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A STABILIZED EQUAL-ORDER FINITE VOLUME METHOD FOR THE STOKES EQUATIONS *

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Abstract

We construct a new stabilized finite volume method on rectangular grids for the Stokes equations. The lowest equal-order conforming finite element pair (piecewise bilinear velocities and pressures) and piecewise constant test spaces for both the velocity and pressure are employed in this method. We show the stability of this method and prove first optimal rate of convergence for the velocity in the H^1 norm and the pressure in the L^2 norm. In addition, a second order optimal error estimate for the velocity in the L^2 norm is derived. Numerical experiments illustrating the theoretical results are included.

Mathematics subject classification: 65N08, 65N15, 65N30, 76D05.
Key words: Stokes equations, Equal-order finite element pair, Finite volume method, Error estimate.

1. Introduction

Finite volume method (FVM) [2,8,9,13,26,27,32], also called generalized difference method, covolume method, or box scheme, has been widely used in computational fluid dynamics and practical fluid mechanics. In general, the programming effort in implementing the finite volume method is usually simpler than the finite element method (FEM). The finite volume method discretizations provide reasonable approximations for the Stokes problems. Many papers were devoted to develop the finite volume method and establish its error analysis for the Stokes equations, for example, see [10–12, 23, 28, 29, 33].

The lowest equal-order finite element pair for the Stokes equations have already attracted much attention [1,3,5,7,16,18,20–24,30] because of their simplicity and attractive computational properties. Since the equal-order finite element pairs hold an identical degree distribution for both the velocity and pressure, they are computationally efficient in multigrids and parallel processing. However, it is well known that the equal-order finite element pairs do not satisfy the inf-sup condition. In order to counteract the lack of inf-sup stability, one possible remedy is to modify the variational formulation associated with the Stokes equations by adding a stabilization term.

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Recently, Li and Chen [23] have developed and analyzed a stabilized finite volume method for the Stokes equations on triangular grids. The lowest equal-order conforming finite element pair (piecewise linear velocities and pressures) are employed in their method. By the relationship between their method and a stabilized finite element method they derived the optimal error estimates for both the velocity and pressure.

In this paper, we study a new stabilized finite volume method for the Stokes equations on rectangular grids with the lowest equal-order conforming finite element pair(piecewise bilinear velocities and pressures). We consider the following stationary Stokes problem in an axiparallel domain $\Omega \subset \mathbb{R}^2$

$$
\begin{aligned}
& -\lambda\Delta\mathbf{u} + \nabla p = \mathbf{f}, && \text{in } \Omega, && (1.1)\\
& \text{div}\mathbf{u} = 0, && \text{in } \Omega, && (1.2)\\
& \mathbf{u} = 0, && \text{on } \partial\Omega, && (1.3)
\end{aligned}
$$

where $\mathbf{u} = (u^1, u^2)$ stands for fluid velocity, p the pressure, $\mathbf{f}$ is a given external force, and $\lambda > 0$ denotes the viscosity of the fluid. Set

$$
\mathbf{V} = H_0^1(\Omega)^2, \quad W = L_0^2(\Omega) = \left\{ q \in L^2(\Omega) : \int_\Omega q d\Omega = 0 \right\}. \tag{1.4}
$$

Define

$$
A(\mathbf{u}, \mathbf{v}) = \lambda \int_\Omega \nabla\mathbf{u} : \nabla\mathbf{v} d\Omega, \tag{1.5}
$$

$$
C(\mathbf{v}, p) = \int_\Omega \mathbf{v} \cdot \nabla p d\Omega, \quad B(\mathbf{u}, q) = -\int_\Omega q \text{div } \mathbf{u} d\Omega. \tag{1.6}
$$

It is well known that $C(\mathbf{v}, q) = B(\mathbf{v}, q)$, then the associated variational formulation of (1.1)-(1.3) is to seek a pair $(\mathbf{u}, p) \in \mathbf{V} \times W$ such that

$$
\begin{aligned}
& A(\mathbf{u}, \mathbf{v}) + B(\mathbf{v}, p) = (\mathbf{f}, \mathbf{v}), && \forall \mathbf{v} \in \mathbf{V}, && (1.7)\\
& B(\mathbf{u}, q) = 0, && \forall q \in W. && (1.8)
\end{aligned}
$$

The above weak formulation (1.1)-(1.3) can be also written as follows:

$$
\begin{aligned}
L(\mathbf{u}, p; \mathbf{v}, q) :=& A(\mathbf{u}, \mathbf{v}) + B(\mathbf{v}, p) + B(\mathbf{u}, q)\\
=& (\mathbf{f}, \mathbf{v}), \quad \forall (\mathbf{v}, q) \in \mathbf{V} \times W.
\end{aligned} \tag{1.9}
$$

In the case of rectangular partition, since the bilinear form C for the finite volume method is no longer equal to the bilinear form B for the finite element method when the lowest equal-order finite element pair are employed, the finite volume method for the Stokes problem (1.1)-(1.3) can not be written as the standard form. To compensate for this deficiency, we discretize Eq. (1.2) using the finite volume method instead of the finite element method, which is different from the classical mixed methods [4, 6, 11, 12, 17, 19, 29]. Moreover, in order to stabilize this system, with the idea of [3, 24] we introduce a stabilization term using a local polynomial pressure projection on dual elements. We will show that our method is unconditionally stable, and achieve optimal accuracy. Moreover, numerical experiments confirm the theoretical results.

The remainder of this paper is organized as follows. In the next section we introduce some notations which will be used throughout the paper and recall the stabilized finite element approximations [3,24] of the Stokes equations. In Section 3, the stabilized finite volume method for the Stokes equations is constructed. Section 4 deals with the stability of this method. The optimal convergence error estimates for the method are derived in Section 5. Finally, we present the numerical experiments illustrating the theoretical results in Section 6.

Throughout this paper the symbol C will denote a generic positive constant independent of the discretization parameters and may have different values at different places.

2. Notations and Preliminaries

For a subdomain $D \subset \mathbb{R}^2$, we denote by $(\cdot,\cdot)_D$ the usual $L^2(D)$ or $L^2(D)^2$ inner product, $\|\cdot\|_{0,D}$ the norm in the space $L^2(D)$ or $L^2(D)^2$. For k a positive integer, let $\|\cdot\|_{k,D}$ and $|\cdot|_{k,D}$ be the norm and the semi-norm of the Sobolev space $H^k(D)$ or $H^k(D)^2$ [14,19,26], respectively. For brevity we omit D in the subscript if $D = \Omega$.

Let $\mathcal{T}_h = \{K_{i,j}, 1 \le i \le M, 1 \le j \le N\}$ be a partition of the domain Ω into a union of rectangles $K_{i,j}$ with centers $c_{i,j} = (x_{i+1/2}, y_{j+1/2})$. Denote by $P_1, P_2, \cdots, P_{\mathring{N}_v}$ those interior vertices and $P_{\mathring{N}_v+1}, \cdots, P_{N_v}$ those on the boundary. Let $h_i^x = x_{i+1} - x_i$, $h_j^y = y_{j+1} - y_j$ and $h = \max_{1\le i\le M, 1\le j\le N}\{h_i^x, h_j^y\}$. We shall assume that the partition $\mathcal{T}_h = \{K_{i,j}\}$ is quasi-uniform, i.e., there exist positive constants C_1 and C_2 independent of h such that

$$C_1 h^2 \le |K_{i,j}| \le C_2 h^2, \quad \forall K_{i,j} \in \mathcal{T}_h, \tag{2.1}$$

where $|K_{i,j}|$ is the area of $Q_{i,j}$.

