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Superconvergence and L^∞-Error Estimates of RT1 Mixed Methods for Semilinear Elliptic Control Problems with an Integral Constraint

Yanping Chen[1,*] and Tianliang Hou[2]

[1] *School of Mathematical Sciences, South China Normal University, Guangzhou 510631, Guangdong, China.*
[2] *Hunan Key Laboratory for Computation and Simulation in Science and Engineering, Department of Mathematics, Xiangtan University, Xiangtan 411105, Hunan, China.*

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Abstract. In this paper, we investigate the superconvergence property and the L^∞-error estimates of mixed finite element methods for a semilinear elliptic control problem with an integral constraint. The state and co-state are approximated by the order one Raviart-Thomas mixed finite element space and the control variable is approximated by piecewise constant functions or piecewise linear functions. We derive some superconvergence results for the control variable and the state variables when the control is approximated by piecewise constant functions. Moreover, we derive L^∞-error estimates for both the control variable and the state variables when the control is discretized by piecewise linear functions. Finally, some numerical examples are given to demonstrate the theoretical results.

AMS subject classifications: 49J20, 65N30

Key words: Semilinear elliptic equations, optimal control problems, superconvergence, L^∞-error estimates, mixed finite element methods, postprocessing.

1. Introduction

The finite element approximation plays an important role in the numerical treatment of optimal control problems. There have been extensive studies in convergence and superconvergence of finite element approximations for optimal control problems, (see, e.g., [1, 6, 11–15, 20–24]). A systematic introduction of finite element methods for PDEs and optimal control problems can be found in, (e.g., [8, 17]). Note that all the above papers aim at the standard finite element methods for optimal controls.

*Corresponding author. *Email addresses:* `yanpingchen@scnu.edu.cn` (Y. Chen), `htlchb@163.com` (T. Hou)

Compared with standard finite element methods, the mixed finite methods have many advantages. When the objective functional contains gradient of the state variable, we will firstly choose the mixed finite element methods. We have done some works on priori error estimates and superconvergence properties of mixed finite elements for optimal control problems [3–5]. In [4], we used the postprocessing projection operator, which was defined by Meyer and Rösch (see [20]) to prove a quadratic superconvergence of the control by mixed finite element methods. Recently, we derived error estimates and superconvergence of mixed methods for convex optimal control problems in [5]. But in that paper, the convergence order is $h^{\frac{3}{2}}$ since the analysis was restricted by the low regularity of the control.

The goal of this paper is to derive the superconvergence property and the L^∞-error estimates of mixed finite element approximation for a semilinear elliptic control problem with an integral constraint. Firstly, when the control is approximated by piecewise constant functions, we derive the superconvergence property between average L^2 projection and the approximation of the control variable, the convergence order is h^2 instead of $h^{\frac{3}{2}}$ in [5], which is caused by the different admissible set. Then, after solving a fully discretized optimal control problem, a control $\hat{u}$ is calculated by the projection of the adjoint state z_h in a postprocessing step. Although the approximation of the discretized solution is only of order h, we will show that this postprocessing step improves the convergence order to h^2. We also derive the L^∞-error estimates for both the control variable and the state variables when the control variable is discretized by piecewise linear functions. Finally, we present two numerical experiments to demonstrate the practical side of the theoretical results about superconvergence and L^∞-error estimates.

We consider the following semilinear optimal control problems for the state variables $\boldsymbol{p}$, y, and the control u with an integral constraint:

$$\min_{u\in U_{ad}} \left\{\frac{1}{2}\|\boldsymbol{p}-\boldsymbol{p}_d\|^2+\frac{1}{2}\|y-y_d\|^2+\frac{\nu}{2}\|u\|^2\right\} \tag{1.1}$$

subject to the state equation

$$-\mathrm{div}(A(x)\mathbf{grad}y)+\phi(y)=u, \quad x\in\Omega, \tag{1.2}$$

which can be written in the form of the first order system

$$\mathrm{div}\boldsymbol{p}+\phi(y)=u, \quad \boldsymbol{p}=-A(x)\mathbf{grad}y, \quad x\in\Omega, \tag{1.3}$$

and the boundary condition

$$y=0, \quad x\in\partial\Omega, \tag{1.4}$$

where Ω is a bounded domain in $\mathbb{R}^2$. U_{ad} denotes the admissible set of the control variable, defined by

$$U_{ad}=\left\{u\in L^\infty(\Omega):\ \int_\Omega u dx\geq 0\right\}. \tag{1.5}$$

We assume that the function $\phi(\cdot) \in W^{2,\infty}(-R,R) \cap H^3(-R,R)$ for any $R > 0$, $\phi'(y) \in L^2(\Omega)$ for any $y \in H^1(\Omega)$, and $\phi' \geq 0$. Moreover, we assume that $y_d \in H^1(\Omega)$ and $\boldsymbol{p}_d \in (H^2(\Omega))^2$. ν is a fixed positive number. The coefficient $A(x) = (a_{ij}(x))$ is a symmetric matrix function with $a_{ij}(x) \in W^{1,\infty}(\Omega)$, which satisfies the ellipticity condition

$$c_*|\xi|^2 \leq \sum_{i,j=1}^{2} a_{ij}(x)\xi_i\xi_j, \quad \forall\ (\xi, x) \in \mathbb{R}^2 \times \bar{\Omega}, \quad c_* > 0.$$

The plan of this paper is as follows. In Section 2, we construct the mixed finite element approximation scheme for the optimal control problem (1.1)-(1.4) and give its equivalent optimality conditions. The main results of this paper are stated in Section 3 and Section 4. In Section 3, when the control variable is discretized by piecewise constant functions, we derive the superconvergence properties between the average L^2 projection and the approximation, as well as between the postprocessing solution and the exact control solution. In Section 4, we will study the L^∞-error estimates for optimal control problem when the control variable is approximated by piecewise linear functions. In Section 5, we present two numerical examples to demonstrate our theoretical results. In the last section, we briefly summarize the results obtained and some possible future extensions.

In this paper, we adopt the standard notation $W^{m,p}(\Omega)$ for Sobolev spaces on Ω with a norm $\|\cdot\|_{m,p}$ given by

$$\|v\|_{m,p}^p = \sum_{|\alpha| \leq m} \|D^\alpha v\|_{L^p(\Omega)}^p,$$

a semi-norm $|\cdot|_{m,p}$ given by

$$|v|_{m,p}^p = \sum_{|\alpha| = m} \|D^\alpha v\|_{L^p(\Omega)}^p.$$

We set $W_0^{m,p}(\Omega) = \{v \in W^{m,p}(\Omega) : v|_{\partial\Omega} = 0\}$. For $p = 2$, we denote $H^m(\Omega) = W^{m,2}(\Omega)$, $H_0^m(\Omega) = W_0^{m,2}(\Omega)$, and $\|\cdot\|_m = \|\cdot\|_{m,2}$, $\|\cdot\| = \|\cdot\|_{0,2}$. In addition C denotes a general positive constant independent of h, where h is the spatial mesh-size for the control and state discretization.

2. Mixed methods of optimal control problems

In this section we shall construct mixed finite element approximation scheme of the control problem (1.1)-(1.4). For sake of simplicity, we assume that the domain Ω is a convex polygon. Now, we introduce the co-state elliptic equation

$$-\mathrm{div}(A(x)(\mathbf{grad} z + \boldsymbol{p} - \boldsymbol{p}_d)) + \phi'(y)z = y - y_d, \quad x \in \Omega, \tag{2.1}$$

which can be written in the form of the first order system

$$\mathrm{div}\boldsymbol{q} + \phi'(y)z = y - y_d, \quad \boldsymbol{q} = -A(x)(\mathbf{grad} z + \boldsymbol{p} - \boldsymbol{p}_d), \quad x \in \Omega, \tag{2.2}$$

and the boundary condition

$$z=0,\quad x\in\partial\Omega. \tag{2.3}$$

Let

$$\boldsymbol{V}=H(\mathrm{div};\Omega)=\left\{\boldsymbol{v}\in(L^2(\Omega))^2,\mathrm{div}\boldsymbol{v}\in L^2(\Omega)\right\},\quad W=L^2(\Omega). \tag{2.4}$$

We recast (1.1)-(1.4) as the following weak form: find $(\boldsymbol{p},y,u)\in\boldsymbol{V}\times W\times U_{ad}$ such that

$$\min_{u\in U_{ad}}\left\{\frac{1}{2}\|\boldsymbol{p}-\boldsymbol{p}_d\|^2+\frac{1}{2}\|y-y_d\|^2+\frac{\nu}{2}\|u\|^2\right\}, \tag{2.5}$$
$$(A^{-1}\boldsymbol{p},\boldsymbol{v})-(y,\mathrm{div}\boldsymbol{v})=0,\qquad \forall\ \boldsymbol{v}\in\boldsymbol{V}, \tag{2.6}$$
$$(\mathrm{div}\boldsymbol{p},w)+(\phi(y),w)=(u,w),\quad \forall\ w\in W. \tag{2.7}$$

It follows from [17] that the optimal control problem (2.5)-(2.7) has a solution $(\boldsymbol{p},y,u)$, and that a triplet $(\boldsymbol{p},y,u)$ is the solution of (2.5)-(2.7) if there is a co-state $(\boldsymbol{q},z)\in\boldsymbol{V}\times W$ such that $(\boldsymbol{p},y,\boldsymbol{q},z,u)$ satisfies the following optimality conditions:

$$(A^{-1}\boldsymbol{p},\boldsymbol{v})-(y,\mathrm{div}\boldsymbol{v})=0,\qquad \forall\ \boldsymbol{v}\in\boldsymbol{V}, \tag{2.8}$$
$$(\mathrm{div}\boldsymbol{p},w)+(\phi(y),w)=(u,w),\qquad \forall\ w\in W, \tag{2.9}$$
$$(A^{-1}\boldsymbol{q},\boldsymbol{v})-(z,\mathrm{div}\boldsymbol{v})=-(\boldsymbol{p}-\boldsymbol{p}_d,\boldsymbol{v}),\qquad \forall\ \boldsymbol{v}\in\boldsymbol{V}, \tag{2.10}$$
$$(\mathrm{div}\boldsymbol{q},w)+(\phi'(y)z,w)=(y-y_d,w),\qquad \forall\ w\in W, \tag{2.11}$$
$$(\nu u+z,\tilde{u}-u)\geq 0,\qquad \forall\ \tilde{u}\in U_{ad}, \tag{2.12}$$

where $(\cdot,\cdot)$ is the inner product of $L^2(\Omega)$.

