Journal of Computational Mathematics
Vol.30, No.4, 2012, 418–432.

http://www.global-sci.org/jcm
doi:10.4208/jcm.1111-m3457

SOLVING THE BACKWARD HEAT CONDUCTION PROBLEM BY DATA FITTING WITH MULTIPLE REGULARIZING PARAMETERS*

Qun Chen
Department of Mathematics, Southeast University, Nanjing 210096, China
College of Mathematics and Statistics, Nanjing University of Information Science and Technology, Nanjing 210044, China
Email: chenqun98@163.com
Jijun Liu
Department of Mathematics, Southeast University, Nanjing 210096, China
Email: jjliu@seu.edu.cn

Abstract

We propose a new reconstruction scheme for the backward heat conduction problem. By using the eigenfunction expansions, this ill-posed problem is solved by an optimization problem, which is essentially a regularizing scheme for the noisy input data with both the number of truncation terms and the approximation accuracy for the final data as multiple regularizing parameters. The convergence rate analysis depending on the strategy of choosing regularizing parameters as well as the computational accuracy of eigenfunctions is given. Numerical implementations are presented to show the validity of this new scheme.

Mathematics subject classification: 34L16, 35K05, 41A25, 65F22.
Key words: Inverse problem, Data fitting, Regularization, Convergence rate, Numerics.

1. Introduction

For a bounded domain $\Omega \subset \mathbb{R}^N (N = 1, 2, 3)$, consider the heat conduction problem

$$\begin{cases} \frac{\partial u}{\partial t} = \nabla \cdot (a(x)\nabla u), & x \in \Omega, t > 0 \\ u(x,t) = 0, & x \in \partial\Omega, t > 0 \\ u(x,0) = u_0(x), & x \in \Omega. \end{cases} \tag{1.1}$$

For given initial data $u_0(x)$, this forward problem is well-posed(Chapter 3, Theorem 3.2, [13]), which defines a map $\mathcal{G} : u_0(\cdot) \in L^2(\Omega) \mapsto u(\cdot, T) \in H_0^1(\Omega)$.

Now assume that $u_0(x)$ is unknown, while the final data is given by $u(x,T) = f(x), x \in \Omega$. The backward problem is to solve $u(x,t)$ for $t \in [0,T)$ from given $f(x)$ or its measurement data $f^\delta(x)$ satisfying $\|f^\delta - f\|_{L^2(\Omega)} \le \delta$ for some known error level $\delta > 0$. It is well-known that this problem is ill-posed due to the irreversibility of heat conduction along time direction.

For this ill-posed problem with wide engineering background [19, 20], many regularizing schemes have been researched thoroughly, which focus on the construction of the approximate solution $u^\delta(x,t)$ from $f^\delta(x)$ and the convergence rate analysis on $\|u^\delta(\cdot,t) - u(\cdot,t)\|$ as $\delta \to 0$. Of course, these two issues depend on the regularizing scheme. One of the well-known scheme

* Received July 4, 2010 / Revised version received November 14, 2011 / Accepted December 14, 2011 / Published online July 6, 2012 /

is the so-called quasi-reversibility method [3], which firstly constructs the regularizing solution $u_0^\delta(x)$ for the initial data and then gets $u^\delta(x,t)$ for $t \in (0,T)$ by solving the direct problem (1.1). The convergence of such kinds of schemes can be established in terms of the convergence of initial data $u_0^\delta(x)$, see [4–6, 9]. For solving $u^\delta(x,t)$ for $t \in (0,T)$ directly from $f^\delta(x)$ with the Hölder stability of order $\frac{t}{T}$, the readers are refereed to [1, 17, 18, 23]. The other work for backward heat problem can be found in [2, 15, 16, 21].

Recently, some attempts to construct a regularizing solution with explicit expression have received much attention. The advantage of this new idea is that the well-posedness of the regularizing problem is guaranteed automatically, provided that the noisy input data be modified appropriately. Then the numerical computation of the regularizing solution for all $t \in [0,T)$ is much easy, for example, see [4, 7, 14, 22] for the mollification method. We call such kind of scheme as data regularization.

In this paper, we propose a new regularizing scheme along this direction. By expanding the noisy data $f^\delta(x)$ in terms of the base functions $\{\varphi_k(x,T) : k \in \mathbb{N}\}$ solved from the heat conduction process, the regularizing data for the noisy measurement are constructed using the finite approximate terms expansion, where both the number of expansion terms and the approximate accuracy are considered as the regularizing parameters simultaneously. Then the regularizing solution $u^\delta(x,t)$ for all $t \in [0,T)$ can be constructed from the approximate final data explicitly. In this regularizing scheme for backward heat problem, all the ill-posedness is concentrated on the final data fitting process. Such a scheme is essentially a regularizing technique for the input data. We analyze the convergence of this new scheme and give some numerical implementations. It is interesting that our regularizing scheme provides the convergence rate of $\|u^\delta(\cdot,t) - u(\cdot,t)\|$ decreasing by the factor $e^{-\lambda_1 t}$ for fixed error level $\delta > 0$, which is physically reasonable from the smoothing property of direct heat conduction process, where $\lambda_1 > 0$ is the minimum eigenvalue of the operator $-\nabla \cdot (a(x)\nabla)$.

We would like to emphasize the difference between our data fitting technique and the classical TSVD method to deal with the linear ill-posed problems. For our problem, $u(x,t)$ for $t \in [0,T)$ satisfies a linear integral equation of the first kind, so the TSVD method can be used to solve this equation, where Tikhonov regularization can be combined together to determine the truncation term from the noise level. In this scheme, the regularization technique is applied at each time $t \in [0,T)$, and therefore the regularization equation should be solved for every time t. However, in our data fitting scheme, we only regularize the final measurement data $u^\delta(x,T)$ by its base function expansion, with both the truncation term and the approximate accuracy as regularizing parameters. Then the approximate solution for any $t \in [0,T)$ can be expressed explicitly using the spatial base function of elliptic operator. In other words, we extract the ill-posedness of the problem from the original parabolic system with the help of the eigensystem of elliptic operator $-\nabla \cdot (a(x)\nabla u)$. Therefore, the novelty of the proposed scheme in this paper compared with the classical TSVD method is that we can decrease the amount of computations by solving the regularizing equation only one times at $t = T$ and then get the regularizing solution for all $t \in [0,T)$ explicitly with convergence rate estimate. Moreover, we also analyze the influence of the computational error for the eigensystem and give an explicit error estimate.

