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SPECTRAL METHOD FOR MIXED INHOMOGENEOUS BOUNDARY VALUE PROBLEMS IN THREE DIMENSIONS*

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Abstract

In this paper, we investigate spectral method for mixed inhomogeneous boundary value problems in three dimensions. Some results on the three-dimensional Legendre approximation in Jacobi weighted Sobolev space are established, which improve and generalize the existing results, and play an important role in numerical solutions of partial differential equations. We also develop a lifting technique, with which we could handle mixed inhomogeneous boundary conditions easily. As examples of applications, spectral schemes are provided for three model problems with mixed inhomogeneous boundary conditions. The spectral accuracy in space of proposed algorithms is proved. Efficient implementations are presented. Numerical results demonstrate their high accuracy, and confirm the theoretical analysis well.

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Key words: Three-dimensional Legendre approximation in Jacobi weighted Sobolev space, Lifting technique, Spectral method for mixed inhomogeneous boundary value problems.

1. Introduction

The spectral method has gained increasing popularity in scientific computations, see [3, 4, 7, 8, 10, 11, 13] and the references therein. Recently, some authors developed the Jacobi spectral approximation, and enlarged the applications of spectral method, see [2, 14, 16–18]. There have been a lot of work on one (or two) -dimensional problems. However, it is also interesting to consider spectral method in three dimensions, cf. [1, 8, 9, 12, 20, 22].

In this paper, we investigate the Legendre spectral method for mixed inhomogeneous boundary value problems in three-dimensional space. In the next section, we first recall some recent results on the one-dimensional Legendre orthogonal approximation presented in [17]. By using those results, together with the interpolation of operators (cf. [6]), we establish the basic results on the three-dimensional Legendre orthogonal approximation in Jacobi weighted Sobolev space. The new results improve and generalize the existing results of [3,4,14,17], and play an important

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role in spectral method for partial differential equations with mixed boundary conditions. In particular, the existence of Jacobi weights in the norms appearing in the error estimates covers certain singularities of approximated functions. In Section 3, we consider spectral method for mixed inhomogeneous boundary value problems. We could handle such problems in two ways. The first way is to approximate the Dirichlet boundary conditions suitably, and then use certain proper approximations, see e.g., [1,15] and the references therein. The second way is to reform the original problems to the related homogeneous boundary value problems, and then solve the resulting ones easily. For instance, the work of [19,21] for Dirichlet boundary value problems and the work of [19] for mixed boundary value problems in two-dimensions. But, it is not easy to generalize this approach to three-dimensional problems. We also refer to the books [4,8,9,20]. We now develop an explicit lifting technique, with which we reformulate three model problems (steady or unsteady) with Dirichlet or mixed boundary conditions to some alternative forms with homogeneous Dirichlet boundary conditions imposed on some parts of the boundary. Then we provide the corresponding spectral schemes and prove their spectral accuracy. In Section 4, we describe the efficient numerical implementations, and present some numerical results demonstrating their high accuracy. The final section is for concluding remarks.

2. Legendre Orthogonal Approximation in Three Dimensions

In this section, we establish the new results on the Legendre orthogonal approximation in three dimensions.

2.1. One-dimensional Legendre orthogonal approximation

We first recall one-dimensional Legendre orthogonal approximation. Let $I = \{x \mid |x| < 1\}$. For $r \geq 0$, we define the Sobolev space $H^r(I)$ and its norm $\|u\|_{r,I}$ as usual. In particular, $L^2(I) = H^0(I)$ with the inner product $(u,v)_I$ and the norm $\|u\|_I$. Let $\mathcal{D}(I)$ be the set consisting of all infinitely differentiable functions with compact supports in I. $H_0^r(I)$ is the closure of $\mathcal{D}(I)$ in $H^r(I)$. For simplicity, we denote $\dfrac{\partial u}{\partial x}$ by $\partial_x u$, etc..

The Legendre polynomial of degree l is given by

$$L_l(x) = \frac{(-1)^l}{2^l l!}\partial_x^l (1-x^2)^l.$$

The set of all Legendre polynomials is a complete $L^2(I)$-orthogonal system.

For positive integer N, $\mathcal{P}_N(I)$ stands for the set of all polynomials of degree at most N. Throughout this paper, we denote by c a generic positive constant which does not depend on N and any function.

The orthogonal projection $P_{N,I}$: $L^2(I) \to \mathcal{P}_N(I)$ is defined by

$$(P_{N,I}u - u, \phi)_I = 0, \qquad \forall \phi \in \mathcal{P}_N(I).$$

By Theorem 2.1 of [17] with $\alpha = \beta = 0$, we know that if $u \in L^2(I), (1-x^2)^{\frac{s}{2}}\partial_x^s u \in L^2(I)$ and integers $0 \leq s \leq N+1$, then

$$||P_{N,I}u - u||_I \leq cN^{-s}||(1-x^2)^{\frac{s}{2}}\partial_x^s u||_I. \tag{2.1}$$

The orthogonal projection $P_{N,I}^1 : H^1(I) \to \mathcal{P}_N(I)$ is defined by

$$(\partial_x(P_{N,I}^1 u - u), \partial_x \phi)_I + (P_{N,I}^1 u - u, \phi)_I = 0, \qquad \forall\ \phi \in \mathcal{P}_N(I).$$

According to Theorem 3.1 of [17], if $u \in H^1(I)$, $(1-x^2)^{\frac{s-1}{2}}\partial_x^s u \in L^2(I)$ and integers $1 \le s \le N+1$, then

$$\|\partial_x^\mu (P_{N,I}^1 u - u)\|_I \le cN^{\mu-s}\|(1-x^2)^{\frac{s-1}{2}}\partial_x^s u\|_I, \qquad \mu = 0,1. \tag{2.2}$$

Let

$$\mathcal{P}_N^0(I) = H_0^1(I) \cap \mathcal{P}_N(I).$$

The orthogonal projection $P_{N,I}^{1,0} : H_0^1(I) \to \mathcal{P}_N^0(I)$ is defined by

$$(\partial_x(P_{N,I}^{1,0}u - u), \partial_x \phi)_I = 0, \qquad \forall\, \phi \in \mathcal{P}_N^0(I).$$

By Theorem 3.4 of [17], if $u \in H_0^1(I)$, $(1-x^2)^{\frac{s-1}{2}}\partial_x^s u \in L^2(I)$ and integers $1 \le s \le N+1$, then

$$\|\partial_x^\mu (P_{N,I}^{1,0} u - u)\|_I \le cN^{\mu-s}\|(1-x^2)^{\frac{s-1}{2}}\partial_x^s u\|_I, \qquad \mu = 0,1. \tag{2.3}$$

Let

$${}^0H^1(I) = \{u \in H^1(I) \mid u(1) = 0\} \quad \text{and} \quad {}^0\mathcal{P}_N(I) = {}^0H^1(I) \cap \mathcal{P}_N(I).$$

The orthogonal projection ${}^0P_{N,I}^1 : {}^0H^1(I) \to {}^0P_N(I)$, is defined by

$$(\partial_x({}^0P_{N,I}^1 u - u), \partial_x \phi)_I = 0, \qquad \forall\, \phi \in {}^0P_N(I).$$

We have from a slight modification of Theorem 3.2 of [17] that if $v \in {}^0H^1(I), (1-x^2)^{\frac{s}{2}}\partial_x^s u \in L^2(I)$ and integers $1 \le s \le N+1$, then

$$\|\partial_x^\mu ({}^0P_{N,I}^1 u - u)\|_I \le cN^{\mu-s}\|(1-x^2)^{\frac{s-1}{2}}\partial_x^s u\|_I, \qquad \mu = 0,1. \tag{2.4}$$

2.2. Three-dimensional Legendre orthogonal approximation

We now turn to the three-dimensional Legendre orthogonal approximation. Let $I_j = \{x_j \mid |x_j| < 1\}$ and $\Omega = \{(x_1, x_2, x_3) \mid x_j \in I_j,\ 1 \le j \le 3\}$. We define the space $H^r(\Omega)$ and its norm $||u||_{r,\Omega}$ in the usual way. The inner product and the norm of $L^2(\Omega)$ are denoted by $(u,v)_\Omega$ and $||u||_\Omega$, respectively.

Let $V_N(\Omega) = \mathcal{P}_N(I_1) \otimes \mathcal{P}_N(I_2) \otimes \mathcal{P}_N(I_3)$. The orthogonal projection $P_{N,\Omega} : L^2(\Omega) \to V_N(\Omega)$ is defined by

$$(P_{N,\Omega}u - u, \phi)_\Omega = 0, \qquad \forall \phi \in V_N(\Omega).$$

For estimating the approximation error, we introduce the following quantity with integer $r \ge 0$,

$$\begin{aligned}\mathbb{A}_{r,\Omega}(u) = &\int_{I_3}\int_{I_2} \|(1-x_1^2)^{\frac{r}{2}}\partial_{x_1}^r u(\cdot, x_2, x_3)\|_{I_1}^2 dx_2 dx_3 \\ &+ \int_{I_3}\int_{I_1} \|(1-x_2^2)^{\frac{r}{2}}\partial_{x_2}^r u(x_1, \cdot, x_3)\|_{I_2}^2 dx_1 dx_3 \\ &+ \int_{I_2}\int_{I_1} \|(1-x_3^2)^{\frac{r}{2}}\partial_{x_3}^r u(x_1, x_2, \cdot)\|_{I_3}^2 dx_1 dx_2.\end{aligned}$$

Theorem 2.1. *If $u \in L^2(\Omega)$, integers $r \ge 0$ and $r \le N+1$, then*

$$\|P_{N,\Omega}u - u\|_\Omega^2 \le cN^{-2r}\mathbb{A}_{r,\Omega}(u), \tag{2.5}$$

provided that $\mathbb{A}_{r,\Omega}(u)$ *is finite.*

Proof. Since $P_{N,\Omega}u = P_{N,I_1} \cdot P_{N,I_2} \cdot P_{N,I_3}u$, we have

$$\begin{aligned}\|P_{N,\Omega}u - u\|_\Omega^2 \le 3(&\|P_{N,I_1}(P_{N,I_2} \cdot P_{N,I_3}u) - P_{N,I_2} \cdot P_{N,I_3}u\|_\Omega^2 \\ &+\|P_{N,I_2}(P_{N,I_3}u) - P_{N,I_3}u\|_\Omega^2 + \|P_{N,I_3}u - u)\|_\Omega^2).\end{aligned} \tag{2.6}$$

