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SUPER-GEOMETRIC CONVERGENCE OF A SPECTRAL ELEMENT METHOD FOR EIGENVALUE PROBLEMS WITH JUMP COEFFICIENTS*

Lin Wang
College of Mathematics and Computer Science, Hunan Normal University, Changsha 410081, China and College of Information Technical Science, Nankai University, Tianjin 300071, China
Email: rinoya@mail.nankai.edu.cn
Ziqing Xie
College of Mathematics and Computer Science, Hunan Normal University, Changsha 410081, China
Email: ziqingxie@hunnu.edu.cn
Zhimin Zhang
Department of Mathematics, Wayne State University, Detroit, MI 48202, USA and College of Mathematics and Computer Science, Hunan Normal University, Changsha 410081, China
Email: zzhang@math.wayne.edu

Abstract

We propose and analyze a C^0 spectral element method for a model eigenvalue problem with discontinuous coefficients in the one dimensional setting. A super-geometric rate of convergence is proved for the piecewise constant coefficients case and verified by numerical tests. Furthermore, the asymptotical equivalence between a Gauss-Lobatto collocation method and a spectral Galerkin method is established for a simplified model.

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Key words: Eigenvalue, Spectral method, Collocation, Galerkin finite element method.

1. Introduction

We often encounter eigenvalue problems with discontinuous coefficients in practice. Examples of such applications may be found in [11]. In this paper, we consider the following one dimensional model problem: Find $(\lambda, u) \in \mathbb{R}^+ \times H^2(-\pi,\pi)$ such that

$$-u''(x) = \lambda c(x) u(x), \qquad u(-\pi) = u(\pi), \quad u'(-\pi) = u'(\pi). \tag{1.1}$$

Here $c(x) \geq c_0 > 0$ is a piecewise constant, or piecewise analytic function. The physics background of this model problem comes from the source-free Maxwell equations describing the transverse-magnetic mode in the one-dimensional periodic media, where the function u represents the electric field pattern, and the dielectric function $c(x)$ describes a unit cell from a multilayer structure with 2π-periodicity. This model problem was discussed by Min and

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Gottlieb in [11] where C^1 conforming spectral collocation methods were constructed on two elements over

$$H^2_{per}(-\pi,\pi) = \big\{v \in H^2(-\pi,\pi):\ \ v(-\pi) = v(\pi),\ v'(-\pi) = v'(\pi)\big\},$$

and error bounds of type $\mathcal{O}(p^{-m})$ were established. Note that the solution of (1.1) belongs to C^1.

It would be interesting to discuss C^0 spectral element methods over

$$H^1_{per}(-\pi,\pi) = \big\{v \in H^1(-\pi,\pi):\ \ v(-\pi) = v(\pi)\big\},$$

since the construction of a C^0 spectral element method is much simpler than that of the global C^1 spectral collocation method proposed in [11]. The idea of the spectral element can be found, e.g., in an early work [12]. Note that the spectral element method is equivalent to the so-called p-version finite element method, see e.g., [3]. Under the finite element variational framework, we are able to prove a super-geometric error bound of type $\mathcal{O}(e^{-2p(\log p-\gamma)})$. In some earlier works of the third author, the super-geometric error bound of type $\mathcal{O}(e^{-p(\log p-\gamma)})$ has been established for some spectral/collocation approximations of the two-point boundary problem [17, 18]. Our error bound for the eigenvalue approximation "doubles" the error bound for the associated eigenfunction approximation, the fact we have known for the h-version finite element method. It is worthy to point out that in the literature of the spectral method, it is a common practice to consider error bounds of type $\mathcal{O}(p^{-m})$, see, e.g., [5–7, 10, 15, 16], and reference therein. To the best of our knowledge, this is the first time that a super-geometric convergence rate is established for the eigenvalue approximation by the spectral method.