This paper is organized as follows. In Section 2, we construct the regularizing solution explicitly. Then in Section 3, we give the convergence analysis on the regularizing solution using the exact eigenfunction expansions. In Section 4, we consider the convergence for the noisy eigensystem, noticing that both the eigenfunctions and the eigenvalues must be computed numerically for general heat conduction system. In this case, the error η in computing the eigenfunctions has essential effects on the regularizing solution. We show that the accuracy of eigensystem should match the noisy level of input data in the sense $\eta = O(\delta)$ in such a practical situation, explaining the optimal balance between the amount of computation for eigensystem and the convergence rate of regularizing solution. Finally we present some numerical results in Section 5 to show the validity of our inversion scheme.

2. The Construction of Regularizing Solution

Denote by $\{\lambda_n, \varphi_n^0(x) : n \in \mathbb{N}\}$ the eigensystem of the operator $\mathcal{A}[\diamond] := -\nabla \cdot (a(x)\nabla\diamond)$, acting on $\mathcal{D}(\mathcal{A}) := \{\psi(x) : \psi \in H^2(\Omega), \psi(x)|_{\partial\Omega} = 0\}$. That is, $\varphi_n^0(x)$ solves

$$\begin{cases} -\nabla \cdot (a(x)\nabla\varphi_n^0(x)) = \lambda_n \varphi_n^0(x), & x \in \Omega \\ \varphi_n^0(x) = 0, & x \in \partial\Omega. \end{cases} \tag{2.1}$$

It is easy to know that $\{\varphi_n^0(x), n \in \mathbb{N}\}$ constitutes the base of $L^2(\Omega)$ and $0 < \lambda_1 < \cdots < \lambda_n \cdots \to +\infty$ (Chapter 2, Theorem 4.1, [13]). We assume that $\{\varphi_n^0(x), n \in \mathbb{N}\}$ is orthogonal with $\|\varphi_n^0\|_{L^2(\Omega)} = 1$ and introduce the functions

$$\varphi_n(x,t) := e^{-\lambda_n t}\varphi_n^0(x), \tag{2.2}$$

which obviously satisfies

$$\begin{cases} \frac{\partial\varphi_n(x,t)}{\partial t} = \nabla \cdot (a(x)\nabla\varphi_n(x,t)), & x \in \Omega, t > 0 \\ \varphi_n(x,t) = 0, & x \in \partial\Omega, t > 0 \\ \varphi_n(x,0) = \varphi_n^0(x), & x \in \Omega. \end{cases} \tag{2.3}$$

Lemma 2.1. $\{\varphi_n(\cdot,t) : n \in \mathbb{N}\}$ *forms the orthogonal base of* $L^2(\Omega)$ *with* $\|\varphi_n(\cdot,t)\| = e^{-\lambda_n t}$ $(n \in \mathbb{N})$ *for any fixed* $t \in [0,T]$.

Now let us expand the initial temperature distribution as

$$u_0(x) = \sum_{m=1}^{\infty} c_m \varphi_m^0(x), \quad x \in \Omega \tag{2.4}$$

with $c_n = \int_\Omega u_0(x)\varphi_n^0(x)dx, n \in \mathbb{N}$. So the exact solution of (1.1) can be expressed as

$$u(x,t) = \sum_{m=1}^{\infty} c_m \varphi_m(x,t). \tag{2.5}$$

Especially, the final value has the expansion

$$f(x) = \sum_{m=1}^{\infty} c_m \varphi_m(x,T). \tag{2.6}$$

In practice, only the noisy data $f^\delta(x)$ of $f(x)$ is given. On the other hand, we can only compute finite terms of the series in (2.6). Therefore, c_m are determined by the following approximation to (2.6):

$$\sum_{m=1}^{M} c_m^\delta \varphi_m(x,T) = f^\delta(x), \tag{2.7}$$

where the truncation term M is unknown in advance which will affect the inversion result. So the solution to (2.7) for determining both M and $\{c_m : m = 1, \cdots, M\}$ is ill-posed, the regularization method should be applied to solve (2.7).

To this end, denote by $C_M^{\varepsilon,\delta} := \{c_m^{\varepsilon,\delta} : m = 1, 2, \cdots, M\} \in \mathbb{R}^M$ the minimum norm solution to the equation (2.7) with discrepancy ε. That is, $C_M^{\varepsilon,\delta}$ satisfies

$$\left\| \sum_{m=1}^{M} c_m^{\varepsilon,\delta} \varphi_m(\cdot, T) - f^\delta(\cdot) \right\|_{L^2(\Omega)} \le \varepsilon, \tag{2.8}$$

$$\|C_M^{\varepsilon,\delta}\|_{\mathbb{R}^M} = \inf \left\{ \|C_M^\delta\|_{\mathbb{R}^M} : \| \sum_{m=1}^{M} c_m^\delta \varphi_m(\cdot, T) - f^\delta(\cdot)\|_{L^2(\Omega)} \le \varepsilon \right\}, \tag{2.9}$$

where $C_M^\delta := \{c_m^\delta : m = 1, 2, \cdots, M\}$. In the following we will choose $\varepsilon = \delta$. The positive integer $M := M(\delta)$ as the regularizing parameter will be specified later.

Define the operator $\mathcal{K} : \mathbb{R}^M \to L^2(\Omega)$ by

$$(\mathcal{K}C_M^\delta)(x) := \sum_{m=1}^{M} c_m^\delta \varphi_m(x, T), \tag{2.10}$$

with its adjoint operator under the dual system $\langle \mathbb{R}^M, \mathbb{R}^M \rangle$ and $\langle L^2(\Omega), L^2(\Omega) \rangle$:

$$\mathcal{K}^* g = \left(\int_\Omega \varphi_1(x, T) g(x) dx, \cdots, \int_\Omega \varphi_M(x, T) g(x) dx \right)^T. \tag{2.11}$$

The minimum norm solution $C_M^{\delta,\delta}$ can be solved by Tikhonov regularization with Morozov principle [10], i.e., $C_M^{\delta,\delta}$ satisfies the following equation

$$(\alpha(\delta) I + \mathcal{K}^*\mathcal{K}) C_M^{\delta,\delta} = \mathcal{K}^* f^\delta(x), \tag{2.12}$$

where the regularizing parameter $\alpha = \alpha(\delta)$ is determined from the implicit system

$$\begin{cases} (\alpha I + \mathcal{K}^*\mathcal{K}) C_M^{\alpha,\delta} = \mathcal{K}^* f^\delta \\ \|\mathcal{K} g_M^{\alpha,\delta} - f^\delta\| = \delta, \end{cases}$$

which can be solved numerically by classical Newton method [10] or recently developed model function method [11, 24]. We remark that the regularizing scheme (2.12) for solving the extremely ill-posed problem (2.7) is valid from the standard theory of Tikhonov regularization in finite dimensional space [10].