In [7], the expression of the control variable is given. Here, we adopt the same method to derive the following operator

$$u=(\max\{0,\bar{z}\}-z)/\nu, \tag{2.13}$$

where $\bar{z}=\int_\Omega z/\int_\Omega 1$ denotes the integral average on Ω of the function z.

Let $\mathcal{T}_h$ denote a regular triangulation of the polygonal domain Ω, h_T denotes the diameter of T and $h=\max h_T$. Let $\boldsymbol{V}_h\times W_h\subset\boldsymbol{V}\times W$ denotes the order $k=1$ Raviart-Thomas mixed finite element space [10, 25], namely,

$$\forall\ T\in\mathcal{T}_h,\ \ \boldsymbol{V}(T)=\boldsymbol{P}_1(T)\oplus\mathrm{span}(xP_1(T)),\ \ W(T)=P_1(T),$$

where $P_1(T)$ denote polynomials of total degree at most 1, $\boldsymbol{P}_1(T)=(P_1(T))^2$, $x=(x_1,x_2)$ which is treated as a vector, and

$$\boldsymbol{V}_h:=\{\boldsymbol{v}_h\in\boldsymbol{V}:\forall\ T\in\mathcal{T}_h,\boldsymbol{v}_h|_T\in\boldsymbol{V}(T)\}, \tag{2.14}$$
$$W_h:=\{w_h\in W:\forall\ T\in\mathcal{T}_h,w_h|_T\in W(T)\}. \tag{2.15}$$

And the approximated space of control is given by

$$U_h := \left\{ \tilde{u}_h \in U_{ad} : \forall\ T \in \mathcal{T}_h,\ \tilde{u}_h|_T = \text{constant} \right\}, \tag{2.16}$$

or

$$U_h := \left\{ \tilde{u}_h \in U_{ad} : \forall\ T \in \mathcal{T}_h,\ \tilde{u}_h|_T \in W(T) \right\}. \tag{2.17}$$

Before the mixed finite element scheme is given, we introduce two operators. Firstly, we define the standard $L^2(\Omega)$-projection [10] $P_h :\ W \to W_h$, which satisfies: for any $\phi \in W$

$$(P_h\phi - \phi, w_h) = 0, \qquad \forall\ w_h \in W_h, \tag{2.18}$$

$$\|\phi - P_h\phi\|_{0,\rho} \le Ch^r\|\phi\|_{r,\rho},\ 1 \le \rho \le \infty,\ \forall\ \phi \in W^{r,\rho}(\Omega),\ \ r = 1,2. \tag{2.19}$$

Next, recall the Fortin projection (see [2] and [10]) $\Pi_h :\ \boldsymbol{V} \to \boldsymbol{V}_h$, which satisfies: for any $\boldsymbol{q} \in \boldsymbol{V}$

$$(\text{div}(\Pi_h\boldsymbol{q} - \boldsymbol{q}), w_h) = 0, \qquad \forall\ w_h \in W_h, \tag{2.20}$$

$$\|\boldsymbol{q} - \Pi_h\boldsymbol{q}\|_0 \le Ch^r\|\boldsymbol{q}\|_r, \qquad \forall\ \boldsymbol{q} \in (H^r(\Omega))^2,\ \ r = 1,2, \tag{2.21}$$

$$\|\text{div}(\boldsymbol{q} - \Pi_h\boldsymbol{q})\|_0 \le Ch^r\|\text{div}\boldsymbol{q}\|_r, \quad \forall\ \text{div}\boldsymbol{q} \in H^r(\Omega),\ \ r = 1,2. \tag{2.22}$$

We have the commuting diagram property

$$\text{div} \circ \Pi_h = P_h \circ \text{div} : \boldsymbol{V} \to W_h \quad \text{and} \quad \text{div}(I - \Pi_h)\boldsymbol{V} \perp W_h, \tag{2.23}$$

where and after, I denotes identity operator.

Furthermore, we also define the standard L^2-orthogonal projection $Q_h : U_{ad} \to U_h$, which satisfies: for any $u \in U_{ad}$

$$(u - Q_h u, u_h) = 0,\ \ \forall\ u_h \in U_h. \tag{2.24}$$

We have the approximation property:

$$\|u - Q_h u\|_{-s,r} \le Ch^{1+s}|\phi|_{1,r},\ s = 0,1,\ \forall\ u \in W^{1,r}(\Omega). \tag{2.25}$$

Then the mixed finite element discretization of (2.5)-(2.7) is as follows: find $(\boldsymbol{p}_h, y_h, u_h) \in \boldsymbol{V}_h \times W_h \times U_h$ such that

$$\min_{u_h \in U_h} \left\{ \frac{1}{2}\|\boldsymbol{p}_h - \boldsymbol{p}_d\|^2 + \frac{1}{2}\|y_h - y_d\|^2 + \frac{\nu}{2}\|u_h\|^2 \right\} \tag{2.26}$$

$$(A^{-1}\boldsymbol{p}_h, \boldsymbol{v}_h) - (y_h, \text{div}\boldsymbol{v}_h) = 0, \qquad \forall\ \boldsymbol{v}_h \in \boldsymbol{V}_h, \tag{2.27}$$

$$(\text{div}\boldsymbol{p}_h, w_h) + (\phi(y_h), w_h) = (u_h, w_h), \quad \forall\ w_h \in W_h. \tag{2.28}$$

The optimal control problem (2.26)-(2.28) again has a solution $(\boldsymbol{p}_h, y_h, u_h)$, and that a triplet $(\boldsymbol{p}_h, y_h, u_h)$ is the solution of (2.26)-(2.28) if there is a co-state $(\boldsymbol{q}_h, z_h) \in \boldsymbol{V}_h \times W_h$ such that $(\boldsymbol{p}_h, y_h, \boldsymbol{q}_h, z_h, u_h)$ satisfies the following optimality conditions:

$$(A^{-1}\boldsymbol{p}_h, \boldsymbol{v}_h) - (y_h, \mathrm{div}\boldsymbol{v}_h) = 0, \qquad \forall\ \boldsymbol{v}_h \in \boldsymbol{V}_h, \tag{2.29}$$

$$(\mathrm{div}\boldsymbol{p}_h, w_h) + (\phi(y_h), w_h) = (u_h, w_h), \qquad \forall\ w_h \in W_h, \tag{2.30}$$

$$(A^{-1}\boldsymbol{q}_h, \boldsymbol{v}_h) - (z_h, \mathrm{div}\boldsymbol{v}_h) = -(\boldsymbol{p}_h - \boldsymbol{p}_d, \boldsymbol{v}_h), \qquad \forall\ \boldsymbol{v}_h \in \boldsymbol{V}_h, \tag{2.31}$$

$$(\mathrm{div}\boldsymbol{q}_h, w_h) + (\phi'(y_h)z_h, w_h) = (y_h - y_d, w_h), \qquad \forall\ w_h \in W_h, \tag{2.32}$$

$$(\nu u_h + z_h, \tilde{u}_h - u_h) \geq 0, \qquad \forall\ \tilde{u}_h \in U_h. \tag{2.33}$$

For the variational inequality (2.33) we have the following conclusion.

Lemma 2.1. *Assume that z_h is known in the variational inequality* (2.33). *The solution of the variational inequality* (2.33) *is*

$$u_h = Q_h\left(-\frac{z_h}{\nu} + \max\left\{0, \frac{\overline{z_h}}{\nu}\right\}\right), \quad \overline{z_h} = \frac{\int_\Omega z_h}{\int_\Omega 1}, \tag{2.34}$$

where the control variable is discretized by piecewise constant functions. When the control variable is approximated by piecewise linear functions, we have

$$u_h = -\frac{z_h}{\nu} + \max\left\{0, \frac{\overline{z_h}}{\nu}\right\}. \tag{2.35}$$

Proof. Here we only give the proof of (2.34). The proof is divided into two steps. We will prove $u_h \in U_h$ at the first step, and then prove u_h is the solution of the variational inequality at the second step.

Step 1. For any $v \in U_{ad}$, we have

$$\int_\Omega (Q_h v - v)\psi = 0, \quad \forall\ \psi \in U_h. \tag{2.36}$$

Since $\phi \equiv 1 \in U_h$ such that

$$\int_\Omega \left[Q_h\left(-\frac{z_h}{\nu} + \max\left\{0, \frac{\overline{z_h}}{\nu}\right\}\right) - \left(-\frac{z_h}{\nu} + \max\left\{0, \frac{\overline{z_h}}{\nu}\right\}\right)\right] = 0, \tag{2.37}$$

hence

$$\int_\Omega u_h = \int_\Omega \left(-\frac{z_h}{\nu} + \max\left\{0, \frac{\overline{z_h}}{\nu}\right\}\right) = -\int_\Omega \frac{z_h}{\nu} + \int_\Omega \max\left\{0, \frac{\overline{z_h}}{\nu}\right\} \geq 0. \tag{2.38}$$

Thus $u_h \in U_h$.