By using (2.1) with $s = r, 0$, we deduce that

$$\begin{aligned}&\|P_{N,I_1}(P_{N,I_2} \cdot P_{N,I_3}u) - P_{N,I_2} \cdot P_{N,I_3}u\|_\Omega^2 \\ &\le cN^{-2r}\int_{I_3}\int_{I_2} \|(1-x_1^2)^{\frac{r}{2}}\partial_{x_1}^r(P_{N,I_2} \cdot P_{M,I_3}u(\cdot, x_2, x_3))\|_{I_1}^2 dx_2dx_3 \\ &\le cN^{-2r}\int_{I_3}\int_{I_2} \|(1-x_1^2)^{\frac{r}{2}}\partial_{x_1}^r u(\cdot, x_2, x_3)\|_{I_1}^2 dx_2dx_3.\end{aligned}$$

We could estimate the other terms at the right side of (2.6) in the same manner. Then the desired result (2.5) follows immediately. □

We now introduce the bilinear form:

$$a(u, v) = (\nabla u, \nabla v)_\Omega + (u, v)_\Omega, \qquad \forall u, v \in H^1(\Omega).$$

The orthogonal projection $P_{N,\Omega}^1 : H^1(\Omega) \to V_N(\Omega)$ is defined by

$$a(P_{N,\Omega}^1 u - u, \phi) = 0, \qquad \forall \phi \in V_N(\Omega).$$

To describe the approximation error, we introduce the quantity $\mathbb{B}_{r,\Omega}(u)$. For $r = 1$ and 2, $\mathbb{B}_{r,\Omega}(u) = ||u||_{r,\Omega}^2$. For $r \ge 3$,

$$\begin{aligned}\mathbb{B}_{r,\Omega}(u) = &\int_\Omega \Big[(1-x_1^2)^{r-1}(\partial_{x_1}^r u)^2 + (1-x_2^2)^{r-1}(\partial_{x_2}^r u)^2 + (1-x_3^2)^{r-1}(\partial_{x_3}^r u)^2\Big]dx_1dx_2dx_3 \\ &+ \int_\Omega \Big[(1-x_1^2)^{r-2}\Big((\partial_{x_1}^{r-1}\partial_{x_2}u)^2 + (\partial_{x_1}^{r-1}\partial_{x_3}u)^2\Big) \\ &\quad + (1-x_2^2)^{r-2}\Big((\partial_{x_1}\partial_{x_2}^{r-1}u)^2 + (\partial_{x_2}^{r-1}\partial_{x_3}u)^2\Big) \\ &\quad + (1-x_3^2)^{r-2}\Big((\partial_{x_2}\partial_{x_3}^{r-1}u)^2 + (\partial_{x_1}\partial_{x_3}^{r-1}u)^2\Big)\Big]dx_1dx_2dx_3 \\ &+ \int_\Omega \Big[(1-x_1^2)^{r-3}(\partial_{x_1}^{r-2}\partial_{x_2}\partial_{x_3}u)^2 + (1-x_2^2)^{r-3}(\partial_{x_1}\partial_{x_2}^{r-2}\partial_{x_3}u)^2 \\ &\quad + (1-x_3^2)^{r-3}(\partial_{x_1}\partial_{x_2}\partial_{x_3}^{r-2}u)^2\Big]dx_1dx_2dx_3.\end{aligned}$$

Theorem 2.2. *If* $u \in H^1(\Omega)$ *and integers* $1 \le r \le N + 1$, *then*

$$\|P_{N,\Omega}^1 u - u\|_{\mu,\Omega}^2 \le cN^{2\mu-2r}\mathbb{B}_{r,\Omega}(u), \qquad \mu = 0, 1, \tag{2.7}$$

provided that $\mathbb{B}_{r,\Omega}(u)$ *is finite.*

Proof. According to the property of the orthogonal projection P_N^1, we have

$$\|P_{N,\Omega}^1 u - u\|_{1,\Omega}^2 \le ||u||_{1,\Omega}^2 = \mathbb{B}_{1,\Omega}(u). \tag{2.8}$$

We next consider the case with $r \ge 3$. By projection theorem,

$$\|\nabla(P_{N,\Omega}^1 u - u)\|_\Omega^2 + \|P_{N,\Omega}^1 u - u\|_\Omega^2 \le \|\nabla(\phi - u)\|_\Omega^2 + \|\phi - u\|_\Omega^2, \qquad \forall \phi \in V_N(\Omega).$$

Take $\phi = P^1_{N,I_1} \cdot P^1_{N,I_2} \cdot P^1_{N,I_3} u$. Then

$$
\begin{aligned}
&\|P^1_{N,\Omega} u - u\|^2_{1,\Omega} \\
&\le \|\nabla(P^1_{N,I_1} \cdot P^1_{N,I_2} \cdot P^1_{N,I_3} u - u)\|^2_{\Omega} + \|P^1_{N,I_1} \cdot P^1_{N,I_2} \cdot P^1_{N,I_3} u - u\|^2_{\Omega}.
\end{aligned} \tag{2.9}
$$

Thus, it suffices to estimate the right side of the above inequality. Obviously, we have

$$
\|\partial_{x_1}(P^1_{N,I_1} \cdot P^1_{N,I_2} \cdot P^1_{N,I_3} u - u)\|^2_{\Omega} \le 3(F_1(u) + F_2(u) + F_3(u))
$$

where

$$
\begin{aligned}
&F_1(u) = \|\partial_{x_1}(P^1_{N,I_1} u - u)\|^2_{\Omega}, \qquad F_2(u) = \|\partial_{x_1} P^1_{N,I_1}(P^1_{N,I_2} u - u)\|^2_{\Omega}, \\
&F_3(u) = \|\partial_{x_1} P^1_{N,I_1}(P^1_{N,I_2} P^1_{N,I_3} u - P^1_{N,I_2} u)\|^2_{\Omega}.
\end{aligned}
$$

Thanks to (2.2) with $\mu = 1$ and $s = r$, we have

$$
F_1(u) \le cN^{2-2r} \int_{I_3} \int_{I_2} \|(1 - x_1^2)^{\frac{r-1}{2}} \partial^r_{x_1} u\|^2_{I_1} dx_2 dx_3.
$$

By virtue of (2.2) with $\mu = s = 1$, we obtain

$$
F_2(u) \le c\|\partial_{x_1}(P^1_{N,I_2} u - u)\|^2_{\Omega}.
$$

Further, we use (2.2) with $\mu = 0$ and $s = r - 1$, to reach that

$$
F_2(u) \le cN^{2-2r} \int_{I_3} \int_{I_1} \|(1 - x_2^2)^{\frac{r-2}{2}} \partial_{x_1} \partial^{r-1}_{x_2} u\|^2_{I_2} dx_1 dx_3.
$$

Furthermore, by using (2.2) with $\mu = s = 1$ again, we obtain

$$
F_3(u) \le c\|\partial_{x_1}(P^1_{N,I_2} P^1_{N,I_3} u - P^1_{N,I_2} u)\|^2_{\Omega}.
$$

On the other hand, the estimate (2.2) with $\mu = 0$ and $s = 1$, implies that

$$
\|P^1_{N,I_2} u\|^2_{I_2} \le 2\|P^1_{N,I_2} u - u\|^2_{I_2} + 2\|u\|^2_{I_2} \le cN^{-2}\|\partial_{x_2} u\|^2_{I_2} + 2\|u\|^2_{I_2}. \tag{2.10}
$$

Thereby, we use (2.10) and (2.2) with $\mu = 0$ and $s = r - 1$ (or $r - 2$), to deduce that

$$
\begin{aligned}
F_3(u) &\le cN^{-2}\|\partial_{x_2}(P^1_{N,I_3} \partial_{x_1} u - \partial_{x_1} u)\|^2_{\Omega} + 2\|P^1_{N,I_3} \partial_{x_1} u - \partial_{x_1} u\|^2_{\Omega} \\
&\le cN^{2-2r} \int_{I_2} \int_{I_1} \|(1 - x_3^2)^{\frac{r-3}{2}} \partial_{x_1} \partial_{x_2} \partial^{r-2}_{x_3} u\|^2_{I_3} dx_1 dx_2 \\
&\quad + cN^{2-2r} \int_{I_2} \int_{I_1} \|(1 - x_3^2)^{\frac{r-2}{2}} \partial_{x_1} \partial^{r-1}_{x_3} u\|^2_{I_3} dx_1 dx_2.
\end{aligned}
$$

We could estimate the terms $\|\partial_{x_2}(P^1_{N,I_1} \cdot P^1_{N,I_2} \cdot P^1_{N,I_3} u - u)\|^2_{\Omega}$ and $\|\partial_{x_3}(P^1_{N,I_1} \cdot P^1_{N,I_2} \cdot P^1_{N,I_3} u - u)\|^2_{\Omega}$ in the same manner.

