum_{n=1}^N\|\sigma^n\|^2+\pi k^4\epsilon_0^2\sum_{n=1}^N\Big\|\frac{\partial^2}{\partial x\partial y}[\partial_t\eta^n-\partial_t u^n]\Big\|^2\bigg].\end{aligned} \tag{4.12}$$

It remains to estimate the terms on the right-hand side of (4.12). First, by Lemma 2.1, we have

$$\|\eta^N\|^2\le\|\eta\|^2_{L^\infty(L^2)}\le ch^{2r}\|u\|^2_{L^\infty(H^r)}. \tag{4.13}$$

To estimate σ_1^n, we use Taylor's theorem with integral remainder and Hölder inequality to obtain

$$\begin{aligned}\|\sigma_1^n\|^2=&\Big\|\frac{1}{k}\int_{t_{n-1}}^{t_n}(t_n-s)u_{tt}(s)ds\Big\|^2=\int_\Omega\Big|\frac{1}{k}\int_{t_{n-1}}^{t_n}(t_n-s)u_{tt}(s)ds\Big|^2d\mu\\ \le&\int_\Omega\Big|\int_{t_{n-1}}^{t_n}u_{tt}(s)ds\Big|^2d\mu\le k\int_{t_{n-1}}^{t_n}\int_\Omega|u_{tt}(s)|^2d\mu ds\\ =&k\int_{t_{n-1}}^{t_n}\|u_{tt}(s)\|^2ds,\end{aligned}$$

where $d\mu=dxdy$. Hence, we have

$$k\sum_{n=1}^N\|\sigma_1^n\|^2\le k^2\sum_{n=1}^N\int_{t_{n-1}}^{t_n}\|u_{tt}(s)\|^2ds\le k^2\|u_{tt}\|^2_{L^2(L^2)}. \tag{4.14}$$

Since

$$\sigma_2^n=\partial_t\eta^n=\frac{1}{k}\int_{t_{n-1}}^{t_n}\eta_t(s)ds,$$

similarly, using Hölder inequality, we obtain

$$\|\sigma_2^n\|^2=\Big\|\frac{1}{k}\int_{t_{n-1}}^{t_n}\eta_t(s)ds\Big\|^2\le\frac{1}{k}\int_{t_{n-1}}^{t_n}\|\eta_t(s)\|^2ds.$$

Then, from Lemma 2.1, we get

$$
\begin{aligned}
k\sum_{n=1}^{N}\|\sigma_2^n\|^2 \leq & \sum_{n=1}^{N}\int_{t_{n-1}}^{t_n}\|\eta_t(s)\|^2 ds \leq \int_0^T \|\eta_t(s)\|^2 ds \\
&= \|\eta_t(s)\|^2_{L^2(L^2)} \leq ch^{2r}\Big\|\frac{\partial u}{\partial t}\Big\|^2_{L^2(H^r)}.
\end{aligned} \tag{4.15}
$$

It follows from (2.6) that

$$
\begin{aligned}
& k\sum_{n=1}^{N}\|\sigma_3^n\|^2 \\
\leq & ck\sum_{n=1}^{N}\Big((k/n)\|\Delta u(0)\|^2 + k^2\int_{t_{n-1}}^{t_n}\|\Delta u_t(s)\|^2 ds + k^2|\ln k|\int_0^{t_{n-1}}\|\Delta u_t(s)\|^2 ds\Big) \\
\leq & c\Big(k^2|\ln k|\|\Delta u(0)\|^2 + k^3\|\Delta u_t\|^2_{L^2(L^2)} + k^2|\ln k|\|\Delta u_t\|^2_{L^2(L^2)}\Big),
\end{aligned} \tag{4.16}
$$

since $1+1/2+\cdots+1/N \leq c|\ln k|$.