Now we choose the lowest equal-order conforming finite element space $\mathbf{V}_h \times W_h \subset \mathbf{V} \times W$ as the trial space:

$$\mathbf{V}_h = \Big\{\mathbf{u} \in \mathbf{V} : \mathbf{u}|_K \in Q_1(K)^2, \forall K \in \mathcal{T}_h\Big\}, \tag{2.2}$$

$$W_h = \Big\{q \in W \cap C^0(\Omega) : q|_K \in Q_1(K), \forall K \in \mathcal{T}_h\Big\}, \tag{2.3}$$

where $Q_1(K) = \{p : p = a + bx + cy + dxy, (x,y) \in K, a,b,c,d \in \mathbb{R}\}$ is the space of bilinear functions.

For $\mathbf{u}_h \in \mathbf{V}_h$, define

$$|\mathbf{u}_h|_2 := \Big(\sum_{K\in\mathcal{T}_h} |\mathbf{u}_h|_{2,K}^2\Big)^{1/2}.$$

Let $I_h\mathbf{u}$ and $J_h q$ be the interpolation projection from $\mathbf{V}$ and W onto $\mathbf{V}_h$ and W_h, respectively:

$$I_h\mathbf{u}|_K \in Q_1(K)^2 \ \text{ and } \ I_h\mathbf{u}|_K(P_i) = \mathbf{u}(P_i), \quad i = 1,2,3,4, \tag{2.4}$$

$$J_h q|_K \in Q_1(K) \ \text{ and } \ J_h q|_K(P_i) = q(P_i), \quad i = 1,2,3,4, \tag{2.5}$$

where $P_i, i = 1,2,3,4$ are the four vertices of the element K.

For $\mathbf{u} \in H^2(\Omega)^2, q \in H^1(\Omega)$, the projection operators I_h and J_h have the following properties [4,14,15]:

$$\|\mathbf{u} - I_h\mathbf{u}\|_i \le Ch^{2-i}|\mathbf{u}|_2, \quad \|q - J_h q\|_0 \le Ch|q|_1, \quad i = 0,1, \tag{2.6}$$

$$|I_h\mathbf{u}|_i \le C|\mathbf{u}|_i, \quad |J_h q|_1 \le C|q|_1, \quad i = 1,2. \tag{2.7}$$

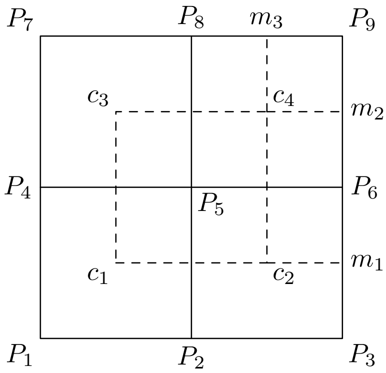

Fig. 2.1. Primal and dual elements.

Moreover, the inverse inequalities hold [4, 14, 15]:

$$|\mathbf{u}_h|_{i+1} \le Ch^{-1}|\mathbf{u}_h|_i, \quad |q_h|_1 \le Ch^{-1}\|q_h\|_0, \quad \forall \mathbf{u}_h \in \mathbf{V}_h, q_h \in W_h, \quad i = 0, 1. \tag{2.8}$$

Denote by M_h the piecewise constant space associated with $\mathcal{T}_h$. Let $\pi_h : L^2(\Omega) \to M_h$ be the standard L^2 projection operator:

$$(p, q_h) = (\pi_h p, q_h), \quad \forall p \in L^2(\Omega), q_h \in M_h. \tag{2.9}$$

The projection operator π_h satisfies [4, 14, 15]:

$$\|\pi_h p\|_0 \le C\|p\|_0, \qquad \forall p \in L^2(\Omega), \tag{2.10}$$

$$\|p - \pi_h p\|_0 \le Ch|p|_1, \qquad \forall p \in H^1(\Omega). \tag{2.11}$$

Now, we can define the bilinear form $G(\cdot,\cdot)$ [3, 23] as follows:

$$G(p_h, q_h) = (p_h - \pi_h p_h, q_h - \pi_h q_h). \tag{2.12}$$

Then the stabilized finite element method for the Stokes problem (1.1)-(1.3) is to seek $(\mathbf{u}_h, p_h) \in \mathbf{V}_h \times W_h$ such that

$$Q(\mathbf{u}_h, p_h; \mathbf{v}_h, q_h) = (\mathbf{f}, \mathbf{v}_h), \quad \forall (\mathbf{v}_h, q_h) \in \mathbf{V}_h \times W_h, \tag{2.13}$$

where

$$Q(\mathbf{u}_h, p_h; \mathbf{v}_h, q_h) = A(\mathbf{u}_h, \mathbf{v}_h) + B(\mathbf{v}_h, p_h) + B(\mathbf{u}_h, q_h) - G(p_h, q_h). \tag{2.14}$$

This bilinear form satisfies the continuity and weak coercivity [3, 24]:

$$Q(\mathbf{u}_h, p_h; \mathbf{v}_h, q_h) \le C\Big(\|\mathbf{u}_h\|_1 + \|p_h\|_0\Big)\Big(\|\mathbf{v}_h\|_1 + \|q_h\|_0\Big), \tag{2.15}$$

$$\sup_{(\mathbf{v}_h, q_h) \in \mathbf{V}_h \times W_h} \frac{Q(\mathbf{u}_h, p_h; \mathbf{v}_h, q_h)}{\|\mathbf{v}_h\|_1 + \|q_h\|_0} \ge \beta\Big(\|\mathbf{u}_h\|_1 + \|p_h\|_0\Big), \tag{2.16}$$

where β is a positive constant independent of h.

Thus, the system (2.13) has unique solution and the following convergence results hold [3,24]:

$$\|\mathbf{u} - \mathbf{u}_h\|_0 + h(\|\mathbf{u} - \mathbf{u}_h\|_1 + \|p - p_h\|_0) \le Ch^2\Big(\|\mathbf{u}\|_2 + \|p\|_1\Big). \tag{2.17}$$

3. The Stabilized Finite Volume Method

In this section we shall present a new stabilized finite volume method for the Stokes problem.

We shall construct the dual partition $\mathcal{T}_h^*$ and the test function spaces. The dual grid is a union of rectangles, can be constructed by the following rule: Referring to Fig.2.1, the interior node P_5 is the common vertex of the rectangles $K_{c_1} = \Box P_1P_2P_5P_4$, $K_{c_2} = \Box P_2P_3P_6P_5$, $K_{c_3} = \Box P_4P_5P_8P_7$ and $K_{c_4} = \Box P_5P_6P_9P_8$, and the rectangle $\Box c_1c_2c_4c_3$ is dual element according to the node P_5, denoted by K_5^*, where c_i is the center of element K_{c_i}. For a boundary node like $P_6(P_9)$ the associated dual element is the rectangle $\Box c_2m_1m_2c_4(\Box c_4m_2P_9m_3)$, denoted by $K_6^*(K_9^*)$, where m_1, m_2, m_3 are the midpoints of the edges P_3P_6, P_6P_9, P_8P_9, respectively.