We now estimate the term $\|P^1_{N,I_1} \cdot P^1_{N,I_2} \cdot P^1_{N,I_3} u - u\|^2_{\Omega}$. Clearly,

$$
\|P^1_{N,I_1} \cdot P^1_{N,I_2} \cdot P^1_{N,I_3} u - u\|^2_{\Omega} \le 3(G_1(u) + G_2(u) + G_3(u))
$$

where

$$
\begin{aligned}
&G_1(u) = \|P^1_{N,I_1} u - u\|^2_{\Omega}, \qquad G_2(u) = \|P^1_{N,I_1}(P^1_{N,I_2} u - u)\|^2_{\Omega}, \\
&G_3(u) = \|P^1_{N,I_1}(P^1_{N,I_2} P^1_{N,I_3} u - P^1_{N,I_2} u)\|^2_{\Omega}.
\end{aligned}
$$

Using (2.2) with $\mu=0$ and $s=r$, we obtain

$$G_1(u) \le cN^{-2r} \int_{I_3}\int_{I_2} \|(1-x_1^2)^{\frac{r-1}{2}}\partial_{x_1}^r u\|_{I_1}^2 dx_2 dx_3.$$

Moreover, by an estimate like (2.10) and the inequality (2.2) with $\mu=0$ and $s=r-1$ (or r), we verify that

$$\begin{aligned} G_2(u) &\le cN^{-2}\|\partial_{x_1}(P^1_{N,I_2}u-u)\|_\Omega^2 + 2\|P^1_{N,I_2}u-u\|_\Omega^2 \\ &\le cN^{-2r}\int_{I_3}\int_{I_1}\Big(\|(1-x_2^2)^{\frac{r-2}{2}}\partial_{x_1}\partial_{x_2}^{r-1}u\|_{I_2}^2 + \|(1-x_2^2)^{\frac{r-1}{2}}\partial_{x_2}^r u\|_{I_2}^2\Big)dx_1dx_3. \end{aligned}$$

Similarly,

$$\begin{aligned} G_3(u) &\le cN^{-2}\|\partial_{x_1}P^1_{N,I_2}(P^1_{N,I_3}u-u)\|_\Omega^2 + 2\|P^1_{N,I_2}(P^1_{N,I_3}u-u)\|_\Omega^2 \\ &\le cN^{-2}(N^{-2}\|\partial_{x_2}(P^1_{N,I_3}\partial_{x_1}u-\partial_{x_1}u)\|_\Omega^2 + 2\|P^1_{N,I_3}\partial_{x_1}u-\partial_{x_1}u\|_\Omega^2) \\ &\quad + cN^{-2}\|\partial_{x_2}(P^1_{N,I_3}u-u)\|_\Omega^2 + 4\|P^1_{N,I_3}u-u\|_\Omega^2 \\ &\le cN^{-2r}\int_{I_2}\int_{I_1}(\|(1-x_3^2)^{\frac{r-3}{2}}\partial_{x_1}\partial_{x_2}\partial_{x_3}^{r-2}u\|_{I_3}^2 + \|(1-x_3^2)^{\frac{r-2}{2}}\partial_{x_1}\partial_{x_3}^{r-1}u\|_{I_3}^2 \\ &\quad + \|(1-x_3^2)^{\frac{r-2}{2}}\partial_{x_2}\partial_{x_3}^{r-1}u\|_{I_3}^2 + \|(1-x_3^2)^{\frac{r-1}{2}}\partial_{x_3}^r u\|_{I_3}^2)dx_1dx_2. \end{aligned}$$

Then, the result (2.7) with $\mu=1$ and $r\ge 3$ comes from a combination of (2.9) with the previous estimates.

In order to derive the result (2.7) with $\mu=1$ and $r=2$, we should use the interpolation of operators, as described in Brenner and Scott [6]. We define the linear operator $\mathcal{L}$, which maps u to the error $P^1_{N,\Omega}u-u$. In other words, $\mathcal{L}u = P^1_{N,\Omega}u-u$. The estimate (2.8) implies that $\mathcal{L}$ maps $H^1(\Omega)$ to $H^1(\Omega)$, with the norm

$$||\mathcal{L}||_{H^1(\Omega)\to H^1(\Omega)} \le c. \tag{2.11}$$

On the other hand, by virtue of (2.7) with $\mu=1$ and $r=3$, we have

$$\|P^1_{N,\Omega}u-u\|_{1,\Omega}^2 \le cN^{-4}\mathbb{B}_{3,\Omega}(u) \le cN^{-4}||u||_{3,\Omega}^2.$$

It means that $\mathcal{L}$ maps $H^3(\Omega)$ to $H^1(\Omega)$, with the norm

$$||\mathcal{L}||_{H^3(\Omega)\to H^1(\Omega)} \le cN^{-2}. \tag{2.12}$$

As is well known, the space $H^2(\Omega)$ is an interpolation between $H^1(\Omega)$ and $H^3(\Omega)$, while the space $H^1(\Omega)$ is an interpolation between $H^1(\Omega)$ and $H^1(\Omega)$. Thus, the operator $\mathcal{L}$ mapping $H^2(\Omega)$ to $H^1(\Omega)$ could be regarded as an interpolation between the operator mapping $H^1(\Omega)$ to $H^1(\Omega)$ and the operator mapping $H^3(\Omega)$ to $H^1(\Omega)$. Accordingly, by virtue of Proposition 14.1.5 with $\theta=\frac{1}{2}$ and $p=2$ of [6], we have

$$||\mathcal{L}||_{H^2(\Omega)\to H^1(\Omega)} \le ||\mathcal{L}||^{\frac{1}{2}}_{H^1(\Omega)\to H^1(\Omega)}||\mathcal{L}||^{\frac{1}{2}}_{H^3(\Omega)\to H^1(\Omega)}.$$

This, along with (2.11) and (2.12), leads to $||\mathcal{L}||_{H^2(\Omega)\to H^1(\Omega)} \le cN^{-1}$. Consequently,

$$\|P^1_{N,\Omega}u-u\|_{1,\Omega}^2 \le cN^{-2}||u||_{2,\Omega}^2 = cN^{-2}\mathbb{B}_{2,\Omega}(u).$$

A combination of the previous statements implies the validity of the desired result (2.7) with $\mu = 1$ and $r \geq 1$.

Finally, we derive the result (2.7) with $\mu = 0$. Let $g \in L^2(\Omega)$ and consider the auxiliary problem

$$a(w, z) = (g, z)_\Omega, \qquad \forall z \in H^1(\Omega). \tag{2.13}$$

In sense of distributions,

$$-\Delta w + w = g.$$

Due to property of elliptic equation, we have $||w||_{2,\Omega} \leq c||g||_\Omega$. Thereby, using (2.7) with $\mu = 1$ and $r = 2$, yields that

$$||P^1_{N,\Omega} w - w||_{1,\Omega} \leq cN^{-1}||w||_{2,\Omega} \leq cN^{-1}||g||_\Omega. \tag{2.14}$$

Now, by taking $z = P^1_{N,\Omega}u - u$ in (2.13), and using (2.7) and (2.14), we verify that

$$|(P^1_{N,\Omega}u - u, g)_\Omega| = |a(P^1_{N,\Omega}u - u, P^1_{N,\Omega}w - w)| \leq cN^{-r}||g||_\Omega \mathbb{B}^{\frac{1}{2}}_{r,\Omega}(u).$$

Consequently,

$$||P^1_{N,\Omega}u - u||_\Omega = \sup_{g \in L^2(\Omega), g \neq 0} \frac{|(P^1_{N,\Omega}u - u, g)_\Omega|}{||g||_\Omega} \leq cN^{-r}\mathbb{B}^{\frac{1}{2}}_{r,\Omega}(u).$$

The proof is completed. □

For numerical solutions of Dirichlet boundary value problems of partial differential equations, we need another projection. Let $V^0_N(\Omega) = H^1_0(\Omega) \cap V_N(\Omega)$. The orthogonal projection $P^{1,0}_{N,\Omega} : H^1_0(\Omega) \to V^0_N(\Omega)$ is defined by

$$(\nabla(P^{1,0}_{N,\Omega}u - u), \nabla\phi)_\Omega = 0, \qquad \forall \phi \in V^0_N(\Omega).$$

Theorem 2.3. *If $u \in H^1_0(\Omega)$ and integers $1 \leq r \leq N + 1$, then*

$$\|P^{1,0}_{N,\Omega}u - u\|^2_{\mu,\Omega} \leq cN^{2\mu-2r}\mathbb{B}_{r,\Omega}(u), \qquad \mu = 0, 1, \tag{2.15}$$

provided that $\mathbb{B}_{r,\Omega}(u)$ is finite.

With the aid of (2.3), we could prove the result (2.15) by the same procedure as in the proof of Theorem 2.2.

Remark 2.1. The result (2.15) generalizes the result (3.33) of [15], which is for two-dimensional Legendre approximation. On the other hand, Bernardi and Mady also considered the n-dimensional Legendre orthogonal approximation, with the estimate (see Remark 2.16 of [3])

$$\|P^{1,0}_{N,\Omega}u - u\|^2_{\mu,\Omega} \leq cN^{2\mu-2r}||u||^2_{H^r(\Omega)}, \qquad \mu = 0, 1, r \geq 1.$$

Since there exist the weights in the quantity $\mathbb{B}_{r,\Omega}(u)$, the new estimate (2.15) improves the existing result with $r \geq 3$.

For numerical solutions of mixed boundary value problems, we introduce certain unusual projections. For instance, we set

$${}^0H^1(\Omega) = \Big\{u \in H^1(\Omega) \mid u(1, x_2, x_3) = u(x_1, 1, x_3) = u(x_1, x_2, 1) = 0\Big\},$$

and ${}^0V_N(\Omega) = {}^0H^1(\Omega) \cap V_N(\Omega)$. The orthogonal projection ${}^0P^1_{N,\Omega} : {}^0H^1(\Omega) \to {}^0V_N(\Omega)$ is defined by

$$(\nabla({}^0P^1_{N,\Omega}u - u), \nabla\phi)_\Omega = 0, \qquad \forall \phi \in {}^0V_N(\Omega).$$

With the aid of (2.4), an argument similar to the proof of Theorem 2.2 leads to the following result.

Theorem 2.4. *If $u \in {}^0H^1(\Omega)$ and integers $1 \le r \le N+1$, then*

$$\|{}^0P^1_{N,\Omega}u - u\|^2_{\mu,\Omega} \le cN^{2\mu-2r}\mathbb{B}_{r,\Omega}(u), \qquad \mu = 0, 1, \tag{2.16}$$

provided that $\mathbb{B}_{r,\Omega}(u)$ is finite.

Remark 2.2. Generally, we denote by $\partial^*\Omega$ a non-empty union of several faces of the cube Ω. Let

$$H^1_{\partial^*\Omega,0}(\Omega) = \Big\{u \in H^1(\Omega) \mid u(x_1, x_2, x_3) = 0 \text{ on } \partial^*\Omega\Big\},$$
$$V_N^{\partial^*\Omega,0}(\Omega) = H^1_{\partial^*\Omega,0}(\Omega) \cap V_N(\Omega).$$

The orthogonal projection $P^{1,\partial^*\Omega,0}_{N,\Omega} : H^1_{\partial^*\Omega,0}(\Omega) \to V_N^{\partial^*\Omega,0}(\Omega)$ is defined by

$$(\nabla(P^{1,\partial^*\Omega,0}_{N,\Omega}u - u), \nabla\phi)_\Omega = 0, \qquad \forall \phi \in V_N^{\partial^*\Omega,0}(\Omega).$$

If $u \in H^1_{\partial^*\Omega,0}(\Omega)$ and integers $1 \le r \le N+1$, then

$$\|P^{1,\partial^*\Omega,0}_{N,\Omega}u - u\|^2_{\mu,\Omega} \le cN^{2\mu-2r}\mathbb{B}_{r,\Omega}(u), \qquad \mu = 0, 1, \tag{2.17}$$

as long as that $\mathbb{B}_{r,\Omega}(u)$ is finite.