We now estimate the last term on the right-hand side of (4.12). Similarly, applying the Hölder inequality, we get

$$
\begin{aligned}
& \Big\|\frac{\partial^2[\partial_t\eta^n]}{\partial x\partial y}\Big\|^2 = \int_\Omega \Big|\frac{\partial^2}{\partial x\partial y}\Big(\frac{1}{k}\int_{t_{n-1}}^{t_n}\eta_t(s)ds\Big)\Big|^2 d\mu \\
\leq & \frac{1}{k^2}\int_\Omega\Big|\int_{t_{n-1}}^{t_n}\frac{\partial^3}{\partial x\partial y\partial t}\eta(s)ds\Big|^2 d\mu \leq \frac{1}{k}\int_{t_{n-1}}^{t_n}\Big\|\frac{\partial^3}{\partial x\partial y\partial t}\eta(s)\Big\|^2 ds.
\end{aligned}
$$

Thus we have

$$
\begin{aligned}
& k\sum_{n=1}^{N}\Big\|\frac{\partial^2[\partial_t\eta^n]}{\partial x\partial y}\Big\|^2 \leq \sum_{n=1}^{N}\int_{t_{n-1}}^{t_n}\Big\|\frac{\partial^3\eta(s)}{\partial x\partial y\partial t}\Big\|^2 ds \\
\leq & \int_0^T\Big\|\frac{\partial^3\eta(s)}{\partial x\partial y\partial t}\Big\|^2 ds = \Big\|\frac{\partial^3\eta}{\partial x\partial y\partial t}\Big\|^2_{L^2(L^2)}.
\end{aligned}
$$

In addition, according to Lemma 2.1 and Lemma 2.4, we obtain

$$
\Big\|\frac{\partial^3\eta}{\partial x\partial y\partial t}\Big\|^2_{L^2(L^2)} \leq ch^{2r-4}\Big\|\frac{\partial u}{\partial t}\Big\|^2_{L^2(H^r)}.
$$

Namely,

$$
\pi k^4\epsilon_0^2\sum_{n=1}^{N}\Big\|\frac{\partial^2[\partial_t\eta^n]}{\partial x\partial y}\Big\|^2 \leq ck^3h^{2r-4}\Big\|\frac{\partial u}{\partial t}\Big\|^2_{L^2(H^r)}. \tag{4.17}
$$

Similarly, we have

$$\begin{aligned}\pi k^4\epsilon_0^2\sum_{n=1}^{N}\left\|\frac{\partial^2[\partial_t u^n]}{\partial x\partial y}\right\|^2 \le& \pi k^3\epsilon_0^2\sum_{n=1}^{N}\int_{t_{n-1}}^{t_n}\left\|\frac{\partial^3 u}{\partial x\partial y\partial t}\right\|^2 ds\\ \le& ck^3\int_0^T\left\|\frac{\partial^3 u}{\partial x\partial y\partial t}\right\|^2 ds = ck^3\left\|\frac{\partial^3 u}{\partial x\partial y\partial t}\right\|^2_{L^2(L^2)}.\end{aligned} \tag{4.18}$$

On substituting (4.13)-(4.18) into (4.12), we obtain

$$\begin{aligned}\|U^N-u^N\|^2 \le& c\Bigg(h^{2r}\left[\|u\|^2_{L^\infty(H^r)}+\left\|\frac{\partial u}{\partial t}\right\|^2_{L^2(H^r)}\right]+k^3h^{2r-4}\left\|\frac{\partial u}{\partial t}\right\|^2_{L^2(H^r)}\\ &+k^3\left[\|\Delta u_t\|^2_{L^2(L^2)}+\left\|\frac{\partial^3 u}{\partial x\partial y\partial t}\right\|^2_{L^2(L^2)}\right]+k^2\Big[|\ln k|\|\Delta u(0)\|^2\\ &+\|u_{tt}\|^2_{L^2(L^2)}+|\ln k|\|\Delta u_t\|^2_{L^2(L^2)}\Big]\Bigg),\end{aligned}$$

which completes the proof. □

Employing a similar technique as in the Theorem 4.1, we can directly derive the stability result.

Theorem 4.2. *The ADI Galerkin finite scheme* (3.5) *is unconditionally stable in the sense that for all $k>0$, it holds*

$$\|U^N\| \le c\left(\|U^0\|+\pi k^3\epsilon_0\left\|\frac{\partial^2 U^0}{\partial x\partial y}\right\|+k\sum_{n=1}^{N}\|f^n\|\right).$$

5. Numerical results

In this section, we present numerical results obtained using the ADI Galerkin finite element scheme (3.5) at $T=1$.