2. Theoretical Setting

The variational formulation of (1.1) is to find $(\lambda, u) \in \mathbb{R}^+ \times H^1_{per}(-\pi,\pi)$ such that

$$(u', v') = \lambda(cu, v), \quad \forall v \in H^1_{per}(-\pi,\pi). \tag{2.1}$$

In this paper, we also consider the Dirichlet problem

$$-u''(x) = \lambda c(x)u(x), \qquad u(0) = 0 = u(1).$$

Its variational formulation is to find $(\lambda, u) \in \mathbb{R}^+ \times H^1_0(0,1)$ such that

$$(u', v') = \lambda(cu, v), \qquad \forall v \in H^1_0(0,1). \tag{2.2}$$

By the general theory [2, 8], both problems (2.1) and (2.2) have countable infinite sequence of eigen-pairs (λ_j, u_j) satisfying

$$0 < \lambda_1 \le \lambda_2 \le \lambda_3 \le \cdots \to \infty, \qquad (u'_i, u'_j) = \lambda_j(cu_i, u_j) = \lambda_j \delta_{ij}.$$

Furthermore, eigenvalues can be characterized as extrema of the Rayleigh quotient $R(u) = (u', u')/(cu, u)$ as follows

$$\lambda_1 = \inf_{u \in S} = R(u_1),$$

$$\lambda_k = \inf_{u\in S,\ (u', u'_j)=0, j=1,\dots,k-1} R(u) = R(u_k), \quad k = 2, 3, \dots,$$

where $S = H^1_{per}(-\pi,\pi)$ or $H^1_0(0,1)$.

Next, we describe the framework of our numerical approximation. We partition the solution interval into m sub-intervals (element) such that $c(x)$ is analytic on each interval. Let h be the maximum length of all elements, we then define a finite dimensional subspace $S^h_p \subset S$, as a piecewise polynomial of degree p on each element. Our spectral element method is to find an eigen-pair $(\lambda(p), w_p) \in R^+ \times S^h_p$ such that

$$(w_p', v') = \lambda(p)(cw_p, v), \qquad \forall v \in S^h_p. \tag{2.3}$$

Note that the partition parameter h is fixed and convergence is achieved by increasing polynomial degree p. Therefore, we may suppress the index h later.

By the general theory [2,8], the problem (2.3) has a finite sequence of eigen-pairs $(\lambda_{j,p}, w_{j,p})$ satisfying

$$0 < \lambda_{1,p} \le \lambda_{2,p} \le \cdots \le \lambda_{N,p}, \quad N = \begin{cases} mp-1 & H^1_0(0,1) \\ mp & H^1_{per}(-\pi,\pi) \end{cases}$$

$$(w_{i,p}' w_{j,p}') = \lambda_{j,p}(cw_{i,p}, w_{j,p}) = \lambda_{j,p}\delta_{ij};$$

$$\lambda_{1,p} = \min_{w\in S_p} = R(w_{1,p}), \tag{2.4}$$

$$\lambda_{k,p} = \min_{w\in S_p,\,(w',w_{j,p}')=0, j=1,\ldots,k-1} R(w) = R(w_{k,p}), \quad k = 2,3,\ldots \tag{2.5}$$

One important observation from the above Minimum-Maximum principle is that the specific eigenvalue approximation is from above in the sense

$$\lambda_k \le \cdots \le \lambda_{k,p+1} \le \lambda_{k,p} \le \lambda_{k,p-1} \cdots \le \lambda_{k,1}.$$

3. A Galerkin Spectral (p-Version) Method

Without loss of generality, we consider piecewise constant $c(x)$ as in [11] with jump at the center of the solution domain. In particular, we take

$$c(x) = \begin{cases} 1 & x \in (-\pi, 0), \\ \omega^2 & x \in (0,\pi). \end{cases}$$