3. Legendre Spectral Method in Three-Dimensions

In this section, we propose the Legendre spectral method for three-dimensional problems.

3.1. Steady problem with inhomogeneous Dirichlet boundary condition

Let $d \ge 0$. We consider the following steady problem,

$$\begin{cases} -\Delta W(x_1, x_2, x_3) + dW(x_1, x_2, x_3) = F(x_1, x_2, x_3), & \text{in } \Omega, \\ W(x_1, x_2, x_3) = g(x_1, x_2, x_3), & \text{on } \partial\Omega, \end{cases} \tag{3.1}$$

where F and g are given functions. More precisely, on the six faces of $\partial\Omega$,

$$\begin{aligned} &g(1, x_2, x_3) = g_1(x_2, x_3), \quad g(x_1, 1, x_3) = g_2(x_1, x_3), \quad g(x_1, x_2, 1) = g_3(x_1, x_2), \\ &g(-1, x_2, x_3) = g_4(x_2, x_3), \quad g(x_1, -1, x_3) = g_5(x_1, x_3), \quad g(x_1, x_2, -1) = g_6(x_1, x_2), \end{aligned} \tag{3.2}$$

while, at the twelve edges of $\partial\Omega$,

$$\begin{aligned} &g(x_1, 1, 1) = g_{11}(x_1), && g(x_1, -1, 1) = g_{12}(x_1), \\ &g(x_1, -1, -1) = g_{13}(x_1), && g(x_1, 1, -1) = g_{14}(x_1), \\ &g(1, x_2, 1) = g_{21}(x_2), && g(-1, x_2, 1) = g_{22}(x_2), \\ &g(-1, x_2, -1) = g_{23}(x_2), && g(1, x_2, -1) = g_{24}(x_2), \\ &g(1, 1, x_3) = g_{31}(x_3), && g(-1, 1, x_3) = g_{32}(x_3), \\ &g(-1, -1, x_3) = g_{33}(x_3), && g(1, -1, x_3) = g_{34}(x_3). \end{aligned} \tag{3.3}$$

Assume that the boundary value $g(x_1,x_2,x_3)$ satisfies the consistent condition, namely,

$$\begin{array}{ll}
g_2(x_1,1)=g_3(x_1,1)=g_{11}(x_1), & g_3(x_1,-1)=g_5(x_1,1)=g_{12}(x_1),\\
g_5(x_1,-1)=g_6(x_1,-1)=g_{13}(x_1), & g_6(x_1,1)=g_2(x_1,-1)=g_{14}(x_1),\\
g_1(x_2,1)=g_3(1,x_2)=g_{21}(x_2), & g_3(-1,x_2)=g_4(x_2,1)=g_{22}(x_2),\\
g_4(x_2,-1)=g_6(-1,x_2)=g_{23}(x_2), & g_6(1,x_2)=g_1(x_2,-1)=g_{24}(x_2),\\
g_1(1,x_3)=g_2(1,x_3)=g_{31}(x_3), & g_2(-1,x_3)=g_4(1,x_3)=g_{32}(x_3),\\
g_4(-1,x_3)=g_5(-1,x_3)=g_{33}(x_3), & g_5(1,x_3)=g_1(-1,x_3)=g_{34}(x_3).
\end{array} \tag{3.4}$$

In other words, $g(x_1,x_2,x_3)$ is continuous on $\partial\Omega$.

We shall reformulate the inhomogeneous boundary value problem (3.1) to a homogeneous boundary value problem. To do this, we introduce three auxiliary functions. The first function W_F corresponds to the six faces,

$$\begin{aligned}
&W_F(x_1,x_2,x_3)\\
=&\frac{1}{2}\Big((1+x_1)W(1,x_2,x_3)+(1+x_2)W(x_1,1,x_3)\\
&+(1+x_3)W(x_1,x_2,1)+(1-x_1)W(-1,x_2,x_3)\\
&+(1-x_2)W(x_1,-1,x_3)+(1-x_3)W(x_1,x_2,-1)\Big).
\end{aligned} \tag{3.5}$$

The second function W_E corresponds to the twelve edges,

$$\begin{aligned}
&W_E(x_1,x_2,x_3)\\
=&-\frac{1}{4}\Big((1+x_1)(1+x_2)W(1,1,x_3)+(1-x_1)(1+x_2)W(-1,1,x_3)\\
&+(1-x_1)(1-x_2)W(-1,-1,x_3)+(1+x_1)(1-x_2)W(1,-1,x_3)\\
&+(1+x_1)(1+x_3)W(1,x_2,1)+(1-x_1)(1+x_3)W(-1,x_2,1)\\
&+(1-x_1)(1-x_3)W(-1,x_2,-1)+(1+x_1)(1-x_3)W(1,x_2,-1)\\
&+(1+x_2)(1+x_3)W(x_1,1,1)+(1-x_2)(1+x_3)W(x_1,-1,1)\\
&+(1-x_2)(1-x_3)W(x_1,-1,-1)+(1+x_2)(1-x_3)W(x_1,1,-1)\Big).
\end{aligned} \tag{3.6}$$

The third function W_V corresponds to the eight vertices,

$$\begin{aligned}
&W_V(x_1,x_2,x_3)\\
=&\frac{1}{8}\Big((1+x_1)(1+x_2)(1+x_3)W(1,1,1)+(1-x_1)(1+x_2)(1+x_3)W(-1,1,1)\\
&+(1-x_1)(1-x_2)(1+x_3)W(-1,-1,1)+(1+x_1)(1-x_2)(1+x_3)W(1,-1,1)\\
&+(1+x_1)(1+x_2)(1-x_3)W(1,1,-1)+(1-x_1)(1+x_2)(1-x_3)W(-1,1,-1)\\
&+(1-x_1)(1-x_2)(1-x_3)W(-1,-1,-1)+(1+x_1)(1-x_2)(1-x_3)W(1,-1,-1)\Big).
\end{aligned} \tag{3.7}$$

Remark 3.1. We have from (3.2), (3.3) and (3.5)-(3.7) that

$$\begin{aligned}
&W_F(x_1,x_2,x_3)\\
=&\frac{1}{2}\Big((1+x_1)g_1(x_2,x_3)+(1+x_2)g_2(x_1,x_3)+(1+x_3)g_3(x_1,x_2)\\
&+(1-x_1)g_4(x_2,x_3)+(1-x_2)g_5(x_1,x_3)+(1-x_3)g_6(x_1,x_2)\Big),
\end{aligned}$$

$$
\begin{aligned}
W_E(x_1,x_2,x_3) =& -\frac{1}{4}\Big((1+x_1)(1+x_2)g_{31}(x_3)+(1-x_1)(1+x_2)g_{32}(x_3)\\
&+(1-x_1)(1-x_2)g_{33}(x_3)+(1+x_1)(1-x_2)g_{34}(x_3)\\
&+(1+x_1)(1+x_3)g_{21}(x_2)+(1-x_1)(1+x_3)g_{22}(x_2)\\
&+(1-x_1)(1-x_3)g_{23}(x_2)+(1+x_1)(1-x_3)g_{24}(x_2)\\
&+(1+x_2)(1+x_3)g_{11}(x_1)+(1-x_2)(1+x_3)g_{12}(x_1)\\
&+(1-x_2)(1-x_3)g_{13}(x_1)+(1+x_2)(1-x_3)g_{14}(x_1)\Big),\\
W_V(x_1,x_2,x_3) =& \frac{1}{8}\Big((1+x_1)(1+x_2)(1+x_3)g_{31}(1)+(1-x_1)(1+x_2)(1+x_3)g_{32}(1)\\
&+(1-x_1)(1-x_2)(1+x_3)g_{33}(1)+(1+x_1)(1-x_2)(1+x_3)g_{34}(1)\\
&+(1+x_1)(1+x_2)(1-x_3)g_{31}(-1)+(1-x_1)(1+x_2)(1-x_3)g_{32}(-1)\\
&+(1-x_1)(1-x_2)(1-x_3)g_{33}(-1)+(1+x_1)(1-x_2)(1-x_3)g_{34}(-1)\Big).
\end{aligned}
$$

Define the function corresponding to the boundary $\partial\Omega$ by

$$W_B(x_1,x_2,x_3)=W_F(x_1,x_2,x_3)+W_E(x_1,x_2,x_3)+W_V(x_1,x_2,x_3). \tag{3.8}$$

According to Remark 3.1 and the consistency (3.4), we verify that

$$W(x_1,x_2,x_3)=W_B(x_1,x_2,x_3), \qquad \text{on } \partial\Omega.$$

We next make the variable transformation

$$
\begin{aligned}
&W(x_1,x_2,x_3)=U(x_1,x_2,x_3)+W_B(x_1,x_2,x_3),\\
&f(x_1,x_2,x_3)=F(x_1,x_2,x_3)+\Delta W_B(x_1,x_2,x_3)-dW_B(x_1,x_2,x_3).
\end{aligned}
$$

Then, (3.1) is changed to

$$
\begin{cases}
-\Delta U(x_1,x_2,x_3)+dU(x_1,x_2,x_3)=f(x_1,x_2,x_3), & \text{in } \Omega,\\
U(x_1,x_2,x_3)=0, & \text{on } \partial\Omega.
\end{cases} \tag{3.9}
$$

A weak formulation of (3.9) is to seek solution $U\in H_0^1(\Omega)$ such that

$$(\nabla U,\nabla v)_\Omega+d(U,v)_\Omega=(f,v)_\Omega, \qquad \forall v\in H_0^1(\Omega). \tag{3.10}$$

The Legendre spectral scheme for (3.10) is to find $u_N\in V_N^0(\Omega)$ such that

$$(\nabla u_N,\nabla\phi)_\Omega+d(u_N,\phi)_\Omega=(f,\phi)_\Omega, \qquad \forall\phi\in V_N^0(\Omega). \tag{3.11}$$

The numerical solution of problem (3.1) is given by

$$w_N(x_1,x_2,x_3)=u_N(x_1,x_2,x_3)+W_B(x_1,x_2,x_3). \tag{3.12}$$

We now deal with the convergence. Let $U_N=P_{N,\Omega}^{1,0}U$. We have from (3.10) that

$$(\nabla U_N,\nabla\phi)_\Omega+d(U_N,\phi)_\Omega=d(U_N-U,\phi)_\Omega+(f,\phi)_\Omega, \qquad \forall\phi\in V_N^0(\Omega). \tag{3.13}$$