Example 5.1. We consider the following nonhomogeneous problem

$$u_t-\int_0^t (t-s)^{-1/2}\Delta u(x,y,s)ds = f(x,y,t), \qquad 0<x,y<1, \quad 0<t\le T, \tag{5.1a}$$

$$u(0,y,t)=u(1,y,t)=u(x,0,t)=u(x,1,t)=0, \qquad 0<t\le T, \tag{5.1b}$$

$$u(x,y,0)=\sin(\pi x)\sin(\pi y), \qquad 0\le x,y\le 1. \tag{5.1c}$$

Then we choose the exact analytical solution:

$$u(x,y,t)=\sin\pi x\sin\pi y(t^{3/2}+1).$$

It can be checked that the associated forcing term is

$$f(x,y,t)=\sin\pi x\sin\pi y\big(3/2t^{1/2}+3/4\pi^3t^2+4\pi^2t^{1/2}\big).$$

In the implementation of the ADI method, we take the linear tensor product bases

$$\varphi_i(x)=\begin{cases}0, & x_0\le x<x_{i-1},\\ \dfrac{x-x_{i-1}}{h_i}, & x_{i-1}\le x<x_i,\\ \dfrac{x_{i+1}-x}{h_{i+1}}, & x_i\le x<x_{i+1},\\ 0, & x_{i+1}\le x<x_{M_x},\end{cases}$$

and

$$\vartheta_j(y)=\begin{cases}0, & y_0\le y<y_{j-1},\\ \dfrac{y-y_{j-1}}{h_j}, & y_{j-1}\le y<y_j,\\ \dfrac{y_{j+1}-y}{h_{j+1}}, & y_j\le y<y_{j+1},\\ 0, & y_{j+1}\le y<y_{M_y},\end{cases}$$

where $i=1,2,\cdots,M_x-1$, $j=1,2,\cdots,M_y-1$, $h_i=x_i-x_{i-1}$, $h_j=y_j-y_{j-1}$. We use the same spacing h in each direction, $h_i=h_j=h$. We present the error in L^2 norm

$$e(k,h)=\|U^N-u^N\|$$

and the convergence rates in the temporal and spatial directions determined by the following formulas

$$\gamma_1\approx\frac{\log(e_l/e_{l+1})}{\log(\tau_l/\tau_{l+1})},\quad \gamma_2\approx\frac{\log(e_m/e_{m+1})}{\log(h_m/h_{m+1})},$$

respectively, where the stepsize $\tau_l=1/N_l$, $h_m=1/M_m$ and e_l, e_m is the norm of the error with $\tau=\tau_l$, $h=h_m$ respectively.

The initial condition can be approximated using $U^0=R_hu^0$. The computational results are displayed in Table 1 when uniform stepsizes $k=h=T/N$ are used. We observe that the numerical convergence order matches well with the theoretical convergence order in temporal direction.

In Table 2, we list the L^2 errors and convergence rates when $k=h^2=T/N$. It can be clearly seen that the second order accuracy in spatial direction is verified, which is in good agreement with the theoretical prediction of Theorem 4.1.

Table 1: The L^2 errors and convergence orders in temporal direction.

h	k	L^2 error	Rate γ_1
1/8	1/8	0.0225	
1/12	1/12	0.0128	1.3912
1/16	1/16	0.0094	1.0732
1/20	1/20	0.0076	0.9526

Table 2: The L^2 errors and convergence orders in spatial direction.

h	k	L^2 error	Rate γ_2
1/4	1/16	0.0401	
1/8	1/64	0.0103	1.9610
1/12	1/144	0.0045	2.0423
1/16	1/256	0.0025	2.0432

6. Concluding remarks

In this paper, we have formulated and analyzed ADI Galerkin finite element method for the two-dimensional fractional evolution equation. It has been shown that the ADI solutions are convergent in the L^2 norm. The theoretical analysis has been verified by some numerical results.

Generally, a method similar to that presented in the paper can be constructed for the equation

$$u_t - \int_0^t (t-s)^{\alpha-1} \Delta u(x,y,s) ds = f(x,y,t), \quad 0 < \alpha < 1.$$

With an analysis parallel to that carried out in the paper, the stability and convergence of the numerical method are preserved.

The problem is linear, and the spatial domain is assumed to be the unit square. It may be possible to extend the present ADI method to nonlinear equations in rectangular polygons.

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