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STABILIZED FEM FOR CONVECTION-DIFFUSION PROBLEMS ON LAYER-ADAPTED MESHES*

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Abstract

The application of a standard Galerkin finite element method for convection-diffusion problems leads to oscillations in the discrete solution, therefore stabilization seems to be necessary. We discuss several recent stabilization methods, especially its combination with a Galerkin method on layer-adapted meshes. Supercloseness results obtained allow an improvement of the discrete solution using recovery techniques.

Mathematics subject classification: 65N30.
Key words: Singular perturbations, Convection-diffusion, Finite element method, Stabilization, Layer-adapted mesh, Superconvergence, Recovery.

1. Introduction

We consider the two-dimensional convection-diffusion problem

$$Lu := -\varepsilon\Delta u - b\cdot\nabla u + cu = f \quad \text{in } \Omega \tag{1.1a}$$

$$u = 0 \quad \text{on } \Gamma = \partial\Omega, \tag{1.1b}$$

where ε is a small positive parameter, b, c are smooth and $f \in L_2(\Omega)$. Assuming

$$c + \frac{1}{2}\operatorname{div} b \geq \alpha_0 > 0, \tag{1.2}$$

the given problem admits a unique solution $u \in H_0^1(\Omega)$.

Let us introduce the ε- weighted H^1norm by

$$\|v\|_\varepsilon := \varepsilon^{1/2}|v|_1 + \|v\|_0.$$

Then for the Galerkin finite element method with piecewise linear or bilinear elements one can prove (C denotes a generic constant that is independent of ε and of the mesh)

$$\|u - u_h\|_\varepsilon \leq Ch|u|_2 \tag{1.3}$$

on quite general triangulations. However, estimate (1.3) is of no worth: in general, $|u|_2$ tends to infinity for $\varepsilon \to 0$ due to the presence of layers. The very weak stability properties of standard Galerkin lead to wild nonphysical oscillations in the discrete solution.

Therefore, stabilized Galerkin methods should be used. For several methods of this type—we shall present a short survey in Section 2—one has stability in a norm $\|\cdot\|_S$ that is stronger than the ε-weighted H^1 norm and, consequently, at most mild oscillations appear in the discrete solution. Moreover, for linear or bilinear elements we have the error estimate

$$\|u - u_h\|_S \leq C(\varepsilon^{1/2} + h^{1/2})\, h|u|_2. \tag{1.4}$$

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In contrast to the Galerkin method stabilized versions allow for local versions of (1.4) which show that stabilized methods provide good approximations in subdomains that exclude layers [40].

If one wants to resolve layers, it is possible to use layer-adapted meshes. These meshes are constructed a priori based on precise information on the structure of the layer (see Section 3). In this paper we mainly discuss problem (1.1) in $\Omega = (0,1)^2$ assuming

$$b = (b_1, b_2) > (\beta_1, \beta_2) > 0 \quad \text{on} \quad \overline{\Omega}. \tag{1.5}$$

Then exponential boundary layers form on

$$\Gamma^+ := \{x \in \Gamma : -b^T n > 0\}.$$

We shall also comment on the important special case $b_1 > \beta_1 > 0$, $b_2 \equiv 0$, which is characterized by the presence of parabolic boundary layers on

$$\Gamma^0 := \{x \in \Gamma : b^T n = 0\}$$

which are also called characteristic layers.

In case of exponential layers one has for standard Galerkin method applied to (1.1) on the simplest layer adapted mesh, the S-mesh (see Section 3), with a number of mesh points proportional to N^2

$$\|u - u^N\|_\varepsilon \le C N^{-1} \ln N, \tag{1.6}$$

see, e.g., [13, 44].

The fine mesh in the layer region induces some stability problems; nonetheless the computed solution exhibits oscillations (see the numerical experiments in [30]). Moreover, the stiffness matrix of the discrete problem generated has eigenvalues with large imaginary parts. Consequently, standard iterative methods are not able to solve the discrete systems efficiently. For the discussion of suitable iterative solvers see [14], Chapter 4. Let us remark, however, that robustness results for iterative solvers in the case of nonsymmetric problems are rare.

Therefore some stabilization seems to be necessary even when the layer-adapted meshes are used. A comparison of (1.3) and (1.6) suggests that for a stabilized method on a S-mesh one has

$$\|u - u^N\|_S \le C\big(\varepsilon^{1/2} + (N^{-1}\ln N)^{1/2}\big) N^{-1} \ln N.$$

But this is impossible because the estimate

$$\varepsilon^{1/2} |u - u^I|_1 \le C N^{-1} \ln N$$

for the interpolation error is optimal. Consequently, to verify the improved properties of stabilized methods on layer-adapted meshes in comparison to standard Galerkin we consider estimates for $\|u^I - u^N\|$ instead of $\|u - u^N\|$. In Section 4 we shall survey results of this type which often turn out to be *supercloseness results*. Furthermore, in many cases supercloseness allows the application of a *recovery procedure*. This yields an approximation Ru^N of u that is better than the approximation u^N computed first. Such recovery techniques are widely used in finite element methods to all kinds of problems including singularly perturbed problems, but for singularly perturbed problems its theoretical justification is especially delicate. Superconvergence and recovery techniques appear in several books of Chinese authors from the 1980s and 1990s but unfortunately these books are available only in Chinese with the recent exception [27].

2. Stabilized FEM on Standard Meshes

Streamline-diffusion FEM (or SDFEM for short) is also known as SUPG: streamline upwind Petrov-Galerkin. It is the most commonly used stabilized FEM for convection-