Instead of constructing C^1 shape functions for eigenvalue problem (2.1), we seek for a C^0 approximation $w_p \in H^1(0,1)$ with traditional expansion

$$w_p(x) = w^0(N_- + N_+) + \sum_{j=1}^{p-1} w^j \phi_{p-j+1}(x) + w^p N(x) + \sum_{j=p+1}^{2p-1} w^j \psi_{j-p+1}(x), \tag{3.1}$$

where $N_-(x), N(x)$, and $N_+(x)$ are linear nodal shape functions at the left end, middle point, and right end of the solution interval, respectively; ϕ_j and ψ_j are bubble functions on the left and right intervals, respectively. The counterpart of ϕ_{k+1} in $[-1,1]$ is defined as

$$\hat\phi_{k+1}(\xi) = \sqrt{\frac{2k+1}{2}} \int_{-1}^{\xi} L_k(t)dt = \frac{1}{\sqrt{2(2k+1)}}\Big(L_{k+1}(\xi) - L_{k-1}(\xi)\Big). \tag{3.2}$$

and the counterpart of ψ_{k+1} in $[-1,1]$ is defined similarly. Note that $w^0 = 0$ for the eigenvalue problem (2.2). With this setting, the resulting stiffness matrix is diagonal and the mass matrix is 5-diagonal, see Appendix.

4. Super-Geometric Convergence Rate

Let (λ_k, u_k) be the kth eigen-pair and $(\lambda_{k,h}, u_{k,h}) \in R \times S^h$ be its h-version finite element approximation. According to [2, p.700],

$$C_1 \epsilon_h^2 \le \lambda_{k,h} - \lambda_k \le C_2 \epsilon_h^2,$$

with

$$\epsilon_h = \inf_{\chi \in S^h} \|u_k - \chi\|_1,$$

for simple eigenvalue λ_k (see [2, p.695 (8.21)]. Transferring this theory to our spectral element method language, we have, for any simple eigen-pair (λ, u),

$$C_1 \epsilon_p^2 \le \lambda_p - \lambda \le C_2 \epsilon_p^2, \tag{4.1}$$

with

$$\epsilon_p = \inf_{\chi \in S_p} \|u - \chi\|_1 \approx \inf_{\chi \in S_p} \|u' - \chi'\| \approx \|u' - u_p'\|,$$

where $u_p \in S_p$ such that u_p' is the piecewise Legendre expansion of u' (not solution of (2.1) or (2.2)). Note that the first "$\approx$" comes from the Poicaré inequality and the last "$\approx$" is based on the fact that the Legendre expansion minimizes the L^2-norm.

Lemma 4.1. *Let u satisfy the regularity assumption*

$$\max_{x \in [-1,1]} |u^{(k)}(x)| \le cM^k$$

for fixed constants c and M, and let $\tilde{u}_p'$ be the Legendre expansion of u' on $[-1,1]$. Then under the assumption $(2p+1)(2p+3) > 2M^2$,

$$\|u' - \tilde{u}_p'\|_{L_2[-1,1]} \le C\sqrt{p}\left(\frac{eM}{2p}\right)^{p+1}, \tag{4.2}$$

where C is independent of p and M.

Proof. The error of $(p-1)$-term Legendre expansion is

$$\|u' - \tilde{u}_p'\|_{L_2[-1,1]}^2 = \sum_{k=p}^{\infty} \frac{2}{2k+1} b_k^2. \tag{4.3}$$

Using the result [13, p.58, Theorem 2.1.6], we have

$$b_k = \frac{2^k k!}{(2k)!} u^{(k+1)}(\eta_k), \quad \eta_k \in (-1,1). \tag{4.4}$$

Note that $(2^k k!)/(2k)! = 1/(2k-1)!!$. Applying the regularity assumption $|u^{(k)}(x)| \le cM^k$, we derive

$$\begin{aligned}
&\|u' - \tilde{u}_p'\|_{L_2[-1,1]}^2 \\
<&2(cM^{p+1})^2 \left(\frac{1}{(2p-1)!!(2p+1)!!} + \frac{M^2}{(2p+1)!!(2p+3)!!} + \frac{M^4}{(2p+3)!!(2p+5)!!} + \cdots\right) \\
=&\frac{2(cM^{p+1})^2}{(2p-1)!!(2p+1)!!}\left(1 + \frac{M^2}{(2p+1)(2p+3)} + \frac{M^4}{(2p+1)(2p+3)^2(2p+5)} + \cdots\right) \\
<&\frac{4(cM^{p+1})^2}{(2p-1)!!(2p+1)!!},
\end{aligned} \tag{4.5}$$

when $(2p+1)(2p+3) > 2M^2$. This last term can be readily estimated by Stirling type formula [1, (4.48)]

$$n! \approx \left(\frac{n}{e}\right)^n \sqrt{2\pi\left(n+\frac{1}{6}\right)}, \tag{4.6}$$

and [4]

$$(2n-1)!! \approx \frac{\sqrt{(2n)!}}{\sqrt[4]{\pi(n+\frac{1}{4})}}. \tag{4.7}$$