Next we define the following two test spaces:

$$\begin{aligned}\tilde{\mathbf{V}}_h := \Big\{\mathbf{v}_h \in L^2(\Omega)^2 : \mathbf{v}_h \text{ is constant vector over } K^* \in \mathcal{T}_h^*,\\ \text{and } \mathbf{v}_h = 0 \text{ on any boundary dual element}\Big\},\end{aligned} \tag{3.1}$$

$$\tilde{W}_h := \Big\{q_h \in L^2(\Omega) : q_h \text{ is constant over } K^* \in \mathcal{T}_h^*\Big\}. \tag{3.2}$$

In addition, define two operators $\Gamma_h : \mathbf{V}_h \to \tilde{\mathbf{V}}_h$ and $\gamma_h : W_h \to \tilde{W}_h$:

$$\Gamma_h\mathbf{v}_h = \sum_{j=1}^{\mathring{N}_v} \mathbf{v}_h(P_j)\chi_{P_j}, \quad \mathbf{v}_h \in \mathbf{V}_h, \tag{3.3}$$

$$\gamma_h q_h = \sum_{j=1}^{N_v} q_h(P_j)\chi_{P_j}, \quad q_h \in W_h, \tag{3.4}$$

where χ_{P_j} is the characteristic function of the dual element K_j^*.

The above idea of connecting the trial and test spaces in the Petrov-Galerkin method through the operator Γ_h or γ_h was first introduced by Li [25] in the context of elliptic problems. The operators Γ_h and γ_h satisfy the following properties [25, 26]:

$$\|\mathbf{v}_h - \Gamma_h\mathbf{v}_h\|_0 \le Ch|\mathbf{v}_h|_1, \quad \mathbf{v}_h \in \mathbf{V}_h, , \tag{3.5}$$

$$\|q_h - \gamma_h q_h\|_0 \le Ch|q_h|_1, \quad q_h \in W_h, \tag{3.6}$$

$$\|\gamma_h q_h\|_0 \le C\|q_h\|_0, \quad q_h \in W_h. \tag{3.7}$$

Noting that $\mathbf{v}_h \in \mathbf{V}_h$ is a piecewise bilinear function, the following lemma which is necessary to derive error estimates can be easily obtained by direct calculation.

Lemma 3.1. *If $K \in \mathcal{T}_h$, then*

$$\int_K (\mathbf{v}_h - \Gamma_h\mathbf{v}_h)dK = 0, \quad \mathbf{v}_h \in \mathbf{V}_h, \tag{3.8}$$

$$\int_K (q_h - \gamma_h q_h)dK = 0, \quad q_h \in W_h. \tag{3.9}$$

Define the following bilinear forms of the finite volume method as follows:

$$\tilde{A}(\mathbf{u}_h, \Gamma_h \mathbf{v}_h) = -\lambda \sum_{j=1}^{\mathring{N}_v} \mathbf{v}_h(P_j) \cdot \int_{\partial K_j^*} \frac{\partial \mathbf{u}_h}{\partial \mathbf{n}} ds, \qquad \mathbf{u}_h, \mathbf{v}_h \in \mathbf{V}_h, \tag{3.10}$$

$$\tilde{B}(\mathbf{v}_h, \gamma_h q_h) = -\sum_{j=1}^{N_v} q_h(P_j) \cdot \int_{\partial K_j^*} \mathbf{v}_h \cdot \mathbf{n} ds, \qquad \mathbf{v}_h \in \mathbf{V}_h, \ \ q_h \in W_h, \tag{3.11}$$

$$\tilde{C}(\Gamma_h \mathbf{v}_h, q_h) = \sum_{j=1}^{\mathring{N}_v} \mathbf{v}_h(P_j) \cdot \int_{\partial K_j^*} q_h \mathbf{n} ds, \qquad \mathbf{v}_h \in \mathbf{V}_h, \ \ q_h \in W_h, \tag{3.12}$$

where $\mathbf{n}$ is the unit normal outward to ∂K_j^*.

Notice that the bilinear form $\tilde{B}$ is different from the definition used in general mixed finite volume methods. Its second argument is now a test function $\gamma_h q_h$ instead of q_h, in other words, it is $\tilde{B}(\cdot, \gamma_h \cdot)$, not $\tilde{B}(\cdot, \cdot)$. We shall show in the following lemma that $\tilde{B}(\mathbf{v}_h, \gamma_h q_h) = \tilde{C}(\Gamma_h \mathbf{v}_h, q_h)$.

Lemma 3.2. *For any* $\mathbf{u}_h, \mathbf{v}_h \in \mathbf{V}_h, q_h \in W_h$, *it holds that*

$$\tilde{A}(\mathbf{u}_h, \Gamma_h \mathbf{u}_h) \geq C \|\mathbf{u}_h\|_1^2, \tag{3.13}$$

$$\tilde{A}(\mathbf{u}_h, \Gamma_h \mathbf{v}_h) \leq C \|\mathbf{u}_h\|_1 \|\mathbf{v}_h\|_1, \tag{3.14}$$

$$\tilde{C}(\Gamma_h \mathbf{v}_h, q_h) = \tilde{B}(\mathbf{v}_h, \gamma_h q_h). \tag{3.15}$$

Proof. Note that (3.13) and (3.14) were shown in [11, 27]. It suffices to prove (3.15). Let $\mathcal{T}_h^* = \{K_{i,j}^*, 1 \leq i \leq M+1, 1 \leq j \leq N+1\}$. For simplicity, we set $h_i^x = h_j^y = h, 1 \leq i \leq M, 1 \leq j \leq N$, and it does not affect the proof. Write $q_{i,j} = q_h(P_{i,j})$, $\mathbf{v}_h = (u, v)$, and $\mathbf{v}_{i,j} = \mathbf{v}_h(P_{i,j}) = (u_{i,j}, v_{i,j})$. Let K be a rectangular and $\mathbf{x}_i = (x_i, y_i), i = 1, 2, 3, 4$ be its four vertices counted anti-clockwise. We take the unit square $\hat{K} = [0,1] \times [0,1]$ as the reference element in the $\xi\eta$-plane with vertices denoted by

$$\hat{\mathbf{x}}_1 = (0,0), \quad \hat{\mathbf{x}}_2 = (1,0), \quad \hat{\mathbf{x}}_3 = (1,1), \quad \hat{\mathbf{x}}_4 = (0,1).$$

Define the bilinear transformation $F_K : \hat{K} \to K$:

$$\mathbf{x} = F_K(\hat{\mathbf{x}}) = \mathbf{x}_1(1-\xi)(1-\eta) + \mathbf{x}_2\xi(1-\eta) + \mathbf{x}_3\xi\eta + \mathbf{x}_4(1-\xi)\eta. \tag{3.16}$$

Then, for any $q_h \in W_h, \mathbf{v}_h \in \mathbf{V}_h$, we have the expressions

$$q_h|_K = q_h(\mathbf{x}_1)(1-\xi)(1-\eta) + q_h(\mathbf{x}_2)\xi(1-\eta) + q_h(\mathbf{x}_3)\xi\eta + q_h(\mathbf{x}_4)(1-\xi)\eta, \tag{3.17}$$

$$\mathbf{v}_h|_K = \mathbf{v}_h(\mathbf{x}_1)(1-\xi)(1-\eta) + \mathbf{v}_h(\mathbf{x}_2)\xi(1-\eta) + \mathbf{v}_h(\mathbf{x}_3)\xi\eta + \mathbf{v}_h(\mathbf{x}_4)(1-\xi)\eta. \tag{3.18}$$