Let $\widetilde{U}_N=u_N-U_N$. By subtracting (3.13) from (3.11), we obtain

$$(\nabla\widetilde{U}_N,\nabla\phi)_\Omega+d(\widetilde{U}_N,\phi)_\Omega=-d(U_N-U,\phi)_\Omega, \qquad \forall\phi\in V_N^0(\Omega).$$

Taking $\phi = \widetilde{U}_N$ in the above equation, we use (2.15) with $\mu = 0$ to deduce that

$$||\nabla \widetilde{U}_N||_\Omega^2 + d||\widetilde{U}_N||_\Omega^2 \le cdN^{2-2r}\mathbb{B}_{r-1,\Omega}(U).$$

The above inequality, together with (2.15) with $\mu = 0, 1$, leads to

$$||\nabla(U - u_N)||_\Omega^2 + d||U - u_N||_\Omega^2 \le cN^{2-2r}\Big(\mathbb{B}_{r,\Omega}(U) + d\mathbb{B}_{r-1,\Omega}(U)\Big). \tag{3.14}$$

By virtue of (3.14), a standard duality argument shows

$$||U - u_N||^2_{H^\mu(\Omega)} \le cN^{2\mu-2r}(\mathbb{B}_{r,\Omega}(U) + d\mathbb{B}_{r-1,\Omega}(U)), \qquad \mu = 0, 1.$$

This, together with (3.12), implies

$$\begin{aligned} &||W - w_N||^2_{H^\mu(\Omega)} \\ \le\ & cN^{2\mu-2r}\Big(\mathbb{B}_{r,\Omega}(W) + \mathbb{B}_{r,\Omega}(W_B) + d\mathbb{B}_{r-1,\Omega}(W) + d\mathbb{B}_{r-1,\Omega}(W_B)\Big), \quad \mu = 0, 1. \end{aligned} \tag{3.15}$$

3.2. Unsteady problem with inhomogeneous Dirichlet boundary condition

We consider the following unsteady problem with constant d,

$$\begin{cases} \partial_t W(x_1, x_2, x_3, t) - \Delta W(x_1, x_2, x_3, t) & \\ \qquad\qquad = dW(x_1, x_2, x_3, t) + F(x_1, x_2, x_3, t), & \text{in } \Omega, \quad 0 < t \le T, \\ W(x_1, x_2, x_3, t) = g(x_1, x_2, x_3, t), & \text{on } \partial\Omega, \quad 0 < t \le T, \\ W(x_1, x_2, x_3, 0) = W_0(x_1, x_2, x_3), & \text{on } \Omega \cup \partial\Omega, \end{cases} \tag{3.16}$$

where F, g and W_0 are given functions. More precisely, like (3.2) and (3.3), we have $g(1, x_2, x_3, t) = g_1(x_2, x_3, t)$ and $g(x_1, 1, 1, t) = g_{11}(x_1, t)$, etc.. We also suppose that g fulfills the consistent condition like (3.4). Furthermore, we define the functions $W_F(x_1, x_2, x_3, t), W_E(x_1, x_2, x_3, t)$ and $W_V(x_1, x_2, x_3, t)$ in the same manner as for (3.5)-(3.7), and

$$W_B(x_1, x_2, x_3, t) = W_F(x_1, x_2, x_3, t) + W_E(x_1, x_2, x_3, t) + W_V(x_1, x_2, x_3, t). \tag{3.17}$$

Clearly, $W(x_1, x_2, x_3, t) = W_B(x_1, x_2, x_3, t)$ on $\partial\Omega$.

Let $W_{0,B} = W_B(x_1, x_2, x_3, 0)$. We make the variable transformation

$$W = U + W_B, \qquad W_0 = U_0 + W_B, \qquad f = F - \partial_t W_B + \Delta W_B + dW_B.$$

Then, (3.17) is reformed to

$$\begin{cases} \partial_t U(x_1, x_2, x_3, t) - \Delta U(x_1, x_2, x_3, t) & \\ \qquad\qquad = dU(x_1, x_2, x_3, t) + f(x_1, x_2, x_3, t), & \text{in } \Omega,\ 0 < t \le T, \\ U(x_1, x_2, x_3, t) = 0, & \text{on } \partial\Omega, \ \ 0 < t \le T, \\ U(x_1, x_2, x_3, 0) = U_0(x_1, x_2, x_3), & \text{on } \Omega \cup \partial\Omega. \end{cases} \tag{3.18}$$

A weak formulation of (3.18) is to seek solution $U \in L^\infty(0, T; L^2(\Omega)) \cap L^2(0, T; H_0^1(\Omega))$, such that for $0 < t \le T$:

$$\begin{cases} (\partial_t U(t), v)_\Omega + (\nabla U(t), \nabla v)_\Omega = d(U(t), v)_\Omega + (f(t), v)_\Omega, & \forall\ v \in H_0^1(\Omega), \\ U(0) = U_0. \end{cases} \tag{3.19}$$

The spectral scheme for (3.19) is to find $u_N \in V_N^0(\Omega)$ for all $0 \le t \le T$, such that

$$\begin{cases} (\partial_t u_N(t), \phi)_\Omega + (\nabla u_N(t), \nabla\phi)_\Omega = d(u_N(t), \phi)_\Omega + (f(t), \phi)_\Omega, & \forall\, \phi \in V_N^0(\Omega), \quad 0 < t \le T \\ u_N(0) = P_{N,\Omega} U_0. \end{cases} \tag{3.20}$$

The numerical solution of problem (3.16) is given by

$$w_N(x_1, x_2, x_3, t) = u_N(x_1, x_2, x_3, t) + W_B(x_1, x_2, x_3, t). \tag{3.21}$$

We now deal with the convergence. Let $U_N = P^{1,0}_{N,\Omega} U$. We obtain from (3.19) that

$$\begin{cases} (\partial_t U_N(t), \phi)_\Omega + (\nabla U_N(t), \nabla\phi)_\Omega \\ \qquad = d(U_N(t), \phi)_\Omega + G_1(t,\phi) + G_2(t,\phi) + (f(t), \phi)_\Omega, \qquad \forall\, \phi \in V_N^0(\Omega), \quad 0 < t \le T \\ U_N(0) = P^{1,0}_{N,\Omega} U_0, \end{cases} \tag{3.22}$$

where

$$G_1(t,\phi) = (\partial_t U_N(t) - \partial_t U(t), \phi)_\Omega, \qquad G_2(t,\phi) = d(U(t) - U_N(t), \phi)_\Omega.$$

Putting $\widetilde{U}_N = u_N - U_N$ and subtracting (3.22) from (3.20), we obtain

$$\begin{cases} (\partial_t \widetilde{U}_N(t), \phi)_\Omega + (\nabla \widetilde{U}_N(t), \nabla\phi)_\Omega \\ \qquad = d(\widetilde{U}_N(t), \phi)_\Omega - G_1(t,\phi) - G_2(t,\phi), \quad \forall\, \phi \in V_N^0(\Omega), \quad 0 < t \le T \\ \widetilde{U}_N(0) = P_{N,\Omega} U_0 - P^{1,0}_{N,\Omega} U_0. \end{cases} \tag{3.23}$$

Take $\phi = 2\widetilde{U}_N$ in (3.23). We use the Cauchy inequality, the Poincar$\acute{e}$ inequality and (2.15) with $\mu = 0$ successively, to deduce that

$$\begin{aligned} &\partial_t ||\widetilde{U}_N(t)||^2_\Omega + ||\nabla \widetilde{U}_N(t)||^2_\Omega \\ &\le 2d||\widetilde{U}_N(t)||^2_\Omega + cN^{2-2r}\Big(\mathbb{B}_{r-1,\Omega}(\partial_t U(t)) + d^2 \mathbb{B}_{r-1,\Omega}(U(t))\Big). \end{aligned} \tag{3.24}$$

For notational convenience, let

$$E(u(t)) = ||u(t)||^2_\Omega + \int_0^t ||\nabla u(\xi)||^2_\Omega d\xi.$$

Then (3.24) reads

$$\partial_t E(\widetilde{U}_N(t)) \le 2dE(\widetilde{U}_N(t)) + cN^{2-2r}\Big(\mathbb{B}_{r-1,\Omega}(\partial_t U(t)) + d^2 \mathbb{B}_{r-1,\Omega}(U(t)\Big),$$

or equivalently,

$$\partial_t (E(\widetilde{U}_N(t)) e^{-2dt}) \le c e^{-2dt} N^{2-2r}\Big(\mathbb{B}_{r-1,\Omega}(\partial_t U(t)) + d^2 \mathbb{B}_{r-1,\Omega}(U(t))\Big). \tag{3.25}$$

Furthermore, we use (2.5) and (2.15) to verify that

$$\begin{aligned} ||\widetilde{U}_N(0)||^2_\Omega &\le 2||P_{N,\Omega} U_0 - U_0||^2_\Omega + 2||U_0 - P^{1,0}_{N,\Omega} U_0||^2_\Omega \\ &\le cN^{2-2r}\Big(\mathbb{A}_{r-1,\Omega}(U_0) + \mathbb{B}_{r-1,\Omega}(U_0)\Big). \end{aligned} \tag{3.26}$$

Integrating (3.25) with respect to t and using (3.26), we obtain

$$E(\widetilde{U}_N(t)) \le cN^{2-2r} R_{r,\Omega}(U, U_0, d, t) \tag{3.27}$$

where

$$\begin{aligned} &R_{r,\Omega}(U, U_0, d, t) \\ &= e^{2dt}\Big(\int_0^t e^{-2d\xi}(\mathbb{B}_{r-1,\Omega}(\partial_\xi U(\xi)) + d^2\mathbb{B}_{r-1,\Omega}(U(\xi)))d\xi + \mathbb{A}_{r-1,\Omega}(U_0) + \mathbb{B}_{r-1,\Omega}(U_0)\Big). \end{aligned}$$

Finally, a combination of (3.27), (2.5) and (2.15) with $\mu = 1$ leads to

$$E(U(t) - u_N(t)) \le cN^{2-2r}\big(R_{r,\Omega}(U, U_0, d, t) + \int_0^t \mathbb{B}_{r,\Omega}(U(\xi))d\xi + \mathbb{B}_{r-1,\Omega}(U(t))\big). \tag{3.28}$$