Now we construct

$$F_M^{\delta,\delta}(x, T) := \sum_{m=1}^{M} c_m^{\delta,\delta} \varphi_m(x, T) \tag{2.13}$$

and consider the problem

$$\begin{cases} \frac{\partial u_M^{\delta,\delta}(x,t)}{\partial t} = \nabla \cdot (a(x) \nabla u_M^{\delta,\delta}(x, t)), & x \in \Omega, t \in (0, T) \\ u_M^{\delta,\delta}(x, t) = 0, & x \in \partial\Omega, t > 0 \\ u_M^{\delta,\delta}(x, T) = F_M^{\delta,\delta}(x, T), & x \in \Omega. \end{cases} \tag{2.14}$$

The unique solution to this backward problem can be expressed explicitly as

$$u_M^{\delta,\delta}(x, t) = \sum_{m=1}^{M} c_m^{\delta,\delta} \varphi_m(x, t), \quad t \in [0, T]. \tag{2.15}$$

The function $u_M^{\delta,\delta}(x,t)$ will be taken as the regularizing solution for $u(x,t)$, with the regularizing parameter M specified in terms of the noise level in the sequel.

3. Convergence Analysis on the Regularizing Solution

Since our scheme regularizes the measurement data and gives an explicit expression of regularizing solution (2.15) uniformly for all $t\in[0,T]$, we can establish the uniform convergence rate for all $t\in[0,T]$. This fact is quite different from the classical regularization where the convergence rate depends on t. Especially, to get the convergence rate at $t=0$, some more strong regularity on $u_0(x)$ should be assumed there [4].

Firstly, we consider the approximation error $u_M^{\delta,\delta}(\cdot,t)-u(\cdot,t)$.

Theorem 3.1. *Assume that $u_0\in H^p$ with $\|u_0\|_{H^p}\le U_p$ for $p=1$ or $p=2$. Then there exists a constant $C_a>0$ such that for arbitrary $\delta>0$ and positive integer M, it holds*

$$\|u_M^{\delta,\delta}(\cdot,t)-u(\cdot,t)\|_{L^2(\Omega)}\le 3e^{-\lambda_1 t}\left[\delta e^{\lambda_M T}+\frac{C_aU_p}{\lambda_M^{p/2}}\right],\quad t\in[0,T]. \tag{3.1}$$

Proof. From direct computations, we have

$$\begin{aligned}|u_M^{\delta,\delta}(x,t)-u(x,t)|^2=&\left(\sum_{m=1}^M(c_m^{\delta,\delta}-c_m)\varphi_m(x,t)\right)^2+\left(\sum_{m=M+1}^\infty c_m\varphi_m(x,t)\right)^2\\&-2\sum_{m=1}^M(c_m^{\delta,\delta}-c_m)\varphi_m(x,t)\sum_{m=M+1}^\infty c_m\varphi_m(x,t).\end{aligned}$$

So it follows from the orthogonality of $\{\varphi_m(\cdot,t):m\in\mathbb{N}\}$ that

$$\begin{aligned}&\|u_M^{\delta,\delta}(\cdot,t)-u(\cdot,t)\|^2_{L^2(\Omega)}\\=&\sum_{m=1}^M(c_m^{\delta,\delta}-c_m)^2e^{-2\lambda_m t}+\sum_{m=M+1}^\infty c_m^2e^{-2\lambda_m t}\\\le&e^{-2\lambda_1 t}\sum_{m=1}^M(c_m^{\delta,\delta}-c_m)^2+e^{-2\lambda_M t}\sum_{m=M+1}^\infty c_m^2.\end{aligned} \tag{3.2}$$

On the other hand, the following identity

$$\begin{aligned}&\sum_{m=1}^M(c_m^{\delta,\delta}-c_m)\varphi_m(x,T)\\\equiv&\sum_{m=1}^M c_m^{\delta,\delta}\varphi_m(x,T)-f^\delta(x)+f^\delta(x)-f(x)+\sum_{m=M+1}^\infty c_m\varphi_m(x,T)\end{aligned}$$

yields

$$\begin{aligned}&\frac{1}{3}\left(\sum_{m=1}^M(c_m^{\delta,\delta}-c_m)\varphi_m(x,T)\right)^2\\\le&\left(\sum_{m=1}^M c_m^{\delta,\delta}\varphi_m(x,T)-f^\delta(x)\right)^2+(f^\delta(x)-f(x))^2+\left(\sum_{m=M+1}^\infty c_m\varphi_m(x,T)\right)^2.\end{aligned}$$

Consequently, it follows from (2.8) that

$$\frac{1}{3}\sum_{m=1}^{M}(c_m^{\delta,\delta}-c_m)^2e^{-2\lambda_m T}\leq 2\delta^2+e^{-2\lambda_M T}\sum_{m=M+1}^{\infty}c_m^2.$$

Therefore

$$\frac{1}{3}\sum_{m=1}^{M}(c_m^{\delta,\delta}-c_m)^2\leq 2e^{2\lambda_M T}\delta^2+\sum_{m=M+1}^{\infty}c_m^2. \tag{3.3}$$

Inserting (3.3) into (3.2) yields

$$\|u_M^{\delta,\delta}(\cdot,t)-u(\cdot,t)\|^2_{L^2(\Omega)}\leq 6\delta^2e^{2\lambda_M T}e^{-2\lambda_1 t}+\left(3e^{-2\lambda_1 t}+e^{-2\lambda_M t}\right)\sum_{m=M+1}^{\infty}c_m^2. \tag{3.4}$$

On the other hand, we have

$$\nabla\cdot(a(x)\nabla u_0(x))=\sum_{m=1}^{\infty}c_m\nabla\cdot(a(x)\nabla\varphi_m^0(x))=-\sum_{m=1}^{\infty}c_m\lambda_m\varphi_m^0(x). \tag{3.5}$$