Combine (3.16)-(3.18) by direct calculation, we obtain

$$\tilde{B}(\mathbf{v}_h,\gamma_h q_h)=-\sum_{j=1}^{N+1}\sum_{i=1}^{M+1}q_{i,j}\int_{\partial K_{i,j}^*}\mathbf{v}_h\cdot\mathbf{n}ds$$
$$=-\sum_{j=1}^{N+1}\sum_{i=1}^{M+1}q_{i,j}\Big(-\int_{1/2}^{1}hq_h(F_{K_{i-1,j-1}}(\frac{1}{2},\eta))d\eta+\int_{1/2}^{1}hq_h(F_{K_{i,j-1}}(\frac{1}{2},\eta))d\eta$$
$$-\int_{0}^{1/2}hq_h(F_{K_{i-1,j}}(\frac{1}{2},\eta))d\eta+\int_{0}^{1/2}hq_h(F_{K_{i,j}}(\frac{1}{2},\eta))d\eta-\int_{1/2}^{1}hq_h(F_{K_{i-1,j-1}}(\xi,\frac{1}{2}))d\xi$$
$$+\int_{1/2}^{1}hq_h(F_{K_{i,j-1}}(\xi,\frac{1}{2}))d\xi-\int_{0}^{1/2}hq_h(F_{K_{i-1,j}}(\xi,\frac{1}{2}))d\xi+\int_{0}^{1/2}hq_h(F_{K_{i,j}}(\xi,\frac{1}{2}))d\xi\Big)$$
$$=-\sum_{j=1}^{N+1}\sum_{i=1}^{M+1}q_{i,j}\Big(-\frac{h}{16}u_{i-1,j-1}+\frac{h}{16}u_{i+1,j-1}-\frac{3h}{8}u_{i-1,j}+\frac{3h}{8}u_{i+1,j}-\frac{h}{16}u_{i-1,j+1}$$
$$+\frac{h}{16}u_{i+1,j+1}-\frac{h}{16}v_{i-1,j-1}-\frac{3h}{8}v_{i,j-1}-\frac{h}{16}v_{i+1,j-1}+\frac{h}{16}v_{i-1,j+1}+\frac{3h}{8}v_{i,j+1}+\frac{h}{16}v_{i+1,j+1}\Big),$$

and

$$\tilde{C}(\Gamma_h\mathbf{v}_h,q_h)=\sum_{j=1}^{N}\sum_{i=1}^{M}\mathbf{v}_{i,j}\cdot\int_{\partial K_{i,j}^*}q_h\mathbf{n}ds$$
$$=\sum_{j=1}^{N}\sum_{i=1}^{M}u_{i,j}\Big(-\frac{h}{16}q_{i-1,j-1}+\frac{h}{16}q_{i+1,j-1}-\frac{3h}{8}q_{i-1,j}+\frac{3h}{8}q_{i+1,j}-\frac{h}{16}q_{i-1,j+1}+\frac{h}{16}q_{i+1,j+1}\Big)$$
$$+v_{i,j}\Big(-\frac{h}{16}q_{i-1,j-1}-\frac{3h}{8}q_{i,j-1}-\frac{h}{16}q_{i+1,j-1}+\frac{h}{16}q_{i-1,j+1}+\frac{3h}{8}q_{i,j+1}+\frac{h}{16}q_{i+1,j+1}\Big)$$
$$=-\sum_{j=1}^{N+1}\sum_{i=1}^{M+1}q_{i,j}\Big(-\frac{h}{16}u_{i-1,j-1}+\frac{h}{16}u_{i+1,j-1}-\frac{3h}{8}u_{i-1,j}+\frac{3h}{8}u_{i+1,j}-\frac{h}{16}u_{i-1,j+1}+\frac{h}{16}u_{i+1,j+1}$$
$$-\frac{h}{16}v_{i-1,j-1}-\frac{3h}{8}v_{i,j-1}-\frac{h}{16}v_{i+1,j-1}+\frac{h}{16}v_{i-1,j+1}+\frac{3h}{8}v_{i,j+1}+\frac{h}{16}v_{i+1,j+1}\Big)$$
$$=\tilde{B}(\mathbf{v}_h,\gamma_h q_h),$$

which gives the desired result (3.15). □

To define the stabilized finite volume method, we need add a stabilization term to the variational formulation associated with the Stokes equations with the idea of [3]. Now we define the following bilinear form:

$$\tilde{G}(p_h,q_h):=(p_h-\gamma_h p_h,q_h-\gamma_h q_h),\quad p_h,q_h\in W_h. \tag{3.19}$$

It is clear that the bilinear form $\tilde{G}(\cdot,\cdot)$ is symmetric and semi-definite form generated on dual elements. It is not like as $G(\cdot,\cdot)$ mentioned above and not equal to $G(\cdot,\gamma_h\cdot)$, but it is still a simple and effective stabilization form.

By Lemma 3.2 and (3.19) we define

$$\tilde{Q}(\tilde{\mathbf{u}}_h,\tilde{p}_h;\mathbf{v}_h,q_h)=\tilde{A}(\tilde{\mathbf{u}}_h,\Gamma_h\mathbf{v}_h)+\tilde{B}(\mathbf{v}_h,\gamma_h\tilde{p}_h)+\tilde{B}(\tilde{\mathbf{u}}_h,\gamma_h q_h)-\tilde{G}(\tilde{p}_h,q_h). \tag{3.20}$$

Then the new stabilized finite volume method for the Stokes problem (1.1)-(1.3) is: Find $(\tilde{\mathbf{u}}_h,\tilde{p}_h)\in\mathbf{V}_h\times W_h$ such that

$$\tilde{Q}(\tilde{\mathbf{u}}_h,\tilde{p}_h;\mathbf{v}_h,q_h)=(\mathbf{f},\Gamma_h\mathbf{v}_h),\quad\forall(\mathbf{v}_h,q_h)\in\mathbf{V}_h\times W_h. \tag{3.21}$$

Remark 3.1. In general, the finite element methods for Stokes problem can not keep the conservativeness. However, the finite volume scheme for the second equation (1.2) is obtained by integrating the equation over a dual element, so the finite volume method should keep the conservation of mass. In this paper, the stabilized method (3.21) is defined by using a stabilization term (3.19) to modify the second variational equation associated with the Stokes equations, therefore, the method keep the approximate local conservation of mass actually.

4. Stability

In this section, we study the stability of the new stabilized finite volume method. The symbols $C_i,\ 1 \le i \le 7$ in this section will be used as a generic positive constant independent of h.

The following continuity and weak coercivity of $\tilde{Q}(\cdot,\cdot;\cdot,\cdot)$ hold.