This estimate, together with (3.21), gives

$$\begin{aligned} &E(W(t) - w_N(t)) \\ &\le cN^{2-2r}\Big(R_{r,\Omega}(W, W_0, d, t) + R_{r,\Omega}(W_B, W_{0,B}, d, t) \\ &\quad + \int_0^t (\mathbb{B}_{r,\Omega}(W(\xi)) + \mathbb{B}_{r,\Omega}(W_B(\xi)))d\xi + \mathbb{B}_{r-1,\Omega}(W(t)) + \mathbb{B}_{r-1,\Omega}(W_B(t))\Big). \end{aligned} \tag{3.29}$$

3.3. Steady problem with mixed inhomogeneous boundary condition

Let $d, \alpha \ge 0$, and

$$\partial^*\Omega = \Big\{\ (x_1, x_2, x_3) \in \partial\Omega \mid x_1 = 1,\ \text{or } x_2 = 1,\ \text{or } x_3 = 1\ \Big\}.$$

We consider the following mixed inhomogeneous boundary value problem,

$$\begin{cases} -\Delta W(x_1, x_2, x_3) + dW(x_1, x_2, x_3) = F(x_1, x_2, x_3), & \text{in } \Omega, \\ W(x_1, x_2, x_3) = g(x_1, x_2, x_3), & \text{on } \partial^*\Omega, \\ \partial_n W(x_1, x_2, x_3) + \alpha W(x_1, x_2, x_3) = H(x_1, x_2, x_3), & \text{on } \partial\Omega \setminus \partial^*\Omega, \end{cases} \tag{3.30}$$

where F, g and H are given functions. More precisely, on the three faces of $\partial^*\Omega$,

$$g(1, x_2, x_3) = g_1(x_2, x_3), \quad g(x_1, 1, x_3) = g_2(x_1, x_3), \quad g(x_1, x_2, 1) = g_3(x_1, x_2), \tag{3.31}$$

while, at the three edges of $\partial^*\Omega$,

$$g(x_1, 1, 1) = g_{11}(x_1), \quad g(1, x_2, 1) = g_{21}(x_2), \quad g(1, 1, x_3) = g_{31}(x_3). \tag{3.32}$$

Assume that the boundary value $g(x_1, x_2, x_3)$ satisfies the consistent condition, namely,

$$\begin{aligned} &g_2(x_1, 1) = g_3(x_1, 1) = g_{11}(x_1), \qquad g_1(x_2, 1) = g_3(1, x_2) = g_{21}(x_2), \\ &g_1(1, x_3) = g_2(1, x_3) = g_{31}(x_3). \end{aligned} \tag{3.33}$$

In other words, $g(x_1, x_2, x_3)$ is continuous on $\partial^*\Omega$.

We shall change the inhomogeneous boundary value problem (3.30) to a boundary value problem with homogeneous Dirichlet boundary condition on $\partial^*\Omega$. For this purpose, we introduce three auxiliary functions. The first function W_F corresponds to the three faces,

$$\begin{aligned} &W_F(x_1, x_2, x_3) \\ &= \frac{1}{2}\Big((1 + x_1)W(1, x_2, x_3) + (1 + x_2)W(x_1, 1, x_3) + (1 + x_3)W(x_1, x_2, 1)\Big). \end{aligned} \tag{3.34}$$

The second function W_E corresponds to the three edges,

$$\begin{aligned} W_E(x_1,x_2,x_3) = -\frac{1}{4}\Big(&(1+x_1)(1+x_2)W(1,1,x_3)+(1+x_1)(1+x_3)W(1,x_2,1)\\ &+(1+x_2)(1+x_3)W(x_1,1,1)\Big). \end{aligned} \tag{3.35}$$

The third function W_V corresponds to the vertex with the coordinates $x_1 = x_2 = x_3 = 1$,

$$W_V(x_1,x_2,x_3) = \frac{1}{8}(1+x_1)(1+x_2)(1+x_3)W(1,1,1). \tag{3.36}$$

Remark 3.2. We have from (3.31), (3.32) and (3.34)-(3.36) that

$$\begin{aligned} W_F(x_1,x_2,x_3) &= \frac{1}{2}\big((1+x_1)g_1(x_2,x_3)+(1+x_2)g_2(x_1,x_3)+(1+x_3)g_3(x_1,x_2)\big),\\ W_E(x_1,x_2,x_3) &= -\frac{1}{4}\big((1+x_1)(1+x_2)g_{31}(x_3)+(1+x_1)(1+x_3)g_{21}(x_2)\\ &\quad +(1+x_2)(1+x_3)g_{11}(x_1)\big),\\ W_V(x_1,x_2,x_3) &= \frac{1}{8}(1+x_1)(1+x_2)(1+x_3)g_{31}(1). \end{aligned}$$

We define the function corresponding to the boundary $\partial\Omega$, by

$$W_B(x_1,x_2,x_3) = W_F(x_1,x_2,x_3)+W_E(x_1,x_2,x_3)+W_V(x_1,x_2,x_3). \tag{3.37}$$

According to Remark 3.2 and the consistency (3.33), we verify that $W(x_1,x_2,x_3) = W_B(x_1, x_2, x_3)$ on $\partial^*\Omega$.

We now make the variable transformation

$$W = U + W_B, \qquad f = F + \Delta W_B - dW_B, \qquad h = H - \partial_n W_B - \alpha W_B.$$

Then, (3.30) is reformed to

$$\begin{cases} -\Delta U(x_1,x_2,x_3) + dU(x_1,x_2,x_3) = f(x_1,x_2,x_3), & \text{in } \Omega,\\ U(x_1,x_2,x_3) = 0, & \text{on } \partial^*\Omega,\\ \partial_n U(x_1,x_2,x_3) + \alpha U(x_1,x_2,x_3) = h(x_1,x_2,x_3), & \text{on } \partial\Omega \setminus \partial^*\Omega. \end{cases} \tag{3.38}$$

A weak formulation of (3.38) is to seek solution $U \in {}^0H^1(\Omega)$ such that

$$\begin{aligned} &(\nabla U,\nabla v)_\Omega + d(U,v)_\Omega + \alpha\int_{\partial\Omega\setminus\partial^*\Omega} UvdS - \int_{\partial\Omega\setminus\partial^*\Omega} hvdS\\ =&(f,v)_\Omega, \quad \forall v \in {}^0H^1(\Omega). \end{aligned} \tag{3.39}$$

The Legendre spectral scheme for (3.39) is to find $u_N \in {}^0V_N(\Omega)$ such that

$$\begin{aligned} &(\nabla u_N,\nabla \phi)_\Omega + d(u_N,\phi)_\Omega + \alpha\int_{\partial\Omega\setminus\partial^*\Omega} u_N\phi dS - \int_{\partial\Omega\setminus\partial^*\Omega} h\phi dS\\ =&(f,\phi)_\Omega, \quad \forall \phi \in {}^0V_N(\Omega). \end{aligned} \tag{3.40}$$

The numerical solution of problem (3.30) is given by

$$w_N(x_1,x_2,x_3) = u_N(x_1,x_2,x_3) + W_B(x_1,x_2,x_3). \tag{3.41}$$

We next analyze the convergence. Let $U_N = {}^0P^1_{N,\Omega}U$. We have from (3.39) that

$$\begin{aligned}&(\nabla U_N, \nabla\phi)_\Omega + d(U_N,\phi)_\Omega + \alpha\int_{\partial\Omega\backslash\partial^*\Omega} U_N\phi dS - \int_{\partial\Omega\backslash\partial^*\Omega} h\phi dS\\ &= d(U_N - U,\phi)_\Omega + \alpha\int_{\partial\Omega\backslash\partial^*\Omega}(U_N - U)\phi dS + (f,\phi)_\Omega, \qquad \forall\phi\in {}^0V_N(\Omega).\end{aligned} \tag{3.42}$$

Let $\widetilde{U}_N = u_N - U_N$. By subtracting (3.42) from (3.40), we obtain

$$\begin{aligned}&(\nabla\widetilde{U}_N, \nabla\phi)_\Omega + d(\widetilde{U}_N,\phi)_\Omega + \alpha\int_{\partial\Omega\backslash\partial^*\Omega} \widetilde{U}_N\phi dS\\ &= -d(U_N - U,\phi)_\Omega - \alpha\int_{\partial\Omega\backslash\partial^*\Omega}(U_N - U)\phi dS, \qquad \forall\phi\in {}^0V^0_N(\Omega).\end{aligned}$$

Take $\phi = \widetilde{U}_N$ in the above equation. We could use (2.16) with $\mu = 0$ to estimate the first term of the right side of the resulting equality as before. Moreover, by using the trace theorem, the Poincaré inequality and (2.16) with $\mu = 1$ successively, we deduce that

$$\begin{aligned}&\left|\alpha\int_{\partial\Omega\backslash\partial^*\Omega}(U_N - U)\widetilde{U}_N\phi dS\right|\\ &\le \frac{1}{2}||\nabla\widetilde{U}_N||^2_\Omega + c\alpha^2||U_N - U||^2_{1,\Omega} \le \frac{1}{2}||\nabla\widetilde{U}_N||^2_\Omega + c\alpha^2 N^{2-2r}\mathbb{B}_{r,\Omega}(U).\end{aligned}$$

Therefore, we obtain

$$\begin{aligned}&||\nabla\widetilde{U}_N||^2_\Omega + d||\widetilde{U}_N||^2_\Omega + \alpha\int_{\partial\Omega\backslash\partial^*\Omega}\widetilde{U}^2_N dS\\ &\le cN^{2-2r}\Big((1+\alpha^2)\mathbb{B}_{r,\Omega}(U) + d\mathbb{B}_{r-1,\Omega}(U)\Big).\end{aligned} \tag{3.43}$$

Finally, with the aid of the trace theorem, the Poincaré inequality and (2.16) with $\mu = 0, 1$, we obtain from (3.43) that

$$\begin{aligned}&||\nabla(U - u_N)||^2_\Omega + d||U - u_N||^2_\Omega + \alpha\int_{\partial\Omega\backslash\partial^*\Omega}(U - U_N)^2 dS\\ &\le cN^{2-2r}((1+\alpha^2)\mathbb{B}_{r,\Omega}(U) + d\mathbb{B}_{r-1,\Omega}(U)).\end{aligned} \tag{3.44}$$

By virtue of the above result, a standard duality argument and (3.41), we obtain

$$\begin{aligned}||W - w_N||^2_{H^\mu(\Omega)} \le & cN^{2\mu-2r}\Big[(1+\alpha)\Big(\mathbb{B}_{r,\Omega}(W) + \mathbb{B}_{r,\Omega}(W_B)\Big)\\ &+ d\Big(\mathbb{B}_{r-1,\Omega}(W) + \mathbb{B}_{r-1,\Omega}(W_B)\Big)\Big], \qquad \mu = 0, 1.\end{aligned} \tag{3.45}$$

Remark 3.3. If $\partial^*\Omega$ is a non-empty union of several faces of the cube Ω, we still can use certain transformation like (3.37), to derive the weak form of mixed inhomogeneous boundary value problem and the corresponding spectral scheme like (3.40). Also, we could use the result (2.17) to obtain the same error estimate as (3.45). The key point is to design different function $W_B(x_1, x_2, x_3)$ for different $\partial^*\Omega$, properly.