Step 2. Noting that for each $v_h \in U_h$,

$$
\begin{aligned}
&\int_\Omega (u_h + z_h/\nu)(v_h - u_h) \\
&= \int_\Omega \left[Q_h \left(-\frac{z_h}{\nu} + \max\left\{0, \frac{\overline{z_h}}{\nu}\right\}\right) - \left(-\frac{z_h}{\nu} + \max\left\{0, \frac{\overline{z_h}}{\nu}\right\}\right) + \max\left\{0, \frac{\overline{z_h}}{\nu}\right\} \right] (v_h - u_h) \\
&= \int_\Omega \max\left\{0, \frac{\overline{z_h}}{\nu}\right\} (v_h - u_h). \qquad (2.39)
\end{aligned}
$$

We see that if $\overline{z_h} \le 0$ then

$$\int_\Omega (\nu u_h + z_h)(v_h - u_h) = 0, \qquad (2.40)$$

and that if $\overline{z_h} \ge 0$ then

$$\int_\Omega (\nu u_h + z_h)(v_h - u_h) \ge 0, \qquad (2.41)$$

as

$$\int_\Omega u_h = \int_\Omega (-\frac{z_h}{\nu} + \max\{0, \frac{\overline{z_h}}{\nu}\} = 0 \quad \text{and} \quad \int_\Omega v_h \ge 0.$$

Therefore it is shown that u_h is the solution of the variational inequality (2.33). □

In the rest of the paper, we shall use some intermediate variables. For any control function $\tilde{u} \in U_{ad}$, we first define the state solution $(\boldsymbol{p}(\tilde{u}), y(\tilde{u}), \boldsymbol{q}(\tilde{u}), z(\tilde{u})) \in (\boldsymbol{V} \times W)^2$ associated with $\tilde{u}$ that satisfies

$$
\begin{aligned}
&(A^{-1}\boldsymbol{p}(\tilde{u}), \boldsymbol{v}) - (y(\tilde{u}), \mathrm{div}\boldsymbol{v}) = 0, && \forall\, \boldsymbol{v} \in \boldsymbol{V}, && (2.42)\\
&(\mathrm{div}\boldsymbol{p}(\tilde{u}), w) + (\phi(y(\tilde{u})), w) = (\tilde{u}, w), && \forall\, w \in W, && (2.43)\\
&(A^{-1}\boldsymbol{q}(\tilde{u}), \boldsymbol{v}) - (z(\tilde{u}), \mathrm{div}\boldsymbol{v}) = -(\boldsymbol{p}(\tilde{u}) - \boldsymbol{p}_d, \boldsymbol{v}), && \forall\, \boldsymbol{v} \in \boldsymbol{V}, && (2.44)\\
&(\mathrm{div}\boldsymbol{q}(\tilde{u}), w) + (\phi'(y(\tilde{u}))z(\tilde{u}), w) = (y(\tilde{u}) - y_d, w), && \forall\, w \in W. && (2.45)
\end{aligned}
$$

Then, we define the discrete state solution $(\boldsymbol{p}_h(\tilde{u}), y_h(\tilde{u}), \boldsymbol{q}_h(\tilde{u}), z_h(\tilde{u})) \in (\boldsymbol{V}_h \times W_h)^2$ associated with $\tilde{u}$ that satisfies

$$
\begin{aligned}
&(A^{-1}\boldsymbol{p}_h(\tilde{u}), \boldsymbol{v}_h) - (y_h(\tilde{u}), \mathrm{div}\boldsymbol{v}_h) = 0, && \forall\, \boldsymbol{v}_h \in \boldsymbol{V}_h, && (2.46)\\
&(\mathrm{div}\boldsymbol{p}_h(\tilde{u}), w_h) + (\phi(y_h(\tilde{u})), w_h) = (\tilde{u}, w_h), && \forall\, w_h \in W_h, && (2.47)\\
&(A^{-1}\boldsymbol{q}_h(\tilde{u}), \boldsymbol{v}_h) - (z_h(\tilde{u}), \mathrm{div}\boldsymbol{v}_h) = -(\boldsymbol{p}_h(\tilde{u}) - \boldsymbol{p}_d, \boldsymbol{v}_h), && \forall\, \boldsymbol{v}_h \in \boldsymbol{V}_h, && (2.48)\\
&(\mathrm{div}\boldsymbol{q}_h(\tilde{u}), w_h) + (\phi'(y_h(\tilde{u}))z_h(\tilde{u}), w_h) = (y_h(\tilde{u}) - y_d, w_h), && \forall w_h \in W_h. && (2.49)
\end{aligned}
$$

Thus, as we defined, the exact solution and its approximation can be written in the following way:

$$
\begin{aligned}
&(\boldsymbol{p}, y, \boldsymbol{q}, z) = (\boldsymbol{p}(u), y(u), \boldsymbol{q}(u), z(u)),\\
&(\boldsymbol{p}_h, y_h, \boldsymbol{q}_h, z_h) = (\boldsymbol{p}_h(u_h), y_h(u_h), \boldsymbol{q}_h(u_h), z_h(u_h)).
\end{aligned}
$$

3. Superconvergence and postprocessing

In this section, we will give a detailed superconvergence analysis. In the rest of the section, the control variable is discretized by piecewise constant functions. Firstly, we recall the following convergence results which are very important for our following work, similar results have been proved in [9].

Lemma 3.1. *Let* $(\boldsymbol{p}(\tilde{u}), y(\tilde{u}), \boldsymbol{q}(\tilde{u}), z(\tilde{u})) \in (\boldsymbol{V}\times W)^2$ *and* $(\boldsymbol{p}_h(\tilde{u}), y_h(\tilde{u}), \boldsymbol{q}_h(\tilde{u}), z_h(\tilde{u})) \in (\boldsymbol{V}_h \times W_h)^2$ *be the solutions of* (2.42)-(2.45) *and* (2.46)-(2.49) *respectively. If the intermediate solutions satisfy*

$$\boldsymbol{p}(\tilde{u}),\ \boldsymbol{q}(\tilde{u}) \in (H^2(\Omega))^2,$$

then we have

$$\|y(\tilde{u}) - y_h(\tilde{u})\| + \|\boldsymbol{p}(\tilde{u}) - \boldsymbol{p}_h(\tilde{u})\| \le Ch^2, \tag{3.1}$$

$$\|z(\tilde{u}) - z_h(\tilde{u})\| + \|\boldsymbol{q}(\tilde{u}) - \boldsymbol{q}_h(\tilde{u})\| \le Ch^2, \tag{3.2}$$

$$\|\mathrm{div}(\boldsymbol{p}(\tilde{u}) - \boldsymbol{p}_h(\tilde{u}))\| + \|\mathrm{div}(\boldsymbol{q}(\tilde{u}) - \boldsymbol{q}_h(\tilde{u}))\| \le Ch. \tag{3.3}$$

By modifying the proof of Theorem 4.3 in [9], we have

Lemma 3.2. *Let u be the solution of* (2.8)-(2.12) *and u_h be the solution of* (2.29)-(2.33)*, respectively. Assume that* $\boldsymbol{p},\ \boldsymbol{q} \in (H^2(\Omega))^2$ *and* $u \in H^1(\Omega)$*. Then, we have*

$$\|u - u_h\| \le Ch. \tag{3.4}$$

Lemma 3.3. *Let* $(\boldsymbol{p}(Q_hu), y(Q_hu), \boldsymbol{q}(Q_hu), z(Q_hu))$ *and* $(\boldsymbol{p}(u), y(u), \boldsymbol{q}(u), z(u))$ *be the solutions of* (2.42)-(2.45) *with* $\tilde{u} = Q_hu$ *and* $\tilde{u} = u$*, respectively. Assume that* $u \in H^1(\Omega)$*. Then we have*

$$\|y(u) - y(Q_hu)\| + \|\boldsymbol{p}(u) - \boldsymbol{p}(Q_hu)\| \le Ch^2, \tag{3.5}$$

$$\|z(u) - z(Q_hu)\| + \|\boldsymbol{q}(u) - \boldsymbol{q}(Q_hu)\| \le Ch^2. \tag{3.6}$$

Proof. First, we choose $\tilde{u} = Q_hu$ and $\tilde{u} = u$ in (2.42)-(2.45) respectively, then we obtain the following error equations

$$(A^{-1}(\boldsymbol{p}(Q_hu) - \boldsymbol{p}(u)), \boldsymbol{v}) - (y(Q_hu) - y(u), \mathrm{div}\boldsymbol{v}) = 0, \tag{3.7}$$

$$(\mathrm{div}(\boldsymbol{p}(Q_hu) - \boldsymbol{p}(u)), w) + (\phi(y(Q_hu)) - \phi(y(u)), w) = (Q_hu - u, w), \tag{3.8}$$

$$(A^{-1}(\boldsymbol{q}(Q_hu) - \boldsymbol{q}(u)), \boldsymbol{v}) - (z(Q_hu) - z(u), \mathrm{div}\boldsymbol{v}) = -(\boldsymbol{p}(Q_hu) - \boldsymbol{p}(u), \boldsymbol{v}), \tag{3.9}$$

$$\begin{aligned}&(\mathrm{div}(\boldsymbol{q}(Q_hu) - \boldsymbol{q}(u)), w) + (\phi'(y(Q_hu))z(Q_hu) - \phi'(y(u))z(u), w)\\ &\qquad = (y(Q_hu) - y(u), w),\end{aligned} \tag{3.10}$$

for any $\boldsymbol{v} \in \boldsymbol{V}$ and $w \in W$.