Theorem 4.1. *The following hold*

$$\tilde{Q}(\tilde{\mathbf{u}}_h,\tilde{p}_h;\mathbf{v}_h,q_h) \le C\Big(\|\tilde{\mathbf{u}}_h\|_1+\|\tilde{p}_h\|_0\Big)\Big(\|\mathbf{v}_h\|_1+\|q_h\|_0\Big),$$
$$\forall(\tilde{\mathbf{u}}_h,\tilde{p}_h),(\mathbf{v}_h,q_h)\in \mathbf{V}_h\times W_h, \tag{4.1}$$

and, there exists a positive constant β independent of h such that

$$\sup_{(\mathbf{v}_h,q_h)\in\mathbf{V}_h\times W_h}\frac{\tilde{Q}(\tilde{\mathbf{u}}_h,\tilde{p}_h;\mathbf{v}_h,q_h)}{\|\mathbf{v}_h\|_1+\|q_h\|_0} \ge \beta(\|\tilde{\mathbf{u}}_h\|_1+\|\tilde{p}_h\|_0), \quad \forall(\tilde{\mathbf{u}}_h,\tilde{p}_h)\in\mathbf{V}_h\times W_h. \tag{4.2}$$

Proof. Using Green's formula and (3.7) gives

$$\tilde{B}(\mathbf{v}_h,\gamma_h q_h) = -(\mathrm{div}\ \mathbf{v}_h,\gamma_h q_h) \le \|\mathrm{div}\ \mathbf{v}_h\|_0\|\gamma_h q_h\|_0 \le C\|\mathbf{v}_h\|_1\|q_h\|_0,$$

we obtain (4.1):

$$\begin{aligned}\tilde{Q}(\tilde{\mathbf{u}}_h,\tilde{p}_h;\mathbf{v}_h,q_h) &= \tilde{A}(\tilde{\mathbf{u}}_h,\Gamma_h\mathbf{v}_h)+\tilde{B}(\mathbf{v}_h,\gamma_h\tilde{p}_h)+\tilde{B}(\tilde{\mathbf{u}}_h,\gamma_h q_h)-\tilde{G}(\tilde{p}_h,q_h)\\ &\le C\Big(\|\tilde{\mathbf{u}}_h\|_1\|\mathbf{v}_h\|_1+\|\mathbf{v}_h\|_1\|\tilde{p}_h\|_0+\|\tilde{\mathbf{u}}_h\|_1\|q_h\|_0+\|\tilde{p}_h\|_0\|q_h\|_0\Big)\\ &= C\Big(\|\tilde{\mathbf{u}}_h\|_1+\|\tilde{p}_h\|_0\Big)\Big(\|\mathbf{v}_h\|_1+\|q_h\|_0\Big).\end{aligned}$$

Next we prove (4.2). For any $\tilde{p}_h\in W_h\subset L^2(\Omega)$, there exists $\mathbf{w}\in H_0^1(\Omega)^2$ [4] satisfying

$$\mathrm{div}\ \mathbf{w}=\tilde{p}_h,\ \text{ and } \|\mathbf{w}\|_1\le C\|\tilde{p}_h\|_0. \tag{4.3}$$

Set $\mathbf{w}_h=I_h\mathbf{w}$, $(\mathbf{v}_h,q_h)=(\tilde{\mathbf{u}}_h-\alpha\mathbf{w}_h,-\tilde{p}_h)$, where α is a positive parameter, yields

$$\begin{aligned}&\tilde{Q}(\tilde{\mathbf{u}}_h,\tilde{p}_h;\tilde{\mathbf{u}}_h-\alpha\mathbf{w}_h,-\tilde{p}_h)\\ &=\tilde{A}(\tilde{\mathbf{u}}_h,\Gamma_h\tilde{\mathbf{u}}_h)-\alpha\tilde{A}(\tilde{\mathbf{u}}_h,\Gamma_h\mathbf{w}_h)-\alpha\tilde{B}(\mathbf{w}_h,\gamma\tilde{p}_h)+\tilde{G}(\tilde{p}_h,\tilde{p}_h).\end{aligned} \tag{4.4}$$

By Lemma 3.2 and Young's inequality, we have that

$$\tilde{A}(\tilde{\mathbf{u}}_h,\Gamma_h\tilde{\mathbf{u}}_h)\ge C_1\|\tilde{\mathbf{u}}_h\|_1^2, \tag{4.5}$$

$$\tilde{A}(\tilde{\mathbf{u}}_h,\Gamma_h\mathbf{w}_h)\le C\|\tilde{\mathbf{u}}_h\|_1\|\mathbf{w}_h\|_1\le C\|\tilde{\mathbf{u}}_h\|_1\|\tilde{p}_h\|_0\le\frac{1}{8}\|\tilde{p}_h\|_0^2+C_2\|\tilde{\mathbf{u}}_h\|_1^2. \tag{4.6}$$

Applying Green's formula, we get

$$\begin{aligned}-\tilde{B}(\mathbf{w}_h,\gamma_h\tilde{p}_h)=&(\text{div }\mathbf{w}_h,\gamma_h\tilde{p}_h)\\=&(\text{div }\mathbf{w},\gamma_h\tilde{p}_h)-(\text{div}(\mathbf{w}-\mathbf{w}_h),\gamma_h\tilde{p}_h-\tilde{p}_h)-(\text{div}(\mathbf{w}-\mathbf{w}_h),\tilde{p}_h).\end{aligned}\tag{4.7}$$

Using the Cauchy-Schwarz inequality, Young's inequality, (2.7) and (4.3) yields

$$\begin{aligned}(\text{div }\mathbf{w},\gamma_h\tilde{p}_h)=&(\tilde{p}_h,\gamma_h\tilde{p}_h)=(\tilde{p}_h,\tilde{p}_h)-(\tilde{p}_h,\tilde{p}_h-\gamma_h\tilde{p}_h)\\\geq&\|\tilde{p}_h\|_0^2-\|\tilde{p}_h\|_0\|\tilde{p}_h-\gamma_h\tilde{p}_h\|_0\geq\frac{3}{4}\|\tilde{p}_h\|_0^2-\|\tilde{p}_h-\gamma_h\tilde{p}_h\|_0^2,\end{aligned}\tag{4.8}$$

and

$$\begin{aligned}(\text{div}(\mathbf{w}-\mathbf{w}_h),\gamma_h\tilde{p}_h-\tilde{p}_h)&\leq\|\text{div}(\mathbf{w}-\mathbf{w}_h)\|_0\|\tilde{p}_h-\gamma_h\tilde{p}_h\|_0\\&\leq C\|\mathbf{w}-\mathbf{w}_h\|_1\|\tilde{p}_h-\gamma_h\tilde{p}_h\|_0\leq C\|\tilde{p}_h\|_0\|\tilde{p}_h-\gamma_h\tilde{p}_h\|_0\\&\leq\frac{1}{4}\|\tilde{p}_h\|_0^2+C_3\|\tilde{p}_h-\gamma_h\tilde{p}_h\|_0^2.\end{aligned}\tag{4.9}$$

Moreover, applying the Green's formula, (2.6), (2.8) and Young's inequality gives

$$\begin{aligned}(\text{div}(\mathbf{w}-\mathbf{w}_h),\tilde{p}_h)\leq& Ch\|\mathbf{w}\|_1|\tilde{p}_h|_1\\\leq& Ch\|\tilde{p}_h\|_0\Big(\sum_{j=1}^{N_v}\sum_{K\in\mathcal{T}_h}|\tilde{p}_h-\gamma_h\tilde{p}_h|^2_{1,K\cap K_j^*}\Big)^{1/2}\\\leq& C\|\tilde{p}_h\|_0\|\tilde{p}_h-\gamma_h\tilde{p}_h\|_0\leq\frac{1}{4}\|\tilde{p}_h\|_0^2+C_4\|\tilde{p}_h-\gamma_h\tilde{p}_h\|_0^2.\end{aligned}\tag{4.10}$$

Combining (4.8)-(4.10) yields

$$\begin{aligned}-\tilde{B}(\mathbf{w}_h,\gamma_h\tilde{p}_h)&\geq\frac{1}{4}\|\tilde{p}_h\|_0^2-(1+C_3+C_4)\|\tilde{p}_h-\gamma_h\tilde{p}_h\|_0^2\\&\geq\frac{1}{4}\|\tilde{p}_h\|_0^2-C_5\|\tilde{p}_h-\gamma_h\tilde{p}_h\|_0^2.\end{aligned}\tag{4.11}$$