(i) If $\partial^*\Omega = \{(x_1, x_2, x_3) \in \partial\Omega \mid x_1 = 1\}$, then

$$W_B(x_1, x_2, x_3) = \frac{1}{2}(1 + x_1)W(1, x_2, x_3). \tag{3.46}$$

(ii) If $\partial^*\Omega = \{(x_1,x_2,x_3)\in\partial\Omega \mid x_1=1 \text{ or } x_2=1\}$, then

$$W_B(x_1,x_2,x_3) = \frac{1}{2}\Big((1+x_1)W(1,x_2,x_3)+(1+x_2)W(x_1,1,x_3)\Big) - \frac{1}{4}(1+x_1)(1+x_2)W(1,1,x_3). \tag{3.47}$$

(iii) If $\partial^*\Omega = \Big\{(x_1,x_2,x_3)\in\partial\Omega \mid x_1=1, \text{ or } x_2=1, \quad \text{or } x_3=-1, \text{ or } x_3=1\Big\}$, then

$$W_B(x_1,x_2,x_3) = W_F(x_1,x_2,x_3)+W_E(x_1,x_2,x_3)+W_V(x_1,x_2,x_3), \tag{3.48}$$

with

$$\begin{aligned} W_F(x_1,x_2,x_3) = \frac{1}{2}\Big(&(1-x_3)W(x_1,x_2,-1)+(1+x_1)W(1,x_2,x_3)\\ &+(1+x_2)W(x_1,1,x_3)+(1+x_3)W(x_1,x_2,1)\Big), \end{aligned}\tag{3.49}$$

$$\begin{aligned} W_E(x_1,x_2,x_3) = -\frac{1}{4}\Big(&(1+x_1)(1-x_3)W(1,x_2,-1)+(1+x_1)(1+x_3)W(1,x_2,1)\\ &+(1+x_2)(1-x_3)W(x_1,1,-1)+(1+x_2)(1+x_3)W(x_1,1,1)\\ &+(1+x_1)(1+x_2)W(1,1,x_3)\Big), \end{aligned}\tag{3.50}$$

$$\begin{aligned} W_V(x_1,x_2,x_3) = \frac{1}{8}\Big(&(1+x_1)(1+x_2)(1+x_3)W(1,1,1)\\ &+(1+x_1)(1+x_2)(1-x_3)W(1,1,-1)\Big). \end{aligned}\tag{3.51}$$

(iv) If $\partial^*\Omega = \{(x_1,x_2,x_3)\in\partial\Omega \mid x_1=1, \text{ or } x_2=-1, \quad \text{or } x_2=1, \quad \text{or } x_3=-1, \text{ or } x_3=1\}$, then

$$W_B(x_1,x_2,x_3) = W_F(x_1,x_2,x_3)+W_E(x_1,x_2,x_3)+W_V(x_1,x_2,x_3), \tag{3.52}$$

with

$$\begin{aligned} &W_F(x_1,x_2,x_3)\\ =\ &\frac{1}{2}\Big((1-x_2)W(x_1,-1,x_3)+(1-x_3)W(x_1,x_2,-1)\\ &\quad +(1+x_1)W(1,x_2,x_3)+(1+x_2)W(x_1,1,x_3)+(1+x_3)W(x_1,x_2,1)\Big).\\ &W_E(x_1,x_2,x_3)\\ =\ &-\frac{1}{4}\Big((1+x_1)(1-x_2)W(1,-1,x_3)+(1+x_1)(1+x_2)W(1,1,x_3)\\ &\quad +(1+x_1)(1-x_3)W(1,x_2,-1)+(1+x_1)(1+x_3)W(1,x_2,1)\\ &\quad +(1+x_2)(1-x_3)W(x_1,1,-1)+(1+x_2)(1+x_3)W(x_1,1,1)\\ &\quad +(1-x_2)(1-x_3)W(x_1,-1,-1)+(1-x_2)(1+x_3)W(x_1,-1,1)\Big).\\ &W_V(x_1,x_2,x_3)\\ =\ &\frac{1}{8}\Big((1+x_1)(1+x_2)(1-x_3)W(1,1,-1)+(1+x_1)(1-x_2)(1-x_3)W(1,-1,-1)\\ &+(1+x_1)(1-x_2)(1+x_3)W(1,-1,1)+(1+x_1)(1+x_2)(1+x_3)W(1,1,1)\Big). \end{aligned}$$

(v) The function $W_B(x_1,x_2,x_3)$ for other two cases were already given by (3.8) and (3.37), respectively.

It is noted that in spectral element method, we need the function (3.8) for the interior elements, while we need the functions (3.37), (3.46) and (3.47) for the elements possessing at least one face on the boundary of the considered rectangular domain. These functions, coupled with domain partitions and variable transformations, are also applicable to the spectral element method for hexahedrons.

4. Numerical Results

In this section, we describe the numerical implementations, and present some numerical results confirming the theoretical analysis in the last section.

We first consider spectral scheme (3.11)-(3.12). Let $\eta_k(x) = L_k(x) - L_{k+2}(x),\ 0 \le k \le N-2$. Obviously, $\eta_k(\pm 1) = 0$. In actual computation, we expand the auxiliary numerical solution as

$$u_N(x_1,x_2,x_3) = \sum_{k=0}^{N-2}\sum_{l=0}^{N-2}\sum_{m=0}^{N-2} a_{k,l,m}\eta_k(x_1)\eta_l(x_2)\eta_m(x_3).$$

Besides, $f_{k,l,m} = (f, \eta_k\eta_l\eta_m)_\Omega,\ 0 \le k,l,m \le N-2$. For deriving a compact matrix form of (3.11), we introduce the vectors

$$\begin{aligned}\mathbf{X} =\Big(&a_{0,0,0},\ a_{1,0,0},\ \cdots,\ a_{N-2,0,0},\ \ a_{0,1,0},\ a_{1,1,0},\ \cdots,\ a_{N-2,1,0},\cdots,\\ &a_{0,N-2,0}, a_{1,N-2,0},\cdots, a_{N-2,N-2,0},\\ &a_{0,0,1},\ a_{1,0,1},\ \cdots,\ a_{N-2,0,1},\ \ a_{0,1,1},\ a_{1,1,1},\ \cdots,\ a_{N-2,1,1},\cdots,\\ &a_{0,N-2,1}, a_{1,N-2,1},\cdots, a_{N-2,N-2,1},\cdots,\cdots,\cdots,\\ &a_{0,0,N-2},\ a_{1,0,N-2},\ \cdots,\ a_{N-2,0,N-2}, a_{0,1,N-2},\ a_{1,1,N-2},\ \cdots,\ a_{N-2,1,N-2},\\ &\cdots, a_{0,N-2,N-2}, a_{1,N-2,N-2},\cdots, a_{N-2,N-2,N-2}\Big)^T,\\ \mathbf{F} =\Big(&f_{0,0,0},\ f_{1,0,0},\ \cdots,\ f_{N-2,0,0},\ \ f_{0,1,0},\ f_{1,1,0},\ \cdots,\ f_{N-2,1,0},\cdots,\\ &f_{0,N-2,0}, f_{1,N-2,0},\cdots, f_{N-2,N-2,0},\\ &f_{0,0,1},\ f_{1,0,1},\ \cdots,\ f_{N-2,0,1},\ \ f_{0,1,1},\ f_{1,1,1},\ \cdots,\ f_{N-2,1,1},\cdots,\\ &f_{0,N-2,1}, f_{1,N-2,1},\cdots, f_{N-2,N-2,1},\cdots,\cdots,\cdots,\\ &f_{0,0,N-2},\ f_{1,0,N-2},\ \cdots,\ f_{N-2,0,N-2}, f_{0,1,N-2},\ f_{1,1,N-2},\cdots,\\ &f_{N-2,1,N-2},\cdots, f_{0,N-2,N-2}, f_{1,N-2,N-2},\cdots, f_{N-2,N-2,N-2}\Big)^T.\end{aligned}$$

By taking $\phi = \eta_{k'}(x_1)\eta_{l'}(x_2)\eta_{m'}(x_3)(0 \le k',l',m' \le N-2)$ in (3.11), we obtain the following compact matrix form,

$$(A\otimes B\otimes B + B\otimes A\otimes B + B\otimes B\otimes A + dB\otimes B\otimes B)\mathbf{X} = \mathbf{F}, \tag{4.1}$$

where the symmetrical and sparse matrices $A = (a_{k'k})$ and $B = (b_{k'k})$, with the following entries,

$$a_{k'k} = \int_I \partial_x\eta_k(x)\partial_x\eta_{k'}(x)dx = \begin{cases} 4k+6, & k'=k,\\ 0, & \text{otherwise},\end{cases}$$

$$b_{k'k}=b_{kk'}=\int_I \eta_k(x)\eta_{k'}(x)dx=\begin{cases}-\dfrac{2}{2k+1}, & k'=k-2,\\ \dfrac{2}{2k+1}+\dfrac{2}{2k+5}, & k'=k,\\ 0, & \text{otherwise.}\end{cases}$$

For raising the numerical accuracy, we evaluate the terms $f_{k'l'm'}$ $(0\le k',l',m'\le N-2)$ by using the Legendre-Gauss-Lobatto quadrature with $2N+1$ nodes.