Setting $\boldsymbol{v} = \boldsymbol{p}(Q_h u) - \boldsymbol{p}(u)$ and $w = y(Q_h u) - y(u)$ in (3.7) and (3.8) respectively and adding the two resulting equations yield

$$\begin{aligned}
&(A^{-1}(\boldsymbol{p}(Q_h u) - \boldsymbol{p}(u)), \boldsymbol{p}(Q_h u) - \boldsymbol{p}(u)) \\
&\qquad + (\phi(y(Q_h u)) - \phi(y(u)), y(Q_h u) - y(u)) \\
&= (Q_h u - u, y(Q_h u) - y(u)). \qquad (3.11)
\end{aligned}$$

Then, we estimate the right hand side of (3.11). Note that $\boldsymbol{p}(Q_h u) - \boldsymbol{p}(u) = -A\mathbf{grad}(y(Q_h u) - y(u))$. By (2.25) and poincare' inequality, we have

$$\begin{aligned}
(Q_h u - u, y(Q_h u) - y(u)) &\leq C\|Q_h u - u\|_{-1}\|y(Q_h u) - y(u))\|_1 \\
&\leq Ch^2|u|_1\|\boldsymbol{p}(Q_h u) - \boldsymbol{p}(u)\|. \qquad (3.12)
\end{aligned}$$

It follows from the assumptions on A and ϕ, (3.11) and (3.12) that

$$\|\boldsymbol{p}(Q_h u) - \boldsymbol{p}(u)\| \leq Ch^2. \qquad (3.13)$$

By the Poincare's inequality, we have

$$\|y(Q_h u) - y(u)\| \leq C\|\boldsymbol{p}(Q_h u) - \boldsymbol{p}(u)\| \leq Ch^2. \qquad (3.14)$$

Similarly, letting $\boldsymbol{v} = \boldsymbol{q}(Q_h u) - \boldsymbol{q}(u)$ and $w = z(Q_h u) - z(u)$ in (3.9) and (3.10), respectively, we have

$$\begin{aligned}
&(A^{-1}(\boldsymbol{q}(Q_h u) - \boldsymbol{q}(u)), \boldsymbol{q}(Q_h u) - \boldsymbol{q}(u)) \\
&\qquad + (\phi'(y(Q_h u))z(Q_h u) - \phi'(y(u))z(u), z(Q_h u) - z(u)) \\
&=(y(Q_h u) - y(u), z(Q_h u) - z(u)) - (\boldsymbol{p}(Q_h u) - \boldsymbol{p}(u), \boldsymbol{q}(Q_h u) - \boldsymbol{q}(u)), \qquad (3.15)
\end{aligned}$$

which gives

$$\begin{aligned}
&(A^{-1}(\boldsymbol{q}(Q_h u) - \boldsymbol{q}(u)), \boldsymbol{q}(Q_h u) - \boldsymbol{q}(u)) + (\phi'(y(Q_h u))(z(Q_h u) - z(u)), z(Q_h u) - z(u)) \\
&=(y(Q_h u) - y(u), z(Q_h u) - z(u)) - (\boldsymbol{p}(Q_h u) - \boldsymbol{p}(u), \boldsymbol{q}(Q_h u) - \boldsymbol{q}(u)) \\
&\qquad + (z(u)(\phi'(y(Q_h u)) - \phi'(y(u))), z(Q_h u) - z(u)). \qquad (3.16)
\end{aligned}$$

Note that

$$-A\mathbf{grad}(z(Q_h u) - z(u)) = \boldsymbol{q}(Q_h u) - \boldsymbol{q}(u) + A(\boldsymbol{p}(Q_h u) - \boldsymbol{p}(u)). \qquad (3.17)$$

It follows from the Poincare's inequality, (3.14) and (3.17) that

$$\begin{aligned}
&(y(Q_h u) - y(u), z(Q_h u) - z(u)) \\
\leq&\|y(Q_h u) - y(u)\| \cdot \|z(Q_h u) - z(u)\| \\
\leq&Ch^2(\|\boldsymbol{q}(Q_h u) - \boldsymbol{q}(u)\| + \|\boldsymbol{p}(Q_h u) - \boldsymbol{p}(u)\|) \\
\leq&Ch^4 + Ch^2\|\boldsymbol{q}(Q_h u) - \boldsymbol{q}(u)\|. \qquad (3.18)
\end{aligned}$$

For the second term on the right side of (3.16), by (3.13) we derive

$$(\boldsymbol{p}(Q_hu)-\boldsymbol{p}(u),\boldsymbol{q}(Q_hu)-\boldsymbol{q}(u))\leq Ch^2\|\boldsymbol{q}(Q_hu)-\boldsymbol{q}(u)\|. \tag{3.19}$$

Using the assumption on ϕ, (3.14) and (3.17), we get

$$\begin{aligned}
&(z(u)(\phi'(y(Q_hu))-\phi'(y(u))),z(Q_hu)-z(u))\\
\leq& C\|z(u)\|_{0,4}\|\phi'(y(Q_hu))-\phi'(y(u))\|\cdot\|z(Q_hu)-z(u)\|_{0,4}\\
\leq& C\|z(u)\|_1\|\phi\|_{2,\infty}\|y(Q_hu)-y(u)\|\cdot\|z(Q_hu)-z(u)\|_1\\
\leq& Ch^2(\|\boldsymbol{q}(Q_hu)-\boldsymbol{q}(u)\|+\|\boldsymbol{p}(Q_hu)-\boldsymbol{p}(u)\|)\\
\leq& Ch^4+Ch^2\|\boldsymbol{q}(Q_hu)-\boldsymbol{q}(u)\|,
\end{aligned} \tag{3.20}$$

where we have used the embedding $\|v\|_{0,4}\leq C\|v\|_1$. Then using (3.16), (3.18)-(3.20) and the assumptions on ϕ and A, we find that

$$\|\boldsymbol{q}(Q_hu)-\boldsymbol{q}(u)\|\leq Ch^2. \tag{3.21}$$

Using (3.13), (3.17), (3.21) and the Poincare's inequality gives

$$\|z(Q_hu)-z(u)\|\leq C(\|\boldsymbol{q}(Q_hu)-\boldsymbol{q}(u)\|+\|\boldsymbol{p}(Q_hu)-\boldsymbol{p}(u)\|)\leq Ch^2. \tag{3.22}$$

Therefore Lemma 3.3 is derived by using (3.13)-(3.14) and (3.21)-(3.22). □

Let $(\boldsymbol{p}(u),y(u))$ be the solutions of (2.5)-(2.7) and $J(\cdot):L^2(\Omega)\to\mathbb{R}$ be a G-differential convex functional near the solution u which satisfies the following form:

$$J(u)=\frac{1}{2}\|\boldsymbol{p}-\boldsymbol{p}_d\|^2+\frac{1}{2}\|y-y_d\|^2+\frac{\nu}{2}\|u\|^2. \tag{3.23}$$

Then we can find that

$$\begin{aligned}
(J'(u),v)&=(\nu u+z,v), &(3.24)\\
(J'(u_h),v)&=(\nu u_h+z(u_h),v), &(3.25)\\
(J'(Q_hu),v)&=(\nu Q_hu+z(Q_hu),v). &(3.26)
\end{aligned}$$

In many applications, $J(\cdot)$ is uniform convex near the solution u. The convexity of $J(\cdot)$ is closely related to the second order sufficient conditions of the control problem, which are assumed in many studies on numerical methods of the problem. Then, there exists a constant $c>0$, independent of h, such that

$$(J'(Q_hu)-J'(u_h),Q_hu-u_h)\geq c\|Q_hu-u_h\|^2, \tag{3.27}$$

where u and u_h are solutions of (2.8)-(2.12) and (2.29)-(2.33) respectively, Q_hu is the orthogonal projection of u which is defined in (2.24). We shall assume the above inequality throughout this paper.

Lemma 3.4. *Let u be the solution of* (2.8)-(2.12) *and u_h be the solution of* (2.29)-(2.33)*, respectively. Assume that $\boldsymbol{p}(u_h)$, $\boldsymbol{q}(u_h)\in(H^1(\Omega))^2$ and $u\in H^1(\Omega)$. Then, we have*

$$\|Q_hu-u_h\|\le Ch^2. \tag{3.28}$$

Proof. We choose $\tilde{u}=u_h$ in (2.12) and $\tilde{u}_h=Q_hu$ in (2.33) to get the following two inequalities:

$$(\nu u+z,u_h-u)\ge 0, \tag{3.29}$$
$$(\nu u_h+z_h,Q_hu-u_h)\ge 0. \tag{3.30}$$

Note that $u_h-u=u_h-Q_hu+Q_hu-u$. Adding the two inequalities (3.29) and (3.30) gives

$$(\nu u_h+z_h-\nu u-z,Q_hu-u_h)+(\nu u+z,Q_hu-u)\ge 0. \tag{3.31}$$

Thus, by (3.27) and (3.31), we find that

$$\begin{aligned}
&c\|Q_hu-u_h\|^2\\
\le&(J'(Q_hu)-J'(u_h),Q_hu-u_h)\\
=&\nu(Q_hu-u_h,Q_hu-u_h)+(z(Q_hu)-z(u_h),Q_hu-u_h)\\
=&\nu(Q_hu-u,Q_hu-u_h)+\nu(u-u_h,Q_hu-u_h)+(z(Q_hu)-z(u_h),Q_hu-u_h)\\
\le&(z_h-z,Q_hu-u_h)+(\nu u+z,Q_hu-u)+(z(Q_hu)-z(u_h),Q_hu-u_h)\\
=&(z_h-z(u_h),Q_hu-u_h)+(\nu u+z,Q_hu-u)+(z(Q_hu)-z(u),Q_hu-u_h).
\end{aligned} \tag{3.32}$$

By Lemma 3.1 and Lemma 3.3, we find that

$$(z_h-z(u_h),Q_hu-u_h)\le Ch^4+\frac{c}{4}\|Q_hu-u_h\|^2, \tag{3.33}$$
$$(z(Q_hu)-z(u),Q_hu-u_h)\le Ch^4+\frac{c}{4}\|Q_hu-u_h\|^2. \tag{3.34}$$

From (2.13), we know that

$$\nu u+z=\max\{0,\bar{z}\}=\text{const}. \tag{3.35}$$

Thus, we have

$$(\nu u+z,Q_hu-u)=(\nu u+z)\int_\Omega(Q_hu-u)=0. \tag{3.36}$$

Combining (3.32), (3.33), (3.34) with (3.36), we derive (3.28).