Then for $u_0(x)\in H^2(\Omega)$, it follows

$$\sum_{m=1}^{\infty}c_m^2\lambda_m^2=C_a\|u_0\|^2_{H^2(\Omega)}\leq C_a^2U_2^2<+\infty \tag{3.6}$$

by using the equivalent norm. While for $u_0\in H^1(\Omega)$, (3.5) yields

$$\int_\Omega a(x)|\nabla u_0(x)|^2dx=\sum_{m=1}^{\infty}c_m^2\lambda_m,$$

by noticing $u_0(x)|_{\partial\Omega}=0$. Then using the Poincare inequality we have

$$\lambda_M^2\sum_{m=M+1}^{\infty}c_m^2\leq\sum_{m=M+1}^{\infty}c_m^2\lambda_m^2\leq\sum_{m=1}^{\infty}c_m^2\lambda_m^2\leq C_a^2U_2^2 \tag{3.7}$$

for $p=2$ and

$$\lambda_M\sum_{m=M+1}^{\infty}c_m^2\leq\sum_{m=M+1}^{\infty}c_m^2\lambda_m\leq\sum_{m=1}^{\infty}c_m^2\lambda_m\leq C_a^2U_1^2 \tag{3.8}$$

for $p=1$. Now it follows from (3.4), (3.7) and (3.8) that

$$\|u_M^{\delta,\delta}(\cdot,t)-u(\cdot,t)\|^2_{L^2(\Omega)}\leq 6\delta^2e^{2\lambda_M T}e^{-2\lambda_1 t}+\left(3e^{-2\lambda_1 t}+e^{-2\lambda_M t}\right)C_a^2U_p^2\lambda_M^{-p}.$$

The proof is complete. □

Now we can establish the optimal convergence rate based on this error estimate.

Theorem 3.2. *Assume that $u_0\in H^p$ with $\|u_0\|_{H^p}\leq U_p$ for $p=1$ or $p=2$. If the truncation term $M=M(\delta)$ satisfies*

$$\lambda_{M(\delta)}\approx\frac{1}{T}\ln\frac{1}{\delta^\beta} \tag{3.9}$$

for any fixed $\beta\in(0,1)$, then as $\delta\to 0$

$$\|u_{M(\delta)}^{\delta,\delta}(\cdot,t)-u(\cdot,t)\|_{L^2(\Omega)}\leq C_0(a,p,T,\beta)e^{-\lambda_1 t}\frac{1}{(-\ln\delta)^{p/2}},\quad t\in[0,T]. \tag{3.10}$$

Proof. For any positive function $h(\delta)$ tending to 0 as $\delta \to 0$, we firstly take $\lambda_{M(\delta)} \approx \frac{1}{T}\frac{1}{h(\delta)}$, then it follows from Theorem 3.1 that for $t \in [0,T]$

$$\begin{aligned}&\|u^{\delta,\delta}_{M(\delta)}(\cdot,t)-u(\cdot,t)\|_{L^2(\Omega)}\\ \le&3e^{-\lambda_1 t}\left[e^{1/h(\delta)}\delta+T^{p/2}C_aU_ph^{\frac{p}{2}}(\delta)\right]\le C(a,p,T)e^{-\lambda_1 t}(e^{1/h(\delta)}\delta+h^{\frac{p}{2}}(\delta))\end{aligned} \tag{3.11}$$

with the constant $C(a,p,T):=3\max\{1,T^{p/2}C_aU_p\}$. By choosing $h(\delta):=1/\ln\frac{1}{\delta^\beta}$ for any fixed $\beta\in(0,1)$, we are led to

$$0<e^{1/h(\delta)}\delta+h^{\frac{p}{2}}(\delta)=\delta^{1-\beta}+\frac{1}{(\ln\frac{1}{\delta^\beta})^{p/2}}\to 0,\quad \delta\to 0.$$

Noticing $\delta^{1-\beta}=o(\frac{1}{(\ln\frac{1}{\delta^\beta})^{p/2}})$ as $\delta\to 0$, the proof is complete. □

Remark 3.1. We cannot take $\beta=1$ in this result. On the other hand, it should be noticed that $\lim_{\beta\to 1}C_0(a,p,T,\beta)=+\infty$. Therefore, a practical convergence rate depending on $\beta\in(0,1)$ should be

$$\|u^{\delta,\delta}_{M(\delta)}(\cdot,t)-u(\cdot,t)\|_{L^2(\Omega)}\le C(a,p,T)e^{-\lambda_1 t}\left(\delta^{1-\beta}+\frac{1}{(\ln\frac{1}{\delta^\beta})^{p/2}}\right),\quad \delta\to 0. \tag{3.12}$$

This observation means that it is impossible to get $O(\frac{1}{(-\ln\delta)^{p/2}})$ convergence in practical computations. The notation $\approx$ in (3.9) means that $\lambda_{M(\delta)}$ has the same order as that of $\frac{1}{T}\ln\frac{1}{\delta^\beta}$, i.e., $\lambda_{M(\delta)}=C\frac{1}{T}\ln\frac{1}{\delta^\beta}$ for some constant $C>0$.

4. The Error Effect of Eigenfunctions

Different from some classical regularizing schemes, here we propose the eigenfunction-based regularizing scheme. In most of the cases, both eigenvalues and eigenfunctions can only be obtained numerically, especially for the operator with variable coefficient and general domain Ω. Therefore we need to consider the error effect arising in computing eigenfunctions on the regularizing solution. On the other hand, since the measurement data contain error, too much accurate computation of eigenfunctions is senseless. We should keep some optimal balance between the accuracy of eigensystem and the noise level of input data.

Assume that the eigensystem (2.1) is solved approximately. Denote by $\{\lambda_{n,\eta},\varphi^0_{n,\eta}:n\in\mathbb{N}\}$ the approximate eigensystem with error level $\eta>0$ measured by

$$\|\varphi^0_{n,\eta}-\varphi^0_n\|_{L^2(\Omega)}\le\eta,\quad |\lambda_{n,\eta}-\lambda_n|\le\eta. \tag{4.1}$$

For the approximation errors of eigensystem for Laplace operator, the readers are refereed to [12]. Our first result is about the linear independence of the noisy eigenfunctions $\{\varphi^0_{n,\eta}:n=1,\cdots,M\}$ for small $\eta>0$.

Theorem 4.1. *For exact eigenfunctions $\{\varphi^0_n:n=1,\cdots,M\}$, the noisy functions $\{\varphi^0_{n,\eta}:n=1,\cdots,M\}$ satisfying (4.1) are linear independent for small $\eta>0$.*

Proof. Note the relation

$$\sum_{k=1}^{M}c_k\varphi^0_{k,\eta}(x)\equiv 0,\quad x\in\Omega. \tag{4.2}$$

We express the noisy functions as $\varphi^0_{k,\eta}(x) = \varphi^0_k(x) + \eta d_k(x)$, with $d_k(x) \in L^2(\Omega)$ satisfying $\|d_k\|_{L^2} \le 1$ and being expanded as $d_k(x) = \sum_{j=1}^{\infty} D_{kj}\varphi^0_j(x)$. Therefore (4.2) becomes

$$\sum_{k=1}^{M} c_k\varphi^0_k(x) + \eta\sum_{k=1}^{M}\sum_{j=1}^{\infty} c_k D_{kj}\varphi^0_j(x) \equiv 0.$$