Therefore,

$$\begin{aligned}(2p-1)!!(2p+1)!! &\approx \frac{\sqrt{(2p)!}}{\sqrt[4]{\pi(p+0.25)}} \frac{\sqrt{(2p+2)!}}{\sqrt[4]{\pi(p+1.25)}} \\ &\approx \frac{(2p)!2p}{\sqrt{\pi p}} \approx \left(\frac{2p}{e}\right)^{2p} 2\sqrt{2}p,\end{aligned}$$

which, combined with (4.5), leads to (4.2) with $C = \sqrt[4]{2}c$. □

Now we are ready to prove our main theorem.

Theorem 4.2. *Let $(\lambda, u) \in R^+ \times S$ be an eigen-pair of problem (2.2), where λ is a simple eigenvalue. Let $\lambda(p)$ be its approximation in the sense of (2.4) or (2.5). Then*

$$\lambda(p) - \lambda \le Cp \left(\frac{e\sqrt{\lambda}}{4p}\right)^{2p+2}, \tag{4.8}$$

where C is independent of p and M.

Proof. Recall that $c(x)$ is constants on $(0,1/2)$ and $(1/2,1)$, we separate

$$\|u' - u'_p\|^2 = \|u' - u'_p\|^2_{L_2(0,1/2)} + \|u' - u'_p\|^2_{L_2(1/2,1)}. \tag{4.9}$$

Recall that u'_p is the piecewise Legendre expansion of u'. The estimate for the first term is as follows,

$$\|u' - u'_p\|^2_{L_2(0,1/2)} = 4\|\hat{u} - \hat{u}'_p\|^2_{L^2(-1,1)}, \tag{4.10}$$

where $\hat{u}(\xi) = u((1+\xi)/4)$. Now we apply Lemma 4.1 to have

$$\|\hat{u} - \hat{u}'_p\|_{L^2(-1,1)} \le C\sqrt{p}\left(\frac{e\hat{M}}{2p}\right)^{p+1} = C\sqrt{p}\left(\frac{e\sqrt{\lambda}}{4p}\right)^{p+1}.$$

Note that

$$\hat{M} = \frac{M}{4} = \frac{\sqrt{\lambda}}{2},$$

where M comes from the condition

$$\max_{x\in[0,1]} |u^k(x)| \le cM^k.$$

In our situation, u contains terms like $\sin 2\sqrt{\lambda}x, \cos 2\sqrt{\lambda}x, \ldots$ $M = 2\sqrt{\lambda}$, and $M = 4\hat{M}$ from

$$\frac{du}{dx}(x) = 4\frac{d\hat{u}}{d\xi}(\xi).$$

The second term in (4.9) can be estimated similarly, and therefore, we have

$$\|u' - u'_p\| \le C\sqrt{p}\left(\frac{e\sqrt{\lambda}}{4p}\right)^{p+1}.$$

The error bound (4.8) follows from (4.1).

Theorem 4.3. *Let* $(\lambda, u) \in R^+ \times S$ *be an eigen-pair of problem* (2.1)*, where* λ *is a simple eigenvalue. Let* $\lambda(p)$ *be its approximation in the sense of* (2.4) *or* (2.5)*. Then*

$$\lambda(p) - \lambda \le Cp\left(\frac{e\pi\sqrt{\lambda}}{2p}\right)^{2p+2}. \tag{4.11}$$

Proof. The proof is the same by the scaling between $(0,1)$ and $(-\pi,\pi)$.