Similar to Lemma 3.3, by Lemma 3.4, we can prove the following estimate.

Lemma 3.5. *Let $(\boldsymbol{p}(u_h),y(u_h),\boldsymbol{q}(u_h),z(u_h))$ and $(\boldsymbol{p}(Q_hu),y(Q_hu),\boldsymbol{q}(Q_hu),z(Q_hu))$ be the solutions of* (2.42)-(2.45) *with $\tilde{u}=u_h$ and $\tilde{u}=Q_hu$, respectively. Assume that all the conditions in Theorem* 3.4 *are valid. Then we have*

$$\|y(Q_hu)-y(u_h)\|+\|\boldsymbol{p}(Q_hu)-\boldsymbol{p}(u_h)\|\le Ch^2, \tag{3.37}$$
$$\|z(Q_hu)-z(u_h)\|+\|\boldsymbol{q}(Q_hu)-\boldsymbol{q}(u_h)\|\le Ch^2. \tag{3.38}$$

Combining Lemmas 3.1, 3.3 and 3.5, we can derive the following error estimate.

Lemma 3.6. *Let* $(\boldsymbol{p}, y, \boldsymbol{q}, z, u)$ *and* $(\boldsymbol{p}_h, y_h, \boldsymbol{q}_h, z_h, u_h)$ *be the solutions of* (2.8)-(2.12) *and* (2.29)-(2.33)*, respectively. Assume that all the conditions in Lemmas* 3.1*,* 3.3 *and* 3.5 *hold. Then we have*

$$\|y - y_h\| + \|z - z_h\| \leq Ch^2. \tag{3.39}$$

Lemma 3.7. *Assume that all the conditions in Lemma 3.4 are valid and* $u \in W^{1,\infty}(\Omega)$*. Let* u *and* u_h *be the solutions of* (2.8)-(2.12) *and* (2.29)-(2.33)*, respectively. Then we have*

$$\|u - u_h\|_{0,\infty} \leq Ch. \tag{3.40}$$

Proof. By (2.25) and the inverse inequality, we arrive at

$$\begin{aligned}\|u - u_h\|_{0,\infty} \leq & C(\|u - Q_h u\|_{0,\infty} + \|Q_h u - u_h\|_{0,\infty}) \\ \leq & C(h\|u\|_{1,\infty} + h^{-1}\|Q_h u - u_h\|).\end{aligned} \tag{3.41}$$

Gathering (3.41) and Lemma 3.4, we derive (3.40). □

Moreover, in order to improve the accuracy of the control approximation on a global scale, similar to the case in [20], we construct the following a postprocessing projection operator of the discrete co-state to the admissible set

$$\hat{u} = (\max\{0, \overline{z_h}\} - z_h)/\nu. \tag{3.42}$$

Now, we can prove the following global superconvergence result.

Theorem 3.1. *Assume that all the conditions in Lemma* 3.6 *hold. Let* u *be the solution of* (2.8)-(2.12) *and* $\hat{u}$ *be the function constructed in* (3.42)*. Then we have*

$$\|u - \hat{u}\| \leq Ch^2. \tag{3.43}$$

Proof. From (2.13) and (3.42), we arrive at

$$|u - \hat{u}| \leq C|z - z_h| + C|\bar{z} - \overline{z_h}|. \tag{3.44}$$

By (3.44) and Lemma 3.6, we have

$$\|u - \hat{u}\| \leq C\|z - z_h\| \leq Ch^2. \tag{3.45}$$

This completes the proof. □

4. L^∞-error estimates

In this section, we will give the L^∞-error estimates both for the control variable and the state, co-state variables. In the rest of the section, the control variable is approximated by piecewise linear functions.

Now, we are in the position of deriving the estimate for $\|P_h z - z_h(u)\|$, we need an a priori regularity estimate for the following auxiliary problems:

$$-\operatorname{div}(A\nabla\xi) + \Phi\xi = F_1, \qquad x \in \Omega, \ \xi|_{\partial\Omega} = 0, \tag{4.1}$$
$$-\operatorname{div}(A\nabla\zeta) + \phi'(y)\zeta = F_2, \quad x \in \Omega, \ \zeta|_{\partial\Omega} = 0, \tag{4.2}$$

where

$$\Phi = \begin{cases} \dfrac{\phi(y) - \phi(y_h(u))}{y - y_h(u)}, & y \neq y_h(u), \\ \phi'(y_h(u)), & y = y_h(u). \end{cases}$$

Lemma 4.1. [18] *Let ξ and ζ be the solutions for* (4.1) *and* (4.2)*, respectively. If Ω is convex, then*

$$\|\xi\|_{H^2(\Omega)} \leq C\|F_1\|_{L^2(\Omega)}, \tag{4.3}$$
$$\|\zeta\|_{H^2(\Omega)} \leq C\|F_2\|_{L^2(\Omega)}. \tag{4.4}$$

Then, we will give the following superconvergence results for the intermediate solutions which are very important for our following work.

Lemma 4.2. *Let $(\boldsymbol{p}, y, \boldsymbol{q}, z) \in (\boldsymbol{V} \times W)^2$ and $(\boldsymbol{p}_h(u), y_h(u), \boldsymbol{q}_h(u), z_h(u)) \in (\boldsymbol{V}_h \times W_h)^2$ be the solutions of* (2.42)-(2.45) *and* (2.46)-(2.49) *with $\tilde{u} = u$ respectively. If the solution satisfies*

$$\boldsymbol{p}, \ \boldsymbol{q} \in (H^2(\Omega))^2, \ y \in W^{2,\infty}(\Omega), \ z \in W^{1,\infty}(\Omega),$$

then we have

$$\|P_h y - y_h(u)\| \leq Ch^3, \tag{4.5}$$
$$\|P_h z - z_h(u)\| \leq Ch^3. \tag{4.6}$$

Proof. From equations (2.42)-(2.45) and (2.46)-(2.49), we can easily obtain the following error equations

$$(A^{-1}(\boldsymbol{p} - \boldsymbol{p}_h(u)), \boldsymbol{v}_h) - (y - y_h(u), \operatorname{div}\boldsymbol{v}_h) = 0, \tag{4.7}$$
$$(\operatorname{div}(\boldsymbol{p} - \boldsymbol{p}_h(u)), w_h) + (\phi(y) - \phi(y_h(u)), w_h) = 0, \tag{4.8}$$
$$(A^{-1}(\boldsymbol{q} - \boldsymbol{q}_h(u)), \boldsymbol{v}_h) - (z - z_h(u), \operatorname{div}\boldsymbol{v}_h) = -(\boldsymbol{p} - \boldsymbol{p}_h(u), \boldsymbol{v}_h), \tag{4.9}$$
$$(\operatorname{div}(\boldsymbol{q} - \boldsymbol{q}_h(u)), w_h) + (\phi'(y)z - \phi'(y_h(u))z_h(u), w_h) = (y - y_h(u), w_h), \tag{4.10}$$

for any $\boldsymbol{v}_h \in \boldsymbol{V}_h$ and $w_h \in W_h$. As a result of (2.18), we can rewrite (4.7)-(4.10) as

$$(A^{-1}(\boldsymbol{p}-\boldsymbol{p}_h(u)),\boldsymbol{v}_h)-(P_hy-y_h(u),\mathrm{div}\boldsymbol{v}_h)=0, \tag{4.11}$$

$$(\mathrm{div}(\boldsymbol{p}-\boldsymbol{p}_h(u)),w_h)+(\phi(y)-\phi(y_h(u)),w_h)=0, \tag{4.12}$$

$$(A^{-1}(\boldsymbol{q}-\boldsymbol{q}_h(u)),\boldsymbol{v}_h)-(P_hz-z_h(u),\mathrm{div}\boldsymbol{v}_h)=-(\boldsymbol{p}-\boldsymbol{p}_h(u),\boldsymbol{v}_h), \tag{4.13}$$

$$(\mathrm{div}(\boldsymbol{q}-\boldsymbol{q}_h(u)),w_h)+(\phi'(y)z-\phi'(y_h(u))z_h(u),w_h)=(P_hy-y_h(u),w_h), \tag{4.14}$$

for any $\boldsymbol{v}_h \in \boldsymbol{V}_h$ and $w_h \in W_h$.

For sake of simplicity, we now denote

$$\tau = P_hy-y_h(u), \qquad e=P_hz-z_h(u). \tag{4.15}$$

Then, we estimate (4.5) and (4.6) in Part I and Part II, respectively.

Part I. As we can see,

$$\|\tau\| = \sup_{\psi\in L^2(\Omega),\psi\neq 0}\frac{(\tau,\psi)}{\|\psi\|}, \tag{4.16}$$

we then need to bound (τ,ψ) for $\psi\in L^2(\Omega)$. Let $\xi\in H^2(\Omega)\cap H_0^1(\Omega)$ be the solution of (4.1). We can see from (2.20) and (4.11)

$$\begin{aligned}(\tau,F_1)&=(\tau,-\mathrm{div}(A\mathbf{grad}\xi))+(\tau,\Phi\xi)\\&=-(\tau,\mathrm{div}(\Pi_h(A\mathbf{grad}\xi)))+(\tau,\Phi\xi)\\&=-(A^{-1}(\boldsymbol{p}-\boldsymbol{p}_h(u)),\Pi_h(A\mathbf{grad}\xi))+(\tau,\Phi\xi).\end{aligned} \tag{4.17}$$

Note that

$$(\mathrm{div}(\boldsymbol{p}-\boldsymbol{p}_h(u)),\xi)+(A^{-1}(\boldsymbol{p}-\boldsymbol{p}_h(u)),A\mathbf{grad}\xi)=0. \tag{4.18}$$