Taking inner product with respect to $\varphi^0_l(x)$ for any $l = 1, \cdots, M$ yields

$$c_l + \eta\sum_{k=1}^{M} c_k D_{kl} = 0, \quad l = 1, \cdots, M.$$

For small $\eta > 0$, this linear system has only trivial solution $c_1 = c_2 = \cdots = c_M = 0$, noticing $|D_{kl}|^2 \le \sum_{j=1}^{\infty} D^2_{kj} = \|d_k\|^2_{L^2} \le 1$, the proof is complete. □

Based on this result, the Schmidt orthogonalization can generate a set of standard orthogonal functions from $\{\varphi^0_{n,\eta} : n = 1, \cdots, M\}$. So we assume that $\{\varphi^0_{n,\eta} : n = 1, \cdots, M\}$ itself is standard orthogonal in the sequel directly. Then the regularizing solution to our backward heat problem can also be constructed from the approximate eigenfunctions. That is, we define

$$\varphi_{n,\eta}(x, t) = e^{-\lambda_{n,\eta}t}\varphi^0_{n,\eta}(x) \tag{4.3}$$

and determine the minimum norm solution to the equation

$$\sum_{m=1}^{M} c^\delta_m \varphi_{m,\eta}(x, T) = f^\delta(x) \tag{4.4}$$

with discrepancy δ. Denote by $c^{\delta,\delta}_{m,\eta}$ the solution of (4.4). Then we construct the regularizing solution by

$$u^{\delta,\delta}_{M,\eta}(x, t) = \sum_{m=1}^{M} c^{\delta,\delta}_{m,\eta}\varphi_{m,\eta}(x, t). \tag{4.5}$$

In this case, we have the following error estimate.

Theorem 4.2. *Assume that $\{\varphi^0_{n,\eta}(x) : n = 1, \cdots, M\}$ is standard orthogonal and $u_0 \in H^p(\Omega)$ for $p = 1$ or $p = 2$. Then there exists a constant $C > 0$ such that for arbitrary $\delta > 0$ and positive integer M, it follows for all $\eta \in [0, 1]$ that*

$$\begin{aligned}\|u^{\delta,\delta}_{M,\eta}(\cdot, t) - u(\cdot, t)\|_{L^2(\Omega)} \le & C(T, u_0, a)e^{-\lambda_1 t}e^{\lambda_M T}\left[\delta + \eta\sqrt{M}e^{\lambda_M T}(2\delta + 1)\right] \\ & + 3e^{-\lambda_1 t}\left[\delta e^{\lambda_M T} + \frac{C_a U_p}{\lambda_M^{p/2}}\right], \quad t \in [0, T].\end{aligned} \tag{4.6}$$

Remark 4.1. The structure of the error in this case is clear: the error order for $\eta = 0$ is just the same as that for the exact eigensystem. Notice, the factor $e^{-\lambda_1 t}$ indicates that numerical error decreases exponentially with respect to $t \in [0, T]$ for fixed δ, η. Once $M(\delta)$ is chosen such that $e^{\lambda_M T}\delta \to 0$ as $\delta \to 0$, the term $e^{-\lambda_1 t}e^{\lambda_M T}\delta$ still decays exponentially with respect to $t \in (0, T)$ for any given small $\delta > 0$.

Proof. From the triangle inequality

$$\begin{aligned}&\|u^{\delta,\delta}_{M,\eta}(\cdot, t) - u(\cdot, t)\|_{L^2(\Omega)} \\ \le & \|u^{\delta,\delta}_{M,\eta}(\cdot, t) - u^{\delta,\delta}_{M}(\cdot, t)\|_{L^2(\Omega)} + \|u^{\delta,\delta}_{M}(\cdot, t) - u(\cdot, t)\|_{L^2(\Omega)}\end{aligned} \tag{4.7}$$

and Theorem 3.1, we only need to estimate the first term in (4.7).

$$\begin{aligned}
&\|u^{\delta,\delta}_{M,\eta}(\cdot,t)-u^{\delta,\delta}_{M}(\cdot,t)\|\\
=&\|\sum_{m=1}^{M}c^{\delta,\delta}_{m,\eta}\varphi_{m,\eta}(\cdot,t)-\sum_{m=1}^{M}c^{\delta,\delta}_{m}\varphi_{m}(\cdot,t)\|\\
\le&\|\sum_{m=1}^{M}c^{\delta,\delta}_{m,\eta}\varphi_{m,\eta}(\cdot,t)-\sum_{m=1}^{M}c^{\delta,\delta}_{m,\eta}\varphi_{m}(\cdot,t)\|+\|\sum_{m=1}^{M}c^{\delta,\delta}_{m,\eta}\varphi_{m}(\cdot,t)-\sum_{m=1}^{M}c^{\delta,\delta}_{m}\varphi_{m}(\cdot,t)\|\\
=:&(I)+(II).
\end{aligned}\tag{4.8}$$

Now we estimate two terms in (4.8) respectively. Observe that

$$\begin{aligned}
(I)\le&\sum_{m=1}^{M}|c^{\delta,\delta}_{m,\eta}|\|\varphi_{m,\eta}(\cdot,t)-\varphi_{m}(\cdot,t)\|\\
\le&\sum_{m=1}^{M}|c^{\delta,\delta}_{m,\eta}|\Big(\|e^{-\lambda_{m,\eta}t}\varphi^0_{m,\eta}-e^{-\lambda_m t}\varphi^0_{m,\eta}\|+\|e^{-\lambda_m t}\varphi^0_{m,\eta}-e^{-\lambda_m t}\varphi^0_{m}\|\Big)\\
\le&\sum_{m=1}^{M}|c^{\delta,\delta}_{m,\eta}||e^{-\lambda_{m,\eta}t}-e^{-\lambda_m t}|\|\varphi^0_{m,\eta}\|+\sum_{m=1}^{M}|c^{\delta,\delta}_{m,\eta}|e^{-\lambda_m t}\|\varphi^0_{m,\eta}-\varphi^0_{m}\|\\
\le&\Big(1+Te^{T}\Big)e^{-\lambda_1 t}\eta\sum_{m=1}^{M}|c^{\delta}_{m,\eta}|,
\end{aligned}\tag{4.9}$$

where we have used the estimate $|e^{-\lambda_{m,\eta}t}-e^{-\lambda_m t}|=e^{-\lambda_m t}|e^{-\eta\theta_m t}-1|\le e^{-\lambda_m t}e^{T}T\eta$ with $\theta_m\in(-1,1)$ due to (4.1). On the other hand, it follows from the definition of $c^{\delta,\delta}_{m,\eta}$ that

$$\Big\|\sum_{m=1}^{M}c^{\delta,\delta}_{m,\eta}\varphi_{m,\eta}(\cdot,T)\Big\|\le\Big\|\sum_{m=1}^{M}c^{\delta,\delta}_{m,\eta}\varphi_{m,\eta}(\cdot,T)-f^{\delta}(\cdot)\Big\|+\|f^{\delta}-f\|+\|f\|\le 2\delta+\|f\|.$$