Thus, it can be deduced from (4.4)-(4.6) and (4.11) that

$$\begin{aligned}&\tilde{Q}(\tilde{\mathbf{u}}_h,\tilde{p}_h;\tilde{\mathbf{u}}_h-\alpha\mathbf{w}_h,-\tilde{p}_h)\\\geq&(C_1-\alpha C_2)\|\tilde{\mathbf{u}}_h\|_1^2+\frac{\alpha}{8}\|\tilde{p}_h\|_0^2+(1-\alpha C_5)\|\tilde{p}_h-\gamma_h\tilde{p}_h\|_0^2.\end{aligned}\tag{4.12}$$

Choose $\alpha=\min\{\frac{2C_1-1}{2C_2},\frac{1}{2C_4}\}$ satisfying

$$C_1-\alpha C_2\geq\frac{1}{2},\quad 1-\alpha C_5\geq\frac{1}{2}.\tag{4.13}$$

With this choice, (4.12) leads to

$$\tilde{Q}(\tilde{\mathbf{u}}_h,\tilde{p}_h;\tilde{\mathbf{u}}_h-\alpha\mathbf{w}_h,-\tilde{p}_h)\geq C_6(\|\tilde{\mathbf{u}}_h\|_1+\|\tilde{p}_h\|_0)^2.\tag{4.14}$$

Also, it is clear that

$$\|\tilde{\mathbf{u}}_h-\alpha\mathbf{w}_h\|_1+\|-\tilde{p}_h\|_0\leq\|\tilde{\mathbf{u}}_h\|_1+\alpha\|\mathbf{w}_h\|_1+\|\tilde{p}_h\|_0\leq C_7(\|\tilde{\mathbf{u}}_h\|_1+\|\tilde{p}_h\|_0).\tag{4.15}$$

Finally, setting $\beta=C_6/C_7$ and combining (4.12) with (4.14), we obtain (4.2). □

We point out that Theorem 4.1 implies the uniqueness and existence of the solution of the new stabilized mixed finite volume system (3.21) [6, 19, 31].

5. Error Estimates

We now prove the main results of this paper.

Theorem 5.1. *Let* $(\mathbf{u},p)$ *and* $(\tilde{\mathbf{u}}_h,\tilde{p}_h)$ *be the solution of the Stokes problem* (1.9) *and the stabilized finite volume system* (3.21), *respectively. Then*

$$\|\mathbf{u}-\tilde{\mathbf{u}}_h\|_1+\|p-\tilde{p}_h\|_0\le Ch(\|\mathbf{u}\|_2+\|p\|_1+\|\mathbf{f}\|_0). \tag{5.1}$$

Proof. Subtracting (3.21) from (2.13) yields

$$\begin{aligned}&\tilde{Q}(\mathbf{u}_h-\tilde{\mathbf{u}}_h,p_h-\tilde{p}_h;\mathbf{v}_h,q_h)\\ =&(f,\mathbf{v}_h-\Gamma_h\mathbf{v}_h)-A(\mathbf{u}_h,\mathbf{v}_h)+\tilde{A}(\mathbf{u}_h,\Gamma_h\mathbf{v}_h)-B(\mathbf{v}_h,p_h)\\ &+\tilde{B}(\mathbf{v}_h,\gamma_hp_h)-B(\mathbf{u}_h,q_h)+\tilde{B}(\mathbf{u}_h,\gamma_hq_h)+G(p_h,q_h)-\tilde{G}(p_h,q_h),\end{aligned} \tag{5.2}$$

where $(\mathbf{u}_h,p_h)$ is the solution of (2.13). Applying the Cauchy-Schwarz inequality and by (3.5) yields

$$(f,\mathbf{v}_h-\Gamma_h\mathbf{v}_h)\le\|\mathbf{f}\|_0\|\mathbf{v}_h-\Gamma_h\mathbf{v}_h\|_0\le Ch\|\mathbf{f}\|_0\|\mathbf{v}_h\|_1. \tag{5.3}$$

The following estimate can be seen from [11, Lemma 2.2] that

$$-A(\mathbf{u}_h,\mathbf{v}_h)+\tilde{A}(\mathbf{u}_h,\Gamma_h\mathbf{v}_h)\le Ch^2|\mathbf{u}_h|_2|\mathbf{v}_h|_2\le Ch|\mathbf{u}_h|_2|\mathbf{v}_h|_1,\quad\forall\mathbf{u}_h,\mathbf{v}_h\in\mathbf{V}_h. \tag{5.4}$$

It follows from Green's formula, (3.6), (3.9) and (2.11) that

$$\begin{aligned}&-B(\mathbf{v}_h,p_h)+\tilde{B}(\mathbf{v}_h,\gamma_hp_h)=\Big(\mathrm{div}\ \mathbf{v}_h,p_h-\gamma_hp_h\Big)\\ &\qquad\le\|\mathrm{div}\ \mathbf{v}_h\|_0\|p_h-\gamma_hp_h\|_0\le Ch\|\mathbf{v}_h\|_1|p_h|_1, &(5.5)\\ &-B(\mathbf{u}_h,q_h)+\tilde{B}(\mathbf{u}_h,\gamma_hq_h)=\Big(\mathrm{div}\ \mathbf{u}_h,q_h-\gamma_hq_h\Big)\\ &\qquad=\Big(\mathrm{div}\ \mathbf{u}_h-\pi_h\mathrm{div}\ \mathbf{u}_h,q_h-\gamma_hq_h\Big)\le Ch|\mathrm{div}\ \mathbf{u}_h|_1\|q_h\|_0\le Ch|\mathbf{u}_h|_2\|q_h\|_0, &(5.6)\\ &G(p_h,q_h)-\tilde{G}(p_h,q_h)\le Ch|p_h|_1\|q_h\|_0. &(5.7)\end{aligned}$$

Combining (5.3)-(5.7) we obtain

$$\tilde{Q}(\mathbf{u}_h-\tilde{\mathbf{u}}_h,p_h-\tilde{p}_h;\mathbf{v}_h,q_h)\le Ch\Big(|\mathbf{u}_h|_2+|p_h|_1+\|\mathbf{f}\|_0\Big)\Big(\|\mathbf{v}_h\|_1+\|q_h\|_0\Big), \tag{5.8}$$

Next, we prove that

$$|\mathbf{u}_h|_2+|p_h|_1\le C(\|\mathbf{u}\|_2+\|p\|_1+\|\mathbf{f}\|_0). \tag{5.9}$$

Let us divide each rectangle $K\in\mathcal{T}_h$ into two triangles T^+ and T^- by connecting its one diagonal. Denote by $\mathcal{T}_h^1$ the resulting triangulation. For $T\in\mathcal{T}_h^1$, denote by $P_1(T)$ the space of all linear polynomials defined on T, and define

$$\mathbf{V}_h^1:=\Big\{\mathbf{v}_h\in\mathbf{V}:\mathbf{v}_h|_T\in P_1(T)^2,\forall T\in\mathcal{T}_h^1\Big\}. \tag{5.10}$$