If follows from (4.12), (4.17) and (4.18), we derive

$$\begin{aligned}(\tau,F_1)=&(A^{-1}(\boldsymbol{p}-\boldsymbol{p}_h(u)),A\mathbf{grad}\xi-\Pi_h(A\mathbf{grad}\xi))\\&+(\mathrm{div}(\boldsymbol{p}-\boldsymbol{p}_h(u)),\xi-P_h\xi)+(\Phi(y-P_hy),\xi)\\&+(\phi(y)-\phi(y_h(u)),\xi-P_h\xi).\end{aligned} \tag{4.19}$$

From Lemma 3.1, (2.19) and (2.21), we have

$$(A^{-1}(\boldsymbol{p}-\boldsymbol{p}_h(u)),A\mathbf{grad}\xi-\Pi_h(A\mathbf{grad}\xi))\le Ch^3\|\xi\|_2, \tag{4.20}$$

$$(\mathrm{div}(\boldsymbol{p}-\boldsymbol{p}_h(u)),\xi-P_h\xi)\le Ch^3\|\xi\|_2. \tag{4.21}$$

For the third term on the right hand side of (4.19), using (2.18), (2.19) and the assumption on ϕ, we get

$$\begin{aligned}(\Phi(y-P_hy),\xi)=&(\Phi(y-P_hy),\xi-P_h\xi)+(y-P_hy,(\Phi-\pi^c\Phi)P_h\xi)\\\le&Ch\|\xi\|_{1,\infty}\|y-P_hy\|\cdot\|\xi\|_1+Ch\|\xi\|_{2,\infty}\|y-P_hy\|\cdot\|\xi\|\\\le&Ch^3\|\xi\|_1,\end{aligned} \tag{4.22}$$

where π^c be the element average operator. By (2.19), Lemma 3.1 and the assumption on ϕ, we find that

$$\begin{aligned}&(\phi(y)-\phi(y_h(u)),\xi-P_h\xi)\\ \le& C\|\phi\|_{1,\infty}\|y-y_h(u)\|\cdot\|\xi-P_h\xi\|\le Ch^3\|\xi\|_1.\end{aligned}\tag{4.23}$$

Thus, (4.5) can be proved by using (4.16) and (4.19)-(4.23).

Part II. Since

$$\|e\|=\sup_{\psi\in L^2(\Omega),\psi\neq 0}\frac{(e,\psi)}{\|\psi\|},\tag{4.24}$$

we then need to bound (e,ψ) for $\psi\in L^2(\Omega)$. Let $\zeta\in H^2(\Omega)\cap H_0^1(\Omega)$ be the solution of (4.2). We can see from (2.20) and (4.13)

$$\begin{aligned}(e,F_2)=&(e,-\mathrm{div}(A\mathbf{grad}\zeta))+(e,\phi'(y)\zeta)\\ =&-(e,\mathrm{div}(\Pi_h(A\mathbf{grad}\zeta)))+(e,\phi'(y)\zeta)\\ =&-(A^{-1}(\boldsymbol{q}-\boldsymbol{q}_h(u)),\Pi_h(A\mathbf{grad}\zeta))+(e,\phi'(y)\zeta)\\ &-(\boldsymbol{p}-\boldsymbol{p}_h(u),\Pi_h(A\mathbf{grad}\zeta)).\end{aligned}\tag{4.25}$$

Note that

$$(\mathrm{div}(\boldsymbol{q}-\boldsymbol{q}_h(u)),\zeta)+(A^{-1}(\boldsymbol{q}-\boldsymbol{q}_h(u)),A\mathbf{grad}\zeta)=0.\tag{4.26}$$

Thus, it follows from (2.19), (2.21), (2.22), (4.14), (4.25) and (4.26), we derive

$$\begin{aligned}(e,F_2)=&(A^{-1}(\boldsymbol{q}-\boldsymbol{q}_h(u)),A\mathbf{grad}\zeta-\Pi_h(A\mathbf{grad}\zeta))\\ &+(\mathrm{div}(\boldsymbol{q}-\boldsymbol{q}_h(u)),\zeta-P_h\zeta)-(P_hy-y_h(u),P_h\zeta)\\ &+(\phi'(y)z-\phi'(y_h(u))z_h(u),\zeta-P_h\zeta)\\ &+(\phi'(y)(P_hz-z),\zeta)+(z_h(u)(\phi'(y_h(u))-\phi'(y)),\zeta)\\ &-(\boldsymbol{p}-\boldsymbol{p}_h(u),\Pi_h(A\mathbf{grad}\zeta))=:\sum_{i=1}^7 I_i.\end{aligned}\tag{4.27}$$

For I_1 and I_2, by Lemma 3.1, (2.19) and (2.21), we have

$$I_1\le C\|\boldsymbol{q}-\boldsymbol{q}_h(u)\|\cdot\|A\mathbf{grad}\zeta-\Pi_h(A\mathbf{grad}\zeta)\|\le Ch^3\|\zeta\|_2,\tag{4.28}$$

$$I_2\le C\|\mathrm{div}(\boldsymbol{q}-\boldsymbol{q}_h(u))\|\cdot\|\zeta-P_h\zeta\|\le Ch^3\|\zeta\|_2.\tag{4.29}$$

It follows from (4.5) that

$$I_3\le C\|P_hy-y_h(u)\|\cdot\|P_h\zeta\|\le Ch^3\|\zeta\|.\tag{4.30}$$

Note that

$$\phi'(y)z-\phi'(y_h(u))z_h(u)=z(\phi'(y)-\phi'(y_h(u)))+\phi'(y_h(u))(z-z_h(u)).\tag{4.31}$$

Then, by Lemma 3.1 and (2.19), and the assumption on ϕ, we find that

$$\begin{aligned} I_4 \leq & C\|z\|_{0,\infty}\|\phi\|_{2,\infty}\|y-y_h(u)\|\cdot\|\zeta-P_h\zeta\| \\ & + C\|\phi\|_{1,\infty}\|z-z_h(u)\|\cdot\|\zeta-P_h\zeta\| \leq Ch^3\|\zeta\|_1. \end{aligned} \tag{4.32}$$

As for I_5, by the assumption on ϕ, (2.18) and (2.19), we derive

$$\begin{aligned} I_5 = & (\phi'(y)(P_hz-z),\zeta-P_h\zeta) + (P_hz-z,(\phi'(y)-\pi^c(\phi'(y)))P_h\zeta) \\ \leq & C\|\phi\|_{1,\infty}\|z-P_hz\|\cdot\|\zeta-P_h\zeta\| + Ch\|\phi\|_{2,\infty}\|z-P_hz\|\cdot\|P_h\zeta\| \\ \leq & Ch^3\|\zeta\|_1. \end{aligned} \tag{4.33}$$

For I_6, by Lemma 3.1, (2.18), (2.19), (4.5), the embedding $\|v\|_{0,\infty} \leq c\|v\|_2$ and the assumption on ϕ, we obtain

$$\begin{aligned} I_6 = & (\phi'(y_h(u))-\phi'(y),(z_h(u)-z)\zeta) + (\phi'(y_h(u))-\phi'(P_hy),z\zeta) \\ & + (\phi''(y)(P_hy-y),z\zeta) + \left(\frac{1}{2}\phi'''(y+\theta(P_hy-y))(P_hy-y)^2,z\zeta\right) \\ = & (\phi'(y_h(u))-\phi'(y),(z_h(u)-z)\zeta) + (\phi'(y_h(u))-\phi'(P_hy),z\zeta) \\ & + (\phi''(y)(P_hy-y),z\zeta-P_h(z\zeta)) + (P_hy-y,(\phi''(y)-\pi^c(\phi''(y)))P_h(z\zeta)) \\ & + \frac{1}{2}(\phi'''(y+\theta(P_hy-y))(P_hy-y)^2,z\zeta) \\ \leq & C\|\phi\|_{2,\infty}\|y-y_h(u)\|\cdot\|z-z_h(u)\|\cdot\|\zeta\|_{0,\infty} + C\|\phi\|_{2,\infty}\|P_hy-y_h(u)\|\cdot\|z\|\cdot\|\zeta\|_{0,\infty} \\ & + Ch\|\phi\|_{2,\infty}\|y-P_hy\|\cdot\|z\|_{1,\infty}\|\zeta\|_1 + Ch\|z\|_{0,\infty}\|\phi\|_3\|y-P_hy\|\cdot\|\zeta\|_{0,\infty} \\ & + C\|\phi\|_3\|y-P_hy\|^2_{0,\infty}\|z\|_{0,\infty}\|\zeta\| \leq Ch^3\|\zeta\|_2, \end{aligned} \tag{4.34}$$

where $0 \leq \theta \leq 1$. Finally, for I_7, from Lemma 3.1, (2.21), (2.22), (4.5) and (4.11), we have

$$\begin{aligned} I_7 = & (\boldsymbol{p}-\boldsymbol{p}_h(u),A\mathbf{grad}\zeta-\Pi_h(A\mathbf{grad}\zeta)) - (A^{-1}(\boldsymbol{p}-\boldsymbol{p}_h(u)),A^2\mathbf{grad}\zeta) \\ = & (\boldsymbol{p}-\boldsymbol{p}_h(u),A\mathbf{grad}\zeta-\Pi_h(A\mathbf{grad}\zeta)) - (A^{-1}(\boldsymbol{p}-\boldsymbol{p}_h(u)),A^2\mathbf{grad}\zeta-\Pi_h(A^2\mathbf{grad}\zeta)) \\ & - (P_hy-y_h(u),\mathrm{div}(\Pi_h(A^2\mathbf{grad}\zeta))) \leq Ch^3\|\zeta\|_2. \end{aligned} \tag{4.35}$$

Substituting the estimates for I_j, $1 \leq j \leq 7$ into (4.27), we derive (4.6) by using (4.24). □