Therefore the orthogonality of $\{\varphi_{m,\eta}(\cdot,T):m=1,\cdots,M\}$ yields

$$e^{-2\lambda_{M,\eta}T}\sum_{m=1}^{M}|c^{\delta,\delta}_{m,\eta}|^2\le\sum_{m=1}^{M}|c^{\delta,\delta}_{m,\eta}|^2e^{-2\lambda_{m,\eta}T}\le 2\Big((2\delta)^2+\|f\|^2\Big).$$

Using this estimate gives

$$\sum_{m=1}^{M}|c^{\delta,\delta}_{m,\eta}|\le\sqrt{M}\sqrt{\sum_{m=1}^{M}|c^{\delta,\delta}_{m,\eta}|^2}\le 2\sqrt{M}e^{\lambda_{M,\eta}T}(2\delta+\|f\|).\tag{4.10}$$

Inserting (4.10) into (4.9) yields

$$(I)\le 2e^{2T}(T+1)e^{-\lambda_1 t}\eta\sqrt{M}e^{\lambda_M T}(2\delta+\|f\|).\tag{4.11}$$

For (II), we have

$$(II)^2=\sum_{m=1}^{M}(c^{\delta,\delta}_{m,\eta}-c^{\delta,\delta}_{m})^2e^{-2\lambda_m t}\le e^{-2\lambda_1 t}\sum_{m=1}^{M}(c^{\delta,\delta}_{m,\eta}-c^{\delta,\delta}_{m})^2.\tag{4.12}$$

On the other hand, we also have

$$\sum_{m=1}^{M}(c_{m,\eta}^{\delta,\delta}-c_m^{\delta,\delta})\varphi_m(x,T)=\sum_{m=1}^{M}c_{m,\eta}^{\delta,\delta}(\varphi_m(x,T)-\varphi_{m,\eta}(x,T))+\sum_{m=1}^{M}c_{m,\eta}^{\delta,\delta}\varphi_{m,\eta}(x,T)$$
$$-f^\delta(x)+f^\delta(x)-\sum_{m=1}^{M}c_m^{\delta,\delta}\varphi_m(x,T), \tag{4.13}$$

which generates by using the same technique in deriving (3.3) that

$$\frac{1}{3}\sum_{m=1}^{M}\left(c_{m,\eta}^{\delta,\delta}-c_m^{\delta,\delta}\right)^2 e^{-2\lambda_m T}\le\left\|\sum_{m=1}^{M}c_{m,\eta}^{\delta,\delta}\left(\varphi_m(x,T)-\varphi_{m,\eta}(x,T)\right)\right\|_{L^2(\Omega)}^2+2\delta^2, \tag{4.14}$$

using the definitions of $c_{m,\eta}^{\delta,\delta}$ and $c_m^{\delta,\delta}$. The first term in the right-hand side of (4.14) is the same as (I) with t replaced by T. So it follows from (4.11) that

$$\frac{1}{3}\sum_{m=1}^{M}(c_{m,\eta}^{\delta,\delta}-c_m^{\delta,\delta})^2e^{-2\lambda_m T}\le 4e^{4T}(T+1)^2\eta^2Me^{2\lambda_M T}(2\delta+\|f\|)^2+2\delta^2. \tag{4.15}$$

Inserting (4.15) into (4.12) gives

$$(II)\le 4e^{-\lambda_1 t}e^{\lambda_M T}[e^{2T}(T+1)\eta\sqrt{M}e^{\lambda_M T}(2\delta+\|f\|)+\delta], \tag{4.16}$$

and inserting (4.11) and (4.16) into (4.8) yields

$$\|u_{M,\eta}^{\delta,\delta}(\cdot,t)-u_M^{\delta,\delta}(\cdot,t)\|\le 4e^{-\lambda_1 t}e^{\lambda_M T}\left(\delta+\eta\sqrt{M}e^{2T}(T+1)(1+e^{\lambda_M T})(2\delta+\|f\|)\right).$$

Noticing the continuous dependence of $\|f\|_{L^2}$ on u_0, we rewrite this estimate as

$$\|u_{M,\eta}^{\delta,\delta}(\cdot,t)-u_M^{\delta,\delta}(\cdot,t)\|_{L^2(\Omega)}\le C(T,u_0,a)e^{-\lambda_1 t}e^{\lambda_M T}\left[\delta+\eta\sqrt{M}e^{\lambda_M T}(2\delta+1)\right] \tag{4.17}$$

for $\eta\in[0,1]$. The proof is complete from (3.1) and (4.17). $\square$

Now we can analyze the convergence rate of regularizing solution in terms of δ,η.

Theorem 4.3. *Under the assumptions of Theorem 3.2, we can choose $M=M(\delta,\eta)$ appropriately such that*

$$\|u_{M(\delta,\eta),\eta}^{\delta,\delta}(\cdot,t)-u(\cdot,t)\|_{L^2(\Omega)}\le C(\beta,T,p,U_p,a)e^{-\lambda_1 t}\frac{1}{(-\ln(\delta+\eta))^{p/2}} \tag{4.18}$$

for all $t\in[0,T]$ uniformly with any fixed $\beta\in(0,1)$.

Remark 4.2. It should be noticed that the analogy of Remark 3.1 also holds for this result. From this convergence rate, it can be seen that we should compute the eigensystem up to the accuracy $\eta=O(\delta)$ for noisy input data with noisy level δ.

Proof. Firstly we rewrite (4.6) for $t\in[0,T]$ as

$$\|u_{M,\eta}^{\delta,\delta}(\cdot,t)-u(\cdot,t)\|\le\tilde{C}^*e^{-\lambda_1 t}\left[e^{2\lambda_M T}\sqrt{M}\eta(1+\delta)+e^{\lambda_M T}\delta+\frac{1}{\lambda_M^{p/2}}\right] \tag{4.19}$$

for all $\eta \in [0,1]$, where the new constant $\tilde{C}^* := \tilde{C}^*(T, u_0, a, p, U_p) > 0$ is obvious from (4.6). The first term in the right-hand side is the error caused from the error η in computing the eigensystem, while the remained two-terms represents the approximation error by our regularizing scheme.