Remark. The error bounds in Theorems 4.2 and Theorem 4.3 are super-geometric type $\mathcal{O}(e^{-(2p+1)(\log p-\gamma)})$ with $\gamma = \ln(e\sqrt{\lambda}/4)$ and $\gamma = \ln(e\pi\sqrt{\lambda}/2)$, respectively. We shall demonstrate in the next section, by numerical tests, that our error bounds are sharp.

Our estimates also indicate that we need higher p for larger λ to realize the convergence. This is consistent with our numerical experiences.

5. Numerical Tests

In this section, we implement the numerical scheme described in Section 3 to solve (1.1) with $\omega = 2$ for the first 14 eigenvalues. We observe convergence for reasonably smaller p. Actually, the error goes to the machine ϵ for $p \le 10$ for the first few eigenvalues. To verify our error bounds, we plot the ratio

$$(\lambda(p) - \lambda)/(p(0.5e\pi\sqrt{\lambda}/p)^{2p+2} \tag{5.1}$$

with some different λs. Here is a list of the square roots of eigenvalues ($\sqrt{\lambda}$ by increasing order):

$$\begin{array}{lllll}
\frac{1}{\pi}\arccos\left(-\frac{1}{3}\right), & \frac{1}{\pi}\arccos\left(-\frac{2}{3}\right), & \frac{1}{\pi}\arccos\frac{2}{3}+1, & \frac{1}{\pi}\arccos\frac{1}{3}+1, & 2, \\
\frac{1}{\pi}\arccos\left(-\frac{1}{3}\right)+2, & \frac{1}{\pi}\arccos\left(-\frac{2}{3}\right)+2, & \frac{1}{\pi}\arccos\frac{2}{3}+3, & \frac{1}{\pi}\arccos\frac{1}{3}+3, & 4, \\
\frac{1}{\pi}\arccos\left(-\frac{1}{3}\right)+4, & \frac{1}{\pi}\arccos\left(-\frac{2}{3}\right)+4, & \frac{1}{\pi}\arccos\frac{2}{3}+5, & \frac{1}{\pi}\arccos\frac{1}{3}+5, & 6, \cdots
\end{array}$$

Figures 1-4 demonstrate the ratio (5.1) associated with $\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5, \lambda_7, \lambda_9, \lambda_{14}$, respectively. We plot the ratio with different range of p. Since for a larger eigenvalue, we need relatively higher p to get into the asymptotic range. On the other hand, when p getting bigger and the error approaching the machine ϵ, the round-off error kicks in. So we can only observe the ratio in a small range of p. Nevertheless, it is sufficient to make our point clear. We see that the ratio (5.1) maintains in a reasonable range for different eigenvalues.

6. A Collocation Method for the Smooth Case

As a special case when $c(x)$ is sufficiently smooth, say a constant, we may use only one element. Without loss of generality, let us consider

$$-u'' = \lambda u \quad \text{in} \quad (-1,1) \qquad u(-1) = 0 = u(1). \tag{6.1}$$

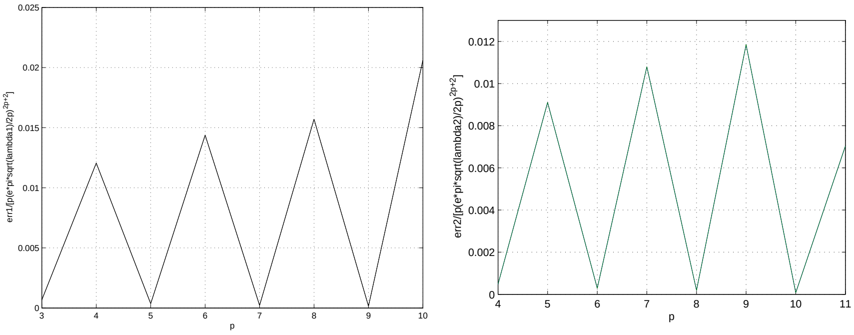

Fig. 5.1. Ratio of the computed errors over the estimated errors for λ_1 and λ_2

In this case, we are seeking an eigen-pair $(\lambda(p), w_p)$ with

$$w_p(\xi) = \sum_{j=2}^{p} w^j \hat{\phi}_j(\xi)$$

to satisfy

$$(w_p', \hat{\phi}_k') = \lambda(p)(w_p, \hat{\phi}_k), \qquad k = 2, \ldots, p. \tag{6.2}$$