Let $(\boldsymbol{p}_h(P_hu), y_h(P_hu), \boldsymbol{q}_h(P_hu), z_h(P_hu))$ and $(\boldsymbol{p}_h(u), y_h(u), \boldsymbol{q}_h(u), z_h(u))$ be the solutions of (2.46)-(2.49) with $\tilde{u} = P_hu$ and $\tilde{u} = u$, respectively. We can get the following error equations

$$(A^{-1}(\boldsymbol{p}_h(P_hu)-\boldsymbol{p}_h(u)),\boldsymbol{v}_h) - (y_h(P_hu)-y_h(u),\mathrm{div}\boldsymbol{v}_h) = 0, \tag{4.36}$$

$$(\mathrm{div}(\boldsymbol{p}_h(P_hu)-\boldsymbol{p}_h(u)),w_h) + (\phi(y_h(P_hu))-\phi(y_h(u)),w_h) = 0, \tag{4.37}$$

$$\begin{aligned} & (A^{-1}(\boldsymbol{q}_h(P_hu)-\boldsymbol{q}_h(u)),\boldsymbol{v}_h) - (z_h(P_hu)-z_h(u),\mathrm{div}\boldsymbol{v}_h) \\ & \qquad = -(\boldsymbol{p}_h(P_hu)-\boldsymbol{p}_h(u),\boldsymbol{v}_h), \end{aligned} \tag{4.38}$$

$$\begin{aligned} & (\mathrm{div}(\boldsymbol{q}_h(P_hu)-\boldsymbol{q}_h(u)),w_h) + (\phi'(y_h(P_hu))z_h(P_hu)-\phi'(y_h(u))z_h(u),w_h) \\ & \qquad = (y_h(P_hu)-y_h(u),w_h), \end{aligned} \tag{4.39}$$

for any $\boldsymbol{v}_h \in \boldsymbol{V}_h$ and $w_h \in W_h$.

Using the assumptions on ϕ and A, we get

Lemma 4.3. *Let* $(\boldsymbol{p}_h(P_h u), y_h(P_h u), \boldsymbol{q}_h(P_h u), z_h(P_h u))$ *and* $(\boldsymbol{p}_h(u), y_h(u), \boldsymbol{q}_h(u), z_h(u))$ *be the solutions of* (2.46)-(2.49) *with* $\tilde{u} = P_h u$ *and* $\tilde{u} = u$*, respectively. Then we have*

$$\|y_h(u) - y_h(P_h u)\| + \|\boldsymbol{p}_h(u) - \boldsymbol{p}_h(P_h u)\| = 0, \tag{4.40}$$

$$\|z_h(u) - z_h(P_h u)\| + \|\boldsymbol{q}_h(u) - \boldsymbol{q}_h(P_h u)\| = 0. \tag{4.41}$$

We assume that we have a sequence of uniform convex functional $J_h(\cdot) : L^2(\Omega) \to \mathbb{R}$:

$$J_h(u) = \frac{1}{2}\|\boldsymbol{p}_h(u) - \boldsymbol{p}_d\|^2 + \frac{1}{2}\|y_h(u) - y_d\|^2 + \frac{\nu}{2}\|u\|^2. \tag{4.42}$$

It is can be shown that

$$(J_h'(u), v) = (\nu u + z_h(u), v), \tag{4.43}$$

$$(J_h'(u_h), v) = (\nu u_h + z_h, v), \tag{4.44}$$

$$(J_h'(P_h u), v) = (\nu P_h u + z_h(P_h u), v). \tag{4.45}$$

Similar to (3.27), there exists a constant $c > 0$ satisfying

$$(J_h'(P_h u) - J_h'(u_h), P_h u - u_h) \geq c\|P_h u - u_h\|^2. \tag{4.46}$$

In the following, we will give the L^∞-error estimate for the control variable.

Theorem 4.1. *Let u and u_h be the solutions of* (2.8)-(2.12) *and* (2.29)-(2.33)*, respectively. If $u \in W^{2,\infty}(\Omega)$, then we have*

$$\|u - u_h\|_{0,\infty} \leq Ch^2. \tag{4.47}$$

Proof. Similar to (3.32), from (2.13), (2.18) and (4.46), we have

$$\begin{aligned}
&c\|P_h u - u_h\|^2 \\
\leq&(z_h(u) - z, P_h u - u_h) + (\nu u + z, P_h u - u) + (z_h(P_h u) - z_h(u), P_h u - u_h) \\
=&(z_h(u) - P_h z, P_h u - u_h) + (z_h(P_h u) - z_h(u), P_h u - u_h).
\end{aligned} \tag{4.48}$$

It follows from Lemma 4.2 and Lemma 4.3 that

$$\|P_h u - u_h\| \leq Ch^3. \tag{4.49}$$

By (2.19) and the inverse inequality, we arrive at

$$\begin{aligned}
\|u - u_h\|_{0,\infty} \leq& C(\|u - P_h u\|_{0,\infty} + \|P_h u - u_h\|_{0,\infty}) \\
\leq& C(h^2\|u\|_{2,\infty} + h^{-1}\|P_h u - u_h\|).
\end{aligned} \tag{4.50}$$

Gathering (4.50) and (4.49), we derive (4.47). ∎

From Eqs. (2.8)-(2.12) and (2.29)-(2.33), we can use (2.18) obtain the following error equations

$$(A^{-1}(\boldsymbol{p}-\boldsymbol{p}_h),\boldsymbol{v}_h)-(P_hy-y_h,\mathrm{div}\boldsymbol{v}_h)=0, \tag{4.51}$$
$$(\mathrm{div}(\boldsymbol{p}-\boldsymbol{p}_h),w_h)+(\phi(y)-\phi(y_h),w_h)=(P_hu-u_h,w_h), \tag{4.52}$$
$$(A^{-1}(\boldsymbol{q}-\boldsymbol{q}_h),\boldsymbol{v}_h)-(P_hz-z_h,\mathrm{div}\boldsymbol{v}_h)=-(\boldsymbol{p}-\boldsymbol{p}_h,\boldsymbol{v}_h), \tag{4.53}$$
$$(\mathrm{div}(\boldsymbol{q}-\boldsymbol{q}_h),w_h)+(\phi'(y)z-\phi'(y_h)z_h,w_h)=(P_hy-y_h,w_h), \tag{4.54}$$

for any $\boldsymbol{v}_h\in\boldsymbol{V}_h$ and $w_h\in W_h$.

Similar to Lemma 4.2, we can obtain

Lemma 4.4. *Let* (y,z) *and* (y_h,z_h) *be the solutions of* (2.8)-(2.12) *and* (2.29)-(2.33) *respectively. Then we have*

$$\|P_hy-y_h\|+\|P_hz-z_h\|\le Ch^3. \tag{4.55}$$

By modifying the proof of Theorem 3.3 in [19], we can derive

Lemma 4.5. *Let* $(\boldsymbol{p},\boldsymbol{q})$ *and* $(\boldsymbol{p}_h,\boldsymbol{q}_h)$ *be the solutions of* (2.8)-(2.12) *and* (2.29)-(2.33) *respectively. Then we have*

$$\|\Pi_h\boldsymbol{p}-\boldsymbol{p}_h\|_{0,\infty}+\|\Pi_h\boldsymbol{q}-\boldsymbol{q}_h\|_{0,\infty}\le Ch^{\frac{3}{2}}|\ln h|^{\frac{1}{2}}. \tag{4.56}$$

Now, combining (2.19), (2.21), Lemmas 4.4 and 4.5 with the inverse inequality, we give the following L^∞-error estimates for the state and the co-state variables.

Theorem 4.2. *Let* $(\boldsymbol{p},y,\boldsymbol{q},z)\in(\boldsymbol{V}\times W)^2$ *and* $(\boldsymbol{p}_h,y_h,\boldsymbol{q}_h,z_h)\in(\boldsymbol{V}_h\times W_h)^2$ *be the solutions of* (2.8)-(2.12) *and* (2.29)-(2.33) *respectively. Then we have*

$$\|y-y_h\|_{0,\infty}+\|z-z_h\|_{0,\infty}\le Ch^2, \tag{4.57}$$
$$\|\boldsymbol{p}-\boldsymbol{p}_h\|_{0,\infty}+\|\boldsymbol{q}-\boldsymbol{q}_h\|_{0,\infty}\le Ch^{\frac{3}{2}}|\ln h|^{\frac{1}{2}}. \tag{4.58}$$

5. Numerical experiments

In this section, we present below two examples to illustrate the theoretical results. The optimization problems were solved numerically by projected gradient methods, with codes developed based on AFEPack [16]. The discretization was already described in previous sections: the control function u was discretized by piecewise constant functions or piecewise linear functions, whereas the state $(y,\boldsymbol{p})$ and the co-state $(z,\boldsymbol{q})$ were approximated by the order $k=1$ Raviart-Thomas mixed finite element functions. In our examples, we choose the domain $\Omega=[0,1]\times[0,1]$, $\nu=1$ and $A=I$.

Example 1. Consider the following two-dimensional elliptic optimal control problem

$$\min_{u\in U_{ad}}\left\{\frac{1}{2}\|\boldsymbol{p}-\boldsymbol{p}_d\|^2+\frac{1}{2}\|y-y_d\|^2+\frac{1}{2}\|u\|^2\right\} \tag{5.1}$$

Table 1: The errors of Example 1 when control was approximated by piecewise constant functions.

Resolution	$\|\|u-u_h\|\|$	$\|\|u-u_h\|\|_{0,\infty}$	$\|\|Q_hu-u_h\|\|$	$\|\|u-\hat{u}\|\|$
16×16	6.5135e-02	1.7887e-01	1.2111e-04	4.9681e-03
32×32	3.2685e-02	9.0705e-02	2.9266e-05	1.2439e-03
64×64	1.6357e-02	4.5511e-02	7.2654e-06	3.1111e-04
128×128	8.1806e-03	2.2776e-02	1.8189e-06	7.7787e-05

Table 2: The errors of Example 1 when control was approximated by piecewise linear functions.