Let I_h^1 be the interpolation operator from $H^2(\Omega)^2$ to $\mathbf{V}_h^1$. Then

$$\begin{aligned}|\mathbf{u}_h|_2&=|\mathbf{u}_h-I_h^1\mathbf{u}|_2\le Ch^{-1}|\mathbf{u}_h-I_h^1\mathbf{u}|_1\\ &\le Ch^{-1}\Big(|\mathbf{u}-\mathbf{u}_h|_1+|\mathbf{u}-I_h^1\mathbf{u}|_1\Big)\le C\Big(\|\mathbf{u}\|_2+\|p\|_1+\|\mathbf{f}\|_0\Big).\end{aligned} \tag{5.11}$$

On the other hand

$$
\begin{aligned}
|p_h|_1 = |p_h - \pi_h p|_1 \le Ch^{-1}\|p_h - \pi_h p\|_0 \\
\le Ch^{-1}\Big(\|p - p_h\|_0 + \|p - \pi_h p\|_0\Big) \le C\Big(\|\mathbf{u}\|_2 + \|p\|_1 + \|\mathbf{f}\|_0\Big).
\end{aligned} \tag{5.12}
$$

Thus, (5.9) holds. Combining (5.8) with (5.9), we obtain

$$
\tilde{Q}(\mathbf{u}_h - \tilde{\mathbf{u}}_h, p_h - \tilde{p}_h; \mathbf{v}_h, q_h) \le Ch\Big(\|\mathbf{u}\|_2 + \|p\|_1 + \|\mathbf{f}\|_0\Big)\Big(\|\mathbf{v}_h\|_1 + \|q_h\|_0\Big), \tag{5.13}
$$

which gives

$$
\begin{aligned}
\beta(\|\mathbf{u}_h - \tilde{\mathbf{u}}_h\|_1 + \|p_h - \tilde{p}_h\|_0) &\le \sup_{(\mathbf{v}_h, q_h)\in V_h\times W_h} \frac{\tilde{Q}(\mathbf{u}_h - \tilde{\mathbf{u}}_h, p_h - \tilde{p}_h; \mathbf{v}_h, q_h)}{\|\mathbf{v}_h\|_1 + \|q_h\|_0} \\
&\le Ch\Big(\|\mathbf{u}\|_2 + \|p\|_1 + \|\mathbf{f}\|_0\Big).
\end{aligned} \tag{5.14}
$$

Finally, applying the triangle inequalities

$$
\|\mathbf{u} - \tilde{\mathbf{u}}_h\|_1 \le \|\mathbf{u} - \mathbf{u}_h\|_1 + \|\mathbf{u}_h - \tilde{\mathbf{u}}_h\|_1, \tag{5.15}
$$

$$
\|p - \tilde{p}_h\|_0 \le \|p - p_h\|_0 + \|p_h - \tilde{p}_h\|_0, \tag{5.16}
$$

we complete the proof. □

Now we employ a duality argument to derive the following theorem. We consider the dual problem: Find $(\mathbf{\Phi}, \Psi) \in \mathbf{V} \times W$ such that

$$
L(\mathbf{v}, q; \mathbf{\Phi}, \Psi) = (\mathbf{v}, \mathbf{u} - \tilde{\mathbf{u}}_h), \quad (\mathbf{v}, q) \in \mathbf{V} \times W. \tag{5.17}
$$

The solution satisfies the regularity condition

$$
\|\mathbf{\Phi}\|_2 + \|\Psi\|_1 \le C\|\mathbf{u} - \tilde{\mathbf{u}}_h\|_0. \tag{5.18}
$$

Theorem 5.2. *Let* $(\mathbf{u}, p)$ *and* $(\tilde{\mathbf{u}}_h, \tilde{p}_h)$ *be the solution of the Stokes problem* (1.9) *and the stabilized finite volume system* (3.21)*, respectively. Then*

$$
\|\mathbf{u} - \tilde{\mathbf{u}}_h\|_0 \le Ch^2(\|\mathbf{u}\|_2 + \|p\|_1 + \|\mathbf{f}\|_1), \tag{5.19}
$$

provided that $\mathbf{f} \in H^1(\Omega)^2$.

Proof. Setting $(\mathbf{v}, q) = (\mathbf{u} - \tilde{\mathbf{u}}_h, p - \tilde{p}_h)$ in (5.17), we have

$$
\|\mathbf{u} - \tilde{\mathbf{u}}_h\|_0^2 = A(\mathbf{u} - \tilde{\mathbf{u}}_h, \mathbf{\Phi}) + B(\mathbf{\Phi}, p - \tilde{p}_h) + B(\mathbf{u} - \tilde{\mathbf{u}}_h, \Psi). \tag{5.20}
$$

Setting $(\mathbf{v}, q) = (I_h\mathbf{\Phi}, J_h\Psi)$ in (1.9) and $(\mathbf{v}_h, q_h) = (I_h\mathbf{\Phi}, J_h\Psi)$ in (3.21), respectively, we have

$$
A(\mathbf{u}, I_h\mathbf{\Phi}) + B(I_h\mathbf{\Phi}, p) + B(\mathbf{u}, J_h\Psi) = (\mathbf{f}, I_h\mathbf{\Phi}), \tag{5.21}
$$

$$
\tilde{A}(\tilde{\mathbf{u}}_h, \Gamma_h I_h\mathbf{\Phi}) + \tilde{B}(I_h\mathbf{\Phi}, \gamma_h\tilde{p}_h) + \tilde{B}(\tilde{\mathbf{u}}_h, \gamma_h J_h\Psi) - \tilde{G}(\tilde{p}_h, J_h\Psi) = (\mathbf{f}, \Gamma_h I_h\mathbf{\Phi}). \tag{5.22}
$$

Combining (5.20)-(5.22), we have

$$
\begin{aligned}
\|\mathbf{u} - \tilde{\mathbf{u}}_h\|_0^2 =& A(\mathbf{u} - \tilde{\mathbf{u}}_h, \mathbf{\Phi}) - A(\mathbf{u}, I_h\mathbf{\Phi}) + \tilde{A}(\tilde{\mathbf{u}}_h, \Gamma_h I_h\mathbf{\Phi}) + B(\mathbf{\Phi}, p - \tilde{p}_h) - B(I_h\mathbf{\Phi}, p) \\
&+ \tilde{B}(I_h\mathbf{\Phi}, \gamma_h\tilde{p}_h) + B(\mathbf{u} - \tilde{\mathbf{u}}_h, \Psi) - B(\mathbf{u}, J_h\Psi) \\
&+ \tilde{B}(\tilde{\mathbf{u}}_h, \gamma_h J_h\Psi) - \tilde{G}(\tilde{p}_h, J_h\Psi) + (\mathbf{f}, I_h\mathbf{\Phi} - \Gamma_h I_h\mathbf{\Phi}).
\end{aligned} \tag{5.23}
$$

A similar argument as for (5.9) yields

$$|\tilde{\mathbf{u}}_h|_2 + |\tilde{p}_h|_1 \le C(\|\mathbf{u}\|_2 + \|p\|_1 + \|\mathbf{f}\|_0). \tag{5.24}$$