Again, this result in an identity matrix on the left and a 5-diagonal matrix on the right. Based on the analysis in Section 4, we have in this case

$$\lambda(p) - \lambda \le Cp \left(\frac{e\sqrt{\lambda}}{2p} \right)^{2p+2}. \tag{6.3}$$

Let us consider a spectral collocation method

$$-w_p''(x_j) = \lambda(p) w_p(x_j), \qquad j = 1, 2, \ldots, p-1, \tag{6.4}$$

where x_js are zeros of L_p' and L_p is the Legendre polynomial of degree p on $[-1, 1]$.

Theorem 6.1. *For the model problem* (6.1)*, the spectral collocation method* (6.4) *is equivalent to replacing all integrations in* (6.2) *by the* $p+1$*-point Gauss-Lobatto quadrature. Furthermore, all numerical integrations are exact except the last term with*

$$(\hat{\phi}_p, \hat{\phi}_p)_* = \sum_{j=0}^{p} \hat{\phi}_p^2(x_j) w_j = (\hat{\phi}_p, \hat{\phi}_p) \frac{3(2p^2 - p - 1)}{2(2p^2 - p)}, \tag{6.5}$$

where $w_j s$ *are weights of the Gauss-Lobatto quadrature.*

Proof. We multiply both sides of (6.4) by $\hat{\phi}_k(x_j) w_j$ and sum up

$$-\sum_{j=0}^{p} w_p''(x_j) \hat{\phi}_k(x_j) w_j = \lambda(p) \sum_{j=0}^{p} w_p(x_j) \hat{\phi}_k(x_j) w_j \tag{6.6}$$

Since the $p+1$-point Gauss-Lobatto quadrature rule is exact for polynomials of degree up to $2p-1$, then we have

$$-\sum_{j=0}^{p} w_p''(x_j) \hat{\phi}_k(x_j) w_j = -(w_p'', \hat{\phi}_k) = (w_p', \hat{\phi}_k'), \quad k = 2, \ldots, p;$$

$$\sum_{j=0}^{p} w_p(x_j) \hat{\phi}_k(x_j) w_j = (w_p, \hat{\phi}_k), \quad k = 2, \ldots, p-1.$$