Resolution	$\|\|u-u_h\|\|_{0,\infty}$	$\|\|y-y_h\|\|_{0,\infty}$	$\|\|z-z_h\|\|_{0,\infty}$	$\|\|\boldsymbol{p}-\boldsymbol{p}_h\|\|_{0,\infty}$	$\|\|\boldsymbol{q}-\boldsymbol{q}_h\|\|_{0,\infty}$
16×16	1.2808e-02	3.1968e-03	1.2808e-02	8.8341e-03	7.1089e-02
32×32	3.2016e-03	7.9961e-04	3.2016e-03	2.2191e-03	1.7715e-02
64×64	8.0015e-04	1.9992e-04	8.0015e-04	5.5551e-04	4.4430e-03
128×128	1.9999e-04	4.9984e-05	1.9999e-04	1.3892e-04	1.1115e-03

subject to the state equation

$$\mathrm{div}\boldsymbol{p}+y^3=f+u,\ \ \boldsymbol{p}=-\mathbf{grad}y, \tag{5.2}$$

where

$$y=\sin(\pi x_1)\sin(\pi x_2),\quad z=\sin(2\pi x_1)\sin(2\pi x_2), \tag{5.3a}$$
$$u=\max(0,\bar{z})-z,\quad f=2\pi^2y+y^3-u,\quad y_d=y-8\pi^2z-3y^2z, \tag{5.3b}$$
$$\boldsymbol{p}_d=-\begin{pmatrix}\pi\cos(\pi x_1)\sin(\pi x_2)\\ \pi\sin(\pi x_1)\cos(\pi x_2)\end{pmatrix}. \tag{5.3c}$$

In the numerical implementation, we choose the solution u which satisfies $\int_\Omega u dx=0$. In the figures, $\hat{u}$ is denoted by u_{proj}. In Table 1, the errors $||u-u_h||$, $||u-u_h||_{0,\infty}$, $||Q_hu-u_h||$ and $||u-\hat{u}||$ obtained on a sequence of uniformly refined meshes are shown when control was approximated by piecewise constant functions. Table 2 displays the errors $||u-u_h||_{0,\infty}$, $||y-y_h||_{0,\infty}$, $||z-z_h||_{0,\infty}$, $||\boldsymbol{p}-\boldsymbol{p}_h||_{0,\infty}$ and $||\boldsymbol{q}-\boldsymbol{q}_h||_{0,\infty}$ when control was approximated by piecewise linear functions. In Figs. 1 and 2, the profile of the numerical solution of u on the 64×64 mesh grid is plotted. Moreover, in Figs. 3 and 4, we show the convergence orders by slopes.

As we can see from the figures that the approximation in the L^∞-norm is of order h^2 for the piecewise linear functions. In contrast to this, the approximations in the L^2-norm and L^∞-norm are only of first order for piecewise constant functions. We also obtain a quadratic approximation rate for $||Q_hu-u_h||$ and $||u-\hat{u}||$. Although the error for piecewise linear approximation is essentially less than that for piecewise constant approximation as provided in Tables 1 and 2, we should keep in mind that we have triplicate the number of degree of freedoms.

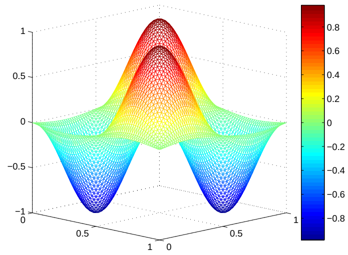

Figure 1: The profile of the numerical solution of Example 1 on 64×64 triangle mesh when control was approximated by piecewise constant functions.

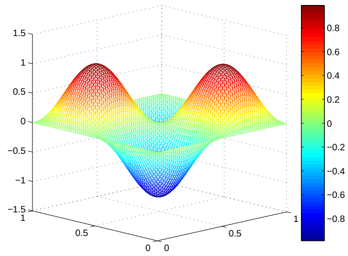

Figure 2: The profile of the numerical solution of Example 1 on 64×64 triangle mesh when control was approximated by piecewise linear functions.

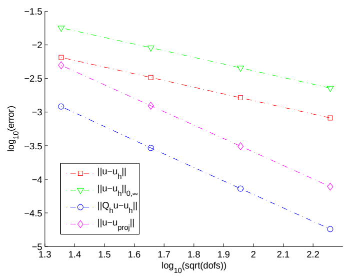

Figure 3: Convergence orders of $u-u_h$, Q_hu-u_h and $u-\hat{u}$ in different norms (piecewise constant).

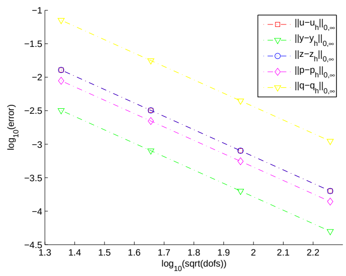

Figure 4: Convergence orders of $u-u_h$, $y-y_h$, $z-z_h$, $\boldsymbol{p}-\boldsymbol{p}_h$ and $\boldsymbol{q}-\boldsymbol{q}_h$ in L^∞-norm (piecewise linear).

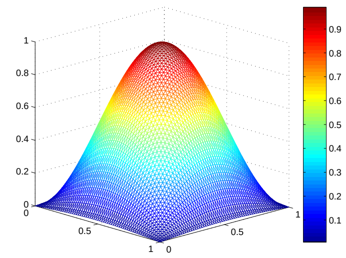

Figure 5: The profile of the numerical solution of Example 2 on 64×64 triangle mesh when control was approximated by piecewise constant functions.

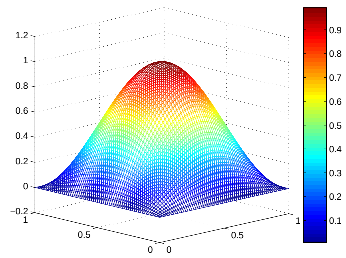

Figure 6: The profile of the numerical solution of Example 2 on 64×64 triangle mesh when control was approximated by piecewise linear functions.

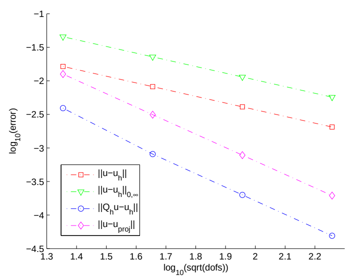

Figure 7: Convergence orders of $u-u_h$, Q_hu-u_h and $u-\hat{u}$ in different norms (piecewise constant).

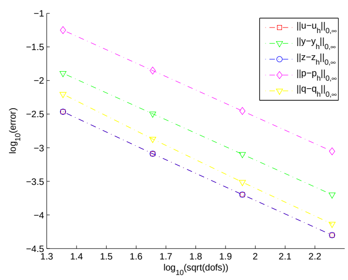

Figure 8: Convergence orders of $u-u_h$, $y-y_h$, $z-z_h$, $\boldsymbol{p}-\boldsymbol{p}_h$ and $\boldsymbol{q}-\boldsymbol{q}_h$ in L^∞-norm (piecewise linear).

Table 3: The errors of Example 2 when control was approximated by piecewise constant functions.

Resolution	$\|\|u-u_h\|\|$	$\|\|u-u_h\|\|_{0,\infty}$	$\|\|Q_hu-u_h\|\|$	$\|\|u-\hat{u}\|\|$
16×16	3.2685e-02	9.0622e-02	1.2387e-04	1.2632e-03
32×32	1.6357e-02	4.5503e-02	2.5658e-05	3.1297e-04
64×64	8.1806e-03	2.2774e-02	6.3101e-06	7.8077e-05
128×128	4.0905e-03	1.1390e-02	1.5538e-06	1.9507e-05

Table 4: The errors of Example 2 when control was approximated by piecewise linear functions.

Resolution	$\|\|u-u_h\|\|_{0,\infty}$	$\|\|y-y_h\|\|_{0,\infty}$	$\|\|z-z_h\|\|_{0,\infty}$	$\|\|\boldsymbol{p}-\boldsymbol{p}_h\|\|_{0,\infty}$	$\|\|\boldsymbol{q}-\boldsymbol{q}_h\|\|_{0,\infty}$
16×16	3.4336e-03	1.2835e-02	3.4336e-03	7.0742e-02	1.2499e-02
32×32	8.1432e-04	3.2016e-03	8.1432e-04	1.7670e-02	2.6833e-03
64×64	2.0085e-04	7.9991e-04	2.0085e-04	4.4383e-03	6.1368e-04
128×128	5.0042e-05	1.9998e-04	5.0042e-05	1.1110e-03	1.4619e-04

Example 2. In this example, we consider the optimal control problem (5.1)-(5.2) with the control u satisfying $\int_\Omega u dx > 0$. The data are as follows:

$$y=\sin(2\pi x_1)\sin(2\pi x_2), \quad z=-\sin(\pi x_1)\sin(\pi x_2), \tag{5.4a}$$

$$u=\max(0,\bar{z})-z, \quad f=8\pi^2 y+y^3-u, \quad y_d=y-2\pi^2 z-3y^2 z, \tag{5.4b}$$

$$\boldsymbol{p}_d=-\begin{pmatrix} 2\pi\cos(2\pi x_1)\sin(2\pi x_2) \\ 2\pi\sin(2\pi x_1)\cos(2\pi x_2) \end{pmatrix}. \tag{5.4c}$$

The profile of the numerical solution of u is presented in Figs. 5 and 6. From the error data on the uniform refined meshes, as listed in Tables 3 and 4, it is easy to see that the numerical results are consistent with our theoretical analysis. We also show the convergence orders by slopes in Figs. 7 and 8.

6. Conclusion and future works

In this paper, we discussed the order $k=1$ Raviart-Thomas mixed finite element methods for the semilinear elliptic optimal control problem (1.1)-(1.4). We derived some superconvergence results of the mixed finite element methods for the control problem when the control was approximated by piecewise constant functions. Moreover, we derived L^∞-error estimates for both the control variable and the state variables when the control was discretized by piecewise linear functions. In our future work, we will investigate the superconvergence of mixed finite element methods for optimal control problems governed by nonlinear parabolic equations.

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