It follows from (5.4), Theorem 5.1, (2.7) and (5.24) that

$$\begin{aligned} & A(\mathbf{u}-\tilde{\mathbf{u}}_h, \mathbf{\Phi}) - A(\mathbf{u}, I_h\mathbf{\Phi}) + \tilde{A}(\tilde{\mathbf{u}}_h, \Gamma_h I_h \mathbf{\Phi}) \\ =& A(\mathbf{u}-\tilde{\mathbf{u}}_h, \mathbf{\Phi} - I_h\mathbf{\Phi}) - A(\tilde{\mathbf{u}}_h, I_h\mathbf{\Phi}) + \tilde{A}(\tilde{\mathbf{u}}_h, \Gamma_h I_h \mathbf{\Phi}) \\ \le& C\|\mathbf{u}-\tilde{\mathbf{u}}_h\|_1 \|\mathbf{\Phi} - I_h\mathbf{\Phi}\|_1 + Ch^2|\tilde{\mathbf{u}}_h|_2 |I_h\mathbf{\Phi}|_2 \\ \le& Ch^2(\|\mathbf{u}\|_2 + \|p\|_1 + \|\mathbf{f}\|_0)\|\mathbf{\Phi}\|_2. \end{aligned} \tag{5.25}$$

Using the Green's formula, (3.9), Theorem 5.1 and (5.24) gives

$$\begin{aligned} & B(\mathbf{\Phi}, p-\tilde{p}_h) - B(I_h\mathbf{\Phi}, p) + \tilde{B}(I_h\mathbf{\Phi}, \gamma_h\tilde{p}_h) \\ =& -(\mathrm{div}(\mathbf{\Phi} - I_h\mathbf{\Phi}), p-\tilde{p}_h) + (\mathrm{div}\ I_h\mathbf{\Phi} - \pi_h \mathrm{div}\ I_h\mathbf{\Phi}, \tilde{p}_h - \gamma_h\tilde{p}_h) \\ \le& C\|\mathbf{\Phi} - I_h\mathbf{\Phi}\|_1 \|p-\tilde{p}_h\|_0 + Ch^2|\mathrm{div}\ I_h\mathbf{\Phi}|_1|\tilde{p}_h|_1 \\ \le& Ch^2(\|\mathbf{u}\|_2 + \|p\|_1 + \|\mathbf{f}\|_0)\|\mathbf{\Phi}\|_2 + Ch^2|I_h\mathbf{\Phi}|_2|\tilde{p}_h|_1 \\ \le& Ch^2(\|\mathbf{u}\|_2 + \|p\|_1 + \|\mathbf{f}\|_0)\|\mathbf{\Phi}\|_2. \end{aligned} \tag{5.26}$$

In view of Green's formula, (3.9), Theorem 5.1 and (5.24), we have

$$\begin{aligned} & B(\mathbf{u}-\tilde{\mathbf{u}}_h, \Psi) - B(\mathbf{u}, J_h\Psi) + \tilde{B}(\tilde{\mathbf{u}}_h, \gamma_h J_h\Psi) \\ =& -(\mathrm{div}(\mathbf{u}-\tilde{\mathbf{u}}_h), \Psi - J_h\Psi) + (\mathrm{div}\ \tilde{\mathbf{u}}_h - \pi_h\mathrm{div}\ \tilde{\mathbf{u}}_h, J_h\Psi - \gamma_h J_h\Psi) \\ \le& C\|\mathbf{u}-\tilde{\mathbf{u}}_h\|_1\|\Psi - J_h\Psi\|_0 + Ch^2|\tilde{\mathbf{u}}_h|_2|J_h\Psi|_1 \\ \le& Ch^2(\|\mathbf{u}\|_2 + \|p\|_1 + \|\mathbf{f}\|_0)\|\Psi\|_1. \end{aligned} \tag{5.27}$$

By (5.24), (2.7), and (3.8), we get

$$-\tilde{G}(\tilde{p}_h, J_h\Psi) \le Ch^2|\tilde{p}_h|_1|J_h\Psi|_1 \le Ch^2(\|\mathbf{u}\|_2 + \|p\|_1 + \|\mathbf{f}\|_0)\|\Psi\|_1, \tag{5.28}$$

$$(\mathbf{f}, I_h\mathbf{\Phi} - \Gamma_h I_h\mathbf{\Phi}) = (\mathbf{f} - \pi_h\mathbf{f}, I_h\mathbf{\Phi} - \Gamma_h I_h\mathbf{\Phi}) \le Ch^2\|\mathbf{f}\|_1|I_h\mathbf{\Phi}|_1 \le Ch^2\|\mathbf{f}\|_1\|\mathbf{\Phi}\|_2. \tag{5.29}$$

Thus, combining (5.25)-(5.29), we obtain

$$\|\mathbf{u}-\tilde{\mathbf{u}}_h\|_0^2 \le Ch^2(\|\mathbf{u}\|_2 + \|p\|_1 + \|\mathbf{f}\|_1)(\|\mathbf{\Phi}\|_2 + \|\Psi\|_1), \tag{5.30}$$

which implies the desired result. □

6. Numerical Experiments

The objective of this section is to confirm the theoretical results obtained in the previous section through numerical experiments. Two examples using the new stabilized finite volume method for the Stokes equations (1.1)-(1.3) on the domain $\Omega = (0,1)\times(0,1)$ with the lowest equal-order finite element pair are considered. In all of the examples below, $\mathbf{u} = (u^1, u^2)$ represents the exact velocity, p the exact pressure, and the right-hand side function f can be computed by using the equation (1.1).

Example 6.1. $u^1 = \frac{1}{\pi}\sin^2(\pi x)\sin(2\pi y)$, $u^2 = -\frac{1}{\pi}\sin(2\pi x)\sin^2(\pi y)$, $p = \cos(\pi x)\cos(\pi y)$, and the viscosity $\lambda = 1$. The results are illustrated in Table 6.1.

Table 6.1: Numerical results for Example 6.1.

1/h	$\|\mathbf{u}-\mathbf{u}_h\|_0$	order	$\|\mathbf{u}-\mathbf{u}_h\|_1$	order	$\|p-p_h\|_0$	order
4	3.27E-2		6.51E-1		3.14E-1	
8	7.59E-3	2.11	3.26E-1	0.997	1.08E-1	1.54
16	1.83E-3	2.05	1.62E-1	1.011	3.56E-2	1.59
32	4.48E-4	2.03	8.06E-2	1.007	1.20E-2	1.56
64	1.11E-4	2.02	4.02E-2	1.004	4.15E-3	1.54
128	2.76E-5	2.01	2.01E-2	1.002	1.44E-3	1.52

Table 6.2: Numerical results for Example 6.2.

1/h	$\|\mathbf{u}-\mathbf{u}_h\|_0$	order	$\|\mathbf{u}-\mathbf{u}_h\|_1$	order	$\|p-p_h\|_0$	order
4	1.58E-3		1.81E-2		3.17E-3	
8	3.97E-4	1.99	8.81E-3	1.04	1.17E-3	1.43
16	8.99E-5	2.14	4.14E-3	1.08	4.72E-4	1.30
32	2.07E-5	2.11	2.00E-3	1.04	1.74E-4	1.43
64	4.92E-6	2.07	9.82E-4	1.02	6.25E-5	1.47
128	1.19E-6	2.04	4.86E-4	1.01	2.22E-5	1.49

Example 6.2. $u^1 = x^2(x-1)^2y(y-1)(2y-1)$, $u^2 = -y^2(y-1)^2x(x-1)(2x-1)$, $p = 2x(x-1)(2x-1)y(y-1)(2y-1)$, and the viscosity $\lambda = 0.1$. Table 6.2 shows the results.

From the results of Tables 6.1 and 6.2 we can see that the stabilized finite volume method for the Stokes equations in this article is effective and the numerical results are consistent with the theoretical analysis.

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