We now use scheme (3.11)-(3.12) to solve (3.1) with $d=0,1$, and the test function

$$W(x_1,x_2,x_3)=(x_1+2x_2+3x_3)\sin(x_1+x_2+x_3). \tag{4.2}$$

We measure the errors of numerical solutions by a discrete norm. Let $\zeta_{N,k}$ and $\rho_{N,k}$ $(0\le k\le N)$ be the nodes and weights of the Legendre-Gauss-Lobatto interpolation, and the quantity

$$\begin{aligned}E_N=&\Big(\sum_{k=0}^{N}\sum_{l=0}^{N}\sum_{m=0}^{N}(W(\zeta_{N,k},\zeta_{N,l},\zeta_{N,m})-w_N(\zeta_{N,k},\zeta_{N,l},\zeta_{N,m}))^2\rho_{N,k}\rho_{N,l}\rho_{N,m}\Big)^{\frac{1}{2}}\\ &\simeq\|W-w_N\|_\Omega.\end{aligned}$$

In Table 4.1, we present the values $\log_{10}E_N$ vs. the mode N. Clearly, the numerical errors decay exponentially as N increases. This fact coincides well with the theoretical analysis, see (3.15).

Table 4.1: Numerical errors of scheme (3.11)-(3.12).

	$N=5$	$N=10$	$N=15$
$d=0$	4.05E-04	7.04E-11	6.70E-15
$d=1$	4.03E-04	7.03E-11	3.27E-15

We next turn to scheme (3.20)-(3.21). We use the Crank-Nicolson discretization in time t, with the mesh step τ. We denote the auxiliary numerical solution by $u_{N,\tau}$. Let $S_\tau=\{\ t\mid t=0,\tau,2\tau,\cdots\}$. The corresponding fully discrete scheme is as follows,

$$\begin{cases}\dfrac{1}{\tau}(u_{N,\tau}(t+\tau)-u_{N,\tau}(t),\phi)_\Omega+\dfrac{1}{2}(\nabla(u_{N,\tau}(t+\tau)+u_{M,\tau}(t)),\nabla\phi)_\Omega\\ =\dfrac{1}{2}d(u_{N,\tau}(t+\tau)+u_{N,\tau}(t),\phi)_\Omega+\dfrac{1}{2}(f(t+\tau)+f(t),\phi)_\Omega, \qquad \phi\in V_N^0(\Omega),\quad t\in S_\tau,\\ u_{N,\tau}(0)=P_{N,\Omega}U_0.\end{cases} \tag{4.3}$$

In actual computation, we expand $u_{N,\tau}(x_1,x_2,x_3,t)$ as

$$u_{N,\tau}(x_1,x_2,x_3,t)=\sum_{k=0}^{N-2}\sum_{l=0}^{N-2}\sum_{m=0}^{N-2}a_{k,l,m}(t)\eta_k(x_1)\eta_l(x_2)\eta_m(x_3). \tag{4.4}$$

Substituting (4.4) into (4.3), and taking $\phi=\eta_{k'}(x_1)\eta_{l'}(x_2)\eta_{m'}(x_3)$ $(0\le k',l',m'\le N-2)$, we obtain the following compact matrix form,

$$\begin{aligned}&(B\otimes B\otimes B+\frac{\tau}{2}(A\otimes B\otimes B+B\otimes A\otimes B+B\otimes B\otimes A-\mu B\otimes B\otimes B))\mathbf{X}(t+\tau)\\ =&\Big(B\otimes B\otimes B-\frac{\tau}{2}(A\otimes B\otimes B+B\otimes A\otimes B+B\otimes B\otimes A-dB\otimes B\otimes B)\Big)\mathbf{X}(t)\\ &+\frac{\tau}{2}\Big(\mathbf{F}(t)+\mathbf{F}(t+\tau)\Big),\end{aligned} \tag{4.5}$$

Table 4.2: Numerical errors of scheme (4.3) with (4.6).

	$\tau = 0.01$	$\tau = 0.005$	$\tau = 0.001$	$\tau = 0.0005$
$N = 5$	4.07E-04	4.07E-04	4.07E-04	4.07E-04
$N = 10$	3.64 E-10	1.15E-10	7.35E-11	1.33E-13
$N = 15$	3.56 E-10	8.91E-11	3.56E-12	5.37E-14

where the matrices A and B are as the same as in (4.1). At each time step, we need to solve a linear system with the unknown coefficients $a_{k,l,m}(t)$. Due to (3.21), the numerical solution of (3.16) is given by

$$w_{N,\tau}(x_1, x_2, x_3, t) = u_{N,\tau}(x_1, x_2, x_3, t) + W_B(x_1, x_2, x_3, t). \tag{4.6}$$

We now use scheme (4.3) with (4.6) to solve (3.16) with $d = 1$, and the test function

$$U(x_1, x_2, x_3, t) = \left(x_1 + 2x_2 + 3x_3 + \sqrt{t+1}\right) \sin(x_1 + x_2 + x_3).$$

The numerical error at time t is measured by the quantity $E_{N,\tau}(t)$, which is similar to E_N. In Table 4.2, we present the values of $\log_{10} E_{N,\tau}(t)$ at $t = 2$, with various values of the mode N and the step size τ. It is shown that the numerical errors decay fast when N increases and τ decreases. This confirms the theoretical analysis, see (3.29). In Fig. 4.1, we plot the values of $\log_{10} E_{N,\tau}(t)$ for $0 \le t \le 20$, with $N = 10$ and $\tau = 0.001$. They indicate the stability of long-time calculation.

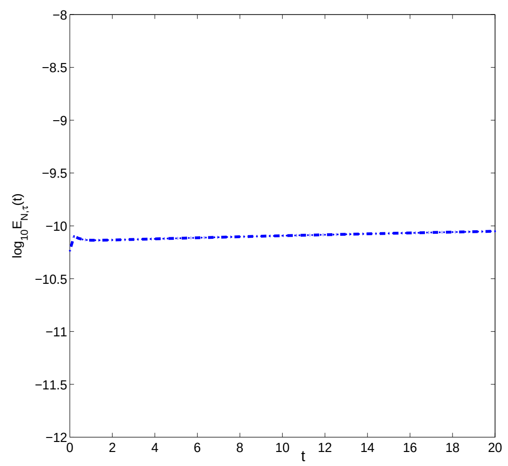

Fig. 4.1. Stability of scheme (4.3) with (4.6).

Finally, we consider the scheme (3.40)-(3.41). Let $\eta_k(x) = L_k(x) - L_{k+1}(x),\ 0 \le k \le N-1$. Obviously, $\eta_k(1) = 0$. In actual computation, we expand the auxiliary numerical solution as

$$u_N(x_1, x_2, x_3) = \sum_{k=0}^{N-1} \sum_{l=0}^{N-1} \sum_{m=0}^{N-1} a_{k,l,m} \eta_k(x_1) \eta_l(x_2) \eta_m(x_3).$$

Besides,

$$f^*_{k',l',m'} = (f, \eta_{k'} \eta_{l'} \eta_{m'})_\Omega + \int_{\partial\Omega \backslash \partial^*\Omega} h \eta_{k'} \eta_{l'} \eta_{m'} dS, \qquad 0 \le k', l', m' \le N-1.$$

By taking $\phi = \eta_{k'}(x_1)\eta_{l'}(x_2)\eta_{m'}(x_3)(0 \le k', l', m' \le N-1)$ in (3.40), we obtain the following compact matrix form,

$$\Big(A \otimes B \otimes B + B \otimes A \otimes B + B \otimes B \otimes A + dB \otimes B \otimes B + \alpha(C \otimes B \otimes B + B \otimes C \otimes B + B \otimes B \otimes C)\Big)\mathbf{X} = \mathbf{F}^*, \tag{4.7}$$

where X and F^* are similar to X and F in (4.1). The matrices $A = (a_{k'k})$, $B = (b_{k'k})$ and $C = (c_{k'k})$, with the following entries,

$$a_{k'k} = a_{kk'} = \int_I \partial_x \eta_k(x) \partial_x \eta_{k'}(x) dx = 2(-1)^{k'+k}(\min(k,k')+1)^2,$$

$$b_{k'k} = b_{kk'} = \int_I \eta_k(x)\eta_{k'}(x)dx = \begin{cases} -\dfrac{2}{2k+1}, & k' = k-1, \\ \dfrac{2}{2k+1} + \dfrac{2}{2k+3}, & k' = k, \\ 0, & \text{otherwise}, \end{cases}$$

$$c_{k'k} = c_{kk'} = \eta_k(-1)\eta_{k'}(-1) = 4(-1)^{k+k'}.$$

The numerical error is measured by the quantity E_N as before.

We now use scheme (3.40)-(3.41) to solve (3.30) with the test function (4.2), with the mixed Dirichlet-Neumann boundary condition ($\alpha = 0$) and the mixed Dirichlet-Robin boundary condition ($\alpha = 1$). In Table 4.3, we present the values $\log_{10} E_N$ vs. the mode N. As is i predicted by (3.45), the numerical errors decay fast as N increases.

Table 4.3: Numerical errors of scheme (3.40)-(3.41).

	$\alpha = 0$			$\alpha = 1$		
	$N = 5$	$N = 10$	$N = 15$	$N = 5$	$N = 10$	$N = 15$
d=0	6.10E-04	1.60E-09	2.52E-14	6.07E-04	1.59E-09	1.57E-14
d=1	6.07E-04	1.60E-09	2.63E-14	6.04E-04	1.59E-09	2.31E-14

5. Concluding Remarks

In this paper, we established some results on the three-dimensional Legendre orthogonal approximation in Jacobi weighted Sobolev space, which improve and generalize the existing ones, and play an important role in spectral method for partial differential equations. We also developed an explicit lifting technique, with which we could reformulate three-dimensional mixed inhomogeneous boundary value problems to alternative formulations with homogeneous Dirichlet boundary conditions imposed on some parts of boundaries. Then we could treat with the resulting problems easily in actual computation and numerical analysis.

We developed the Legendre spectral method for mixed inhomogeneous boundary value problems in three-dimensional space. We provided spectral schemes for two model problems (steady and unsteady) with inhomogeneous Dirichlet boundary conditions, and a steady problem with mixed inhomogeneous Dirichlet-Robin boundary conditions. All of them possess the spectral accuracy in space. The numerical results demonstrated their high accuracy. Moreover, we could combine the idea of [15, 19] with the techniques in this paper, to design and analyze Petrov-Galerkin spectral element methods for various mixed inhomogeneous boundary value problems defined on hexahedrons. We shall report the related results in the future.

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