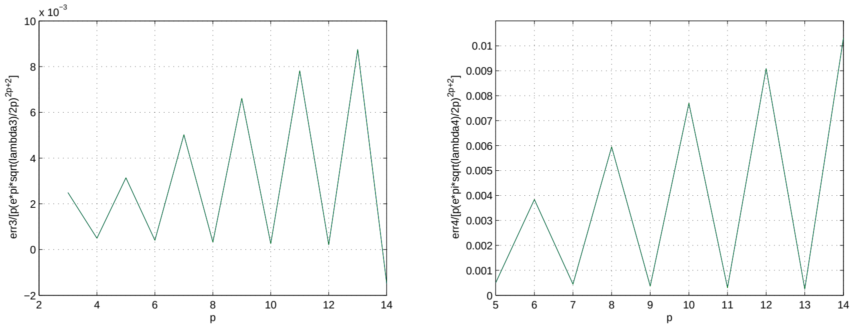

Fig. 5.2. Ratio of the computed errors over the estimated errors for λ_3 and λ_4

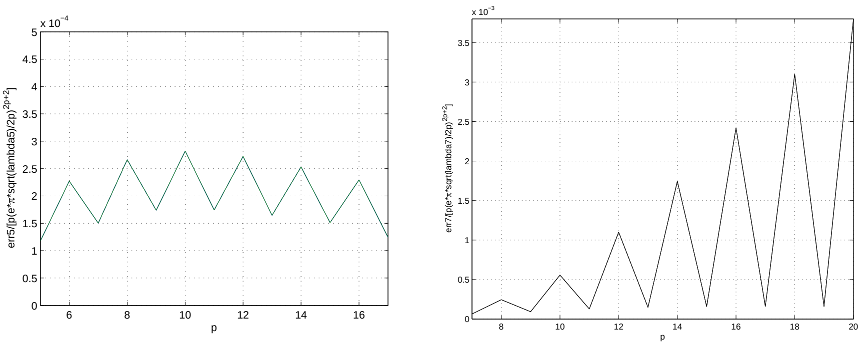

Fig. 5.3. Ratio of the computed errors over the estimated errors for λ_5 and λ_7

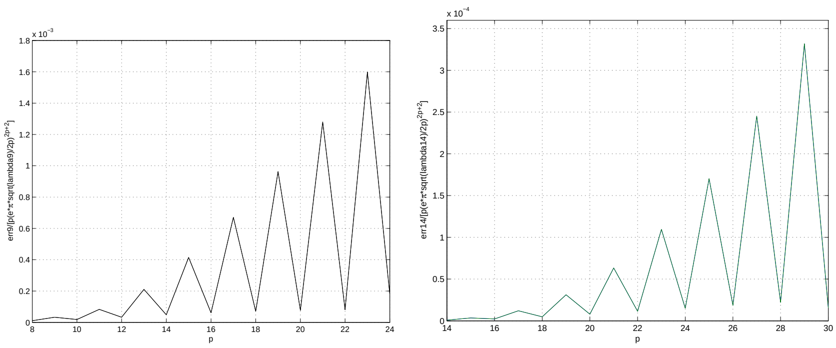

Fig. 5.4. Ratio of the computed errors over the estimated errors for λ_9 and λ_{14}

We see that the collocation method (6.4) is almost identical to the spectral method (6.2) except one term

$$(w_p, \hat{\phi}_p) \neq \sum_{j=0}^{p} w_p(x_j)\hat{\phi}_p(x_j)w_j = (w_p, \hat{\phi}_p)_*.$$

and their difference is

$$(w_p, \hat{\phi}_p) - (w_p, \hat{\phi}_p)_* = w^p[(\hat{\phi}_p, \hat{\phi}_p) - \hat{\phi}_p, \hat{\phi}_p)_*].$$

Using the fact $(L_p, L_p)_* = 2/p$, a direct calculation yields,

$$\begin{aligned}(\hat{\phi}_p, \hat{\phi}_p) =& \frac{1}{2(2p-1)}(L_p - L_{p-2}, L_p - L_{p-2}) \\ =& \frac{1}{2(2p-1)}\left((L_p, L_p) + (L_{p-2}, L_{p-2})\right) \\ =& \frac{1}{2(2p-1)}\left(\frac{2}{2p+1} + \frac{2}{2p-3}\right),\end{aligned} \tag{6.7}$$

and

$$\begin{aligned}(\hat{\phi}_p, \hat{\phi}_p)_* =& \frac{1}{2(2p-1)}\left((L_p, L_p)_* + (L_{p-2}, L_{p-2})\right) \\ =& \frac{1}{2(2p-1)}\left(\frac{2}{p} + \frac{2}{2p-3}\right).\end{aligned} \tag{6.8}$$

Therefore,

$$\frac{(\hat{\phi}_p, \hat{\phi}_p)_*}{(\hat{\phi}_p, \hat{\phi}_p)} = \frac{3(2p+1)(p-1)}{2p(2p-1)},$$

and (6.5) follows.

We see that the $p+1$-points Gauss-Lobatto quadrature has a 50% over-shoot asymptotically in calculating $(\hat{\phi}_p, \hat{\phi}_p)$. Nevertheless, the last coefficient w^p decays fast in general and the collocation method (6.4) is asymptotically equivalent to the spectral method (6.2). As a consequence, the spectral collocation method (6.4) also enjoys the super-geometric convergence rate (6.3).

Remark. It is feasible that a parallel result may be developed for the Chebysheve spectral/collocation methods. It is also feasible that the error bounds in [11] may be improved to the similar super-geometric rate as in this paper.