Noticing $\delta \to 0$, (4.19) yields for all $\eta, \delta \in (0,1)$ that

$$\|u^{\delta,\delta}_{M,\eta}(\cdot,t) - u(\cdot,t)\| \le C^* e^{-\lambda_1 t}\left(e^{2\lambda_M T}\sqrt{M}\eta + e^{\lambda_M T}\delta + \frac{1}{\lambda_M^{p/2}}\right). \tag{4.20}$$

On the other hand, we can estimate the eigenvalue λ_M from the asymptotic behavior $\sqrt{M} \le C_0 e^{\lambda_M T}$ with $C_0 = C_0(a, \Omega, T)$. Therefore (4.20) becomes

$$\|u^{\delta,\delta}_{M,\eta}(\cdot,t) - u(\cdot,t)\| \le C_0^* e^{-\lambda_1 t}\left(e^{3\lambda_M T}(\eta + \delta) + \frac{1}{\lambda_M^{p/2}}\right). \tag{4.21}$$

Using the same technique as that in the proof of Theorem 3.2 for the right-hand side, we know that for $M = M(\delta, \eta)$ choosing

$$3\lambda_{M(\delta,\eta)} \approx \frac{1}{T}\ln\frac{1}{(\delta+\eta)^\beta} \tag{4.22}$$

leads to (4.18). This completes the proof. $\square$

5. Numerical Examples

We present three numerical examples to show the validity of our inversion scheme.

Example 5.1. Consider the following 1-dimensional heat conduction problem

$$\frac{\partial u}{\partial t} = u_{xx}, \qquad x \in (0,\pi), t \in (0,T] \tag{5.1a}$$

$$u(0,t) = u(\pi,t) = 0, \qquad t \in (0,T] \tag{5.1b}$$

$$u(x,0) = \begin{cases} \frac{2}{\pi}x, & x \in [0, \frac{\pi}{2}] \\ 2 - \frac{2}{\pi}x, & x \in (\frac{\pi}{2}, \pi], \end{cases} \tag{5.1c}$$

where the measurement data is given at $T = 0.4$.

For this model, the eigensystem has the exact representation $\lambda_n = n^2$, $\varphi_n^0(x) = \sin nx$, for $n = 1, 2, \cdots$. The exact solution is

$$u(x,t) = \sum_{m=1}^{+\infty} \frac{8}{m^2\pi^2}\sin\frac{m\pi}{2}e^{-m^2 t}\sin mx.$$

We approximate the above infinite series by its first 20-terms, and the noisy data of $u(x,T)$ is simulated by

$$u^\delta(x,T) = \sum_{m=1}^{20} \frac{8}{m^2\pi^2}\sin\frac{m\pi}{2}e^{-m^2 T}\sin mx + \delta \times randn(x),$$

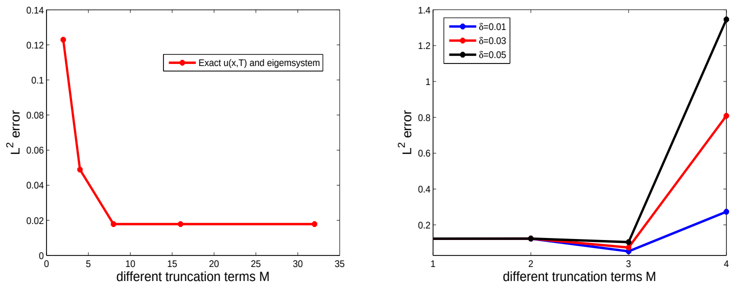

Fig. 5.1. Example 5.1: errors with respect to truncation term M.

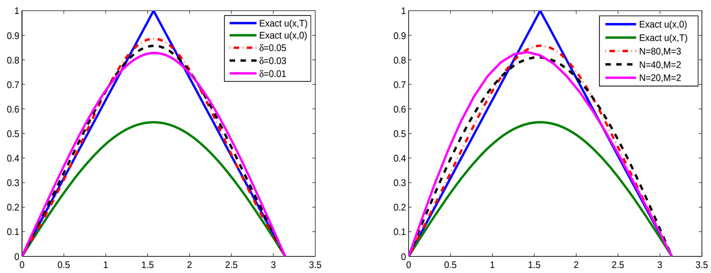

Fig. 5.2. Example 5.1: inversion results with different δ (left) and different nodes for $\delta = 0.03$ (right).

where the random number $randn(x)$ is a normal distribution with mean 0 and standard deviation 1 uniformly distributed in $[-1,1]$. So the L^2-error is

$$||u^\delta(\cdot,T) - u(\cdot,T)||_{L^2(\Omega)} \doteq \sqrt{\pi}\delta := \delta_1.$$

In our numerical procedure, we must solve the minimum norm solution $C_M^{\delta,\delta}$ to the equation (2.7) or $C_{M,\eta}^{\delta,\delta}$ to (4.4), where the regularizing parameter M can be chosen in terms of (3.9) or (4.22) from our result. However, this choice strategy is numerically not easy, since the theoretical relation between M, δ, η given in (3.9) and (4.22) contains some implicit constant C. In the following numerical procedure, M is given by minimizing the error between the exact initial temperature $u(x,0)$ and the inversion result $u_{M,\eta}^{\delta,\delta}(x,0)$. In this example with $\delta = 0.01$,

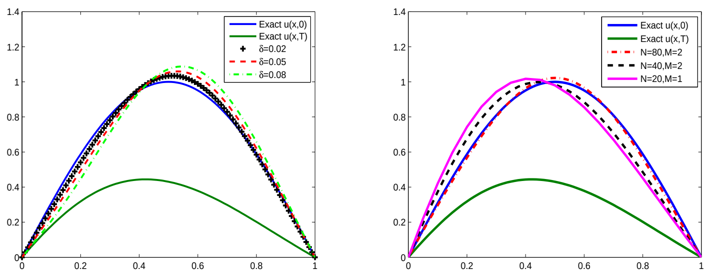

Fig. 5.3. Example 5.2: inversion result with different δ (left) and Inversion result with different nodes (right).

we get the optimal value $M = 3$ by trials and errors for the node number $N = 80$ of the interval $[0, \pi]$. In this case, the corresponding λ_M approximates $\frac{C}{T} \ln \frac{1}{\delta^\beta}$ for $\beta = \frac{1}{2}$ with $C = 10$.

Fig. 5.1 (left) shows the truncation-error curves for exact data $u(x, T)$ and exact eigensystem. It is consistent with our theoretical analysis that, the larger M is, the better the inversion result is for exact input data, although the L^2-error keeps almost a small constant for $M > 8$ due to the truncation error from the computer itself. However we can see from Fig. 5.1 (right) that the optimal parameter $M = 3$ for noisy data $u^\delta(x, T)$ and exact eigensystem.