7. Appendix. The property of stiffiness matrix

Since $(\hat{\phi}'_i, \hat{\phi}'_j) = \delta_{ij}$, it is straightforward to verify that

$$(\phi'_i, \phi'_j) = 4\delta_{ij} = (\psi'_i, \psi'_j), \quad (N', \phi'_j) = 0 = (N', \psi'_j), \quad (N', N') = 4.$$

Furthermore, observe that

$$\begin{aligned}&4(\phi_i, \phi_j) = (\hat{\phi}_i, \hat{\phi}_j) = 4(\psi_i, \psi_j), \\ &(\phi_{k+1}, N) = 0 = (\psi_{k+1}, N) \quad k > 2, \\ &(N, N) = \frac{1}{4}\int_{-1}^{1}\left(\frac{1+\xi}{2}\right)^2 + \left(\frac{1-\xi}{2}\right)^2 = \frac{1}{16}\left(4 + \frac{4}{3}\right) = \frac{1}{3}, \\ &(\phi_2, N) = \frac{1}{4}(\hat{\phi}_2, \hat{N}_2) = \frac{1}{4\sqrt{6}}(L_2 - L_0, \hat{N}_2) = \frac{-1}{4\sqrt{6}}\int_{-1}^{1}\frac{1+\xi}{2} = \frac{-1}{4\sqrt{6}}, \\ &(\phi_3, N) = \frac{1}{4}(\hat{\phi}_3, \hat{N}_2) = \frac{1}{4\sqrt{10}}(L_3 - L_1, \hat{N}_2) = \frac{-1}{4\sqrt{10}}\int_{-1}^{1}\xi\frac{1+\xi}{2} = \frac{-1}{12\sqrt{10}}, \\ &(\psi_2, N) = \frac{1}{4}(\hat{\phi}_2, \hat{N}_1) = \frac{1}{4\sqrt{6}}(L_2 - L_0, \hat{N}_1) = \frac{-1}{4\sqrt{6}}\int_{-1}^{1}\frac{1-\xi}{2}\frac{-1}{4\sqrt{6}}, \\ &(\psi_3, N) = \frac{1}{4}(\hat{\phi}_3, \hat{N}_1) = \frac{1}{4\sqrt{10}}(L_3 - L_1, \hat{N}_1) = \frac{-1}{4\sqrt{10}}\int_{-1}^{1}\xi\frac{1-\xi}{2} = \frac{1}{12\sqrt{10}}.\end{aligned}$$

The special arrangement of ordering has advantage of symmetric structure in the resulting matrix. It is straightforward to verify that the weak formulation

$$(u_p', v') = \lambda(cu_p, v), \qquad \forall v \in S_p$$

yields a diagonal stiffness matrix $4I$ and a 5-diagonal mass matrix $\frac{\lambda}{4}A$, where the upper half of A is of the form

$$\begin{pmatrix}
\frac{2}{(2p+1)(2p-3)} & 0 & \frac{-(2p-3)^{-1}}{(2p-1)(2p-5)} & & & & & \\
0 & \frac{2}{(2p-1)(2p-5)} & 0 & & & & & \\
\frac{-(2p-3)^{-1}}{(2p-1)(2p-5)} & 0 & \frac{2}{(2p-3)(2p-7)} & \ddots & & & & \\
 & \ddots & \ddots & \ddots & \ddots & & & \\
 & & & 0 & \frac{2}{45} & 0 & \frac{-1}{5\sqrt{21}} & \\
 & & & \ddots & 0 & \frac{2}{21} & 0 & \frac{1}{3\sqrt{10}} \\
 & & & & -\frac{1}{5\sqrt{21}} & 0 & \frac{2}{5} & -\frac{1}{\sqrt{6}} \\
 & & & & & \frac{1}{3\sqrt{10}} & -\frac{1}{\sqrt{6}} & \frac{2}{3}
\end{pmatrix},$$

and the lower half of A is obtained by taking the above expression up-side-down and multiplied by ω^2. The bottom right entry $2/3$ of the upper part and the top left entry $2\omega^2/3$ of the lower part is the only overlap of the two parts.

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