In Fig. 5.2 (left), we present the exact data $u(x, 0)$, $u(x, T)$ and inversion results of $u(x, 0)$ for different noise level δ of $u^\delta(x, T)$ using the exact eigensystem. It can be seen that the inversion results are satisfactory with the exact eigensystem expansion.

To show the effect of computational error of the eigensystem, we yield the eigenfunction and eigenvalue by FEM and the computational error is controlled by the node number N of the interval $[0, \pi]$. It should be pointed out that the optimal value of truncation term M also changes for different node number N in numerical implementations. Fig. 5.2 (right) gives the inversion results of $u(x, 0)$ and the according optimal value of M with different node number N for fixed $\delta = 0.03$. It can be seen that the accurate computation of eigensystem is necessary for our inversion scheme.

Example 5.2. Consider the following 1-dimensional heat conduction problem with variable coefficient $a(x) = e^x/20$:

$$\frac{\partial u}{\partial t} = \nabla \cdot (a(x)\nabla u), \quad x \in (0, 1), t \in (0, T] \tag{5.2a}$$

$$u(0, t) = u(\pi, t) = 0, \quad t \in (0, T] \tag{5.2b}$$

$$u(x, 0) = \sin(\pi x), \qquad x \in [0, 1], \tag{5.2c}$$

where the measurement data is given at $T = 1$.

In this example, both exact solution and the eigensystem have to be computed numerically. The FEM is used here to solve the direct problem and the eigenvalue problems. The optimal value of parameter M is 2 for node number $N = 80$ in this case. Fig. 5.3 (left) shows the exact data $u(x, 0)$, $u(x, T)$ and the inversion results with different noise level δ. Similarly the inversion results are satisfactory.

Now we consider the effect of the error in computing eigensystem on the inversion results. Similarly we represent the accuracy of eigensystem by using different node number N of the interval $[0, 1]$ in computing (5.2) and the eigensystem (2.1). In Fig. 5.3 (right), we show the inversion results and the corresponding optimal value of M with $\delta = 0.03$ for different node number N. Similarly the truncation term M is also different for different N. From Fig. 5.3 (right), we can get the same conclusion as that for Example 5.1.

Example 5.3. Consider the following 2-dimensional heat conduction problem

$$\frac{\partial u}{\partial t} = \nabla \cdot (\sigma \nabla u), \qquad x \in \Omega = (0, \pi) \times (0, \pi), t \in (0, T] \tag{5.3a}$$

$$u(x, t) = 0, \qquad x \in \partial\Omega, t \in (0, T] \tag{5.3b}$$

$$u(x, 0) = \sin x_1 \sin x_2, \quad x \in (0, \pi) \times (0, \pi), \tag{5.3c}$$

where the measurement data is given at $T = 0.2$ and $\sigma(x) = 0.3x_1^2 + x_2 + 1$.

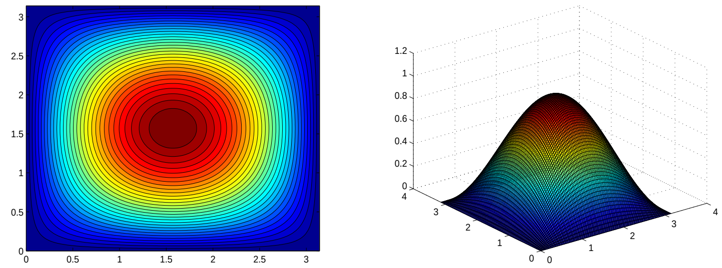

Fig. 5.4. Example 5.3: isolines(left) and three-dimensional shaded surface (right) of $u(x,0)$.

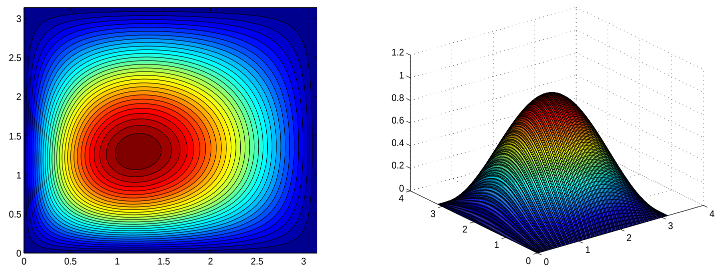

Fig. 5.5. Example 5.3: isolines of $u(x,T)$ (left) and Inversion result for $\delta = 0.02$ (right).

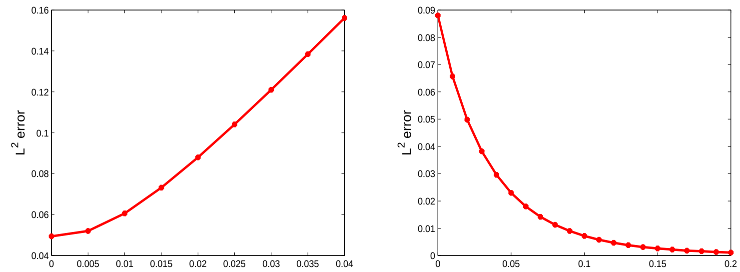

Fig. 5.6. L^2-error curve with different δ (left) and with different time (right).

Both exact solution and the eigensystem are computed numerically by FEM. And the noisy data of $u(x,T)$ is given by $u^\delta(x,T) = u(x,T) + \delta \times randn(x)$. So $||u^\delta(\cdot,T) - u(\cdot,T)||_{L^2(\Omega)} \doteq \pi\delta := \delta_2$. In this case, the optimal M is 4 for node number $N = 100$. Fig. 5.4 shows the isolines and three-dimensional shaded surface of exact initial data $u(x,0)$ which is symmetric from Fig. 5.4 (left). Due to the general form of $\sigma(x)$, the final data $u(x,T)$ becomes dissymmetric as shown in Fig. 5.5 (left). However the inversion results are satisfactory for $\delta = 0.02$ even we apply the noisy data $u^\delta(x,T)$, see Fig. 5.5 (right).

Fig. 5.6 (left) gives the L^2-error curve between $u^\delta(\cdot,0)$ and $u(\cdot,0)$ with different noise level δ. Fig. 5.6 (right) shows the L^2-error between $u^\delta(\cdot,t)$ and $u(\cdot,t)$ depending on time t for $\delta = 0.02$ and node number $N = 100$, which shows the exponentially decreasing with respect to time t, as given in our theoretical analysis in Section 4.

Acknowledgments. This work is supported by NSFC (No.11071039) and Natural Science Foundation of Jiangsu Province (No.BK2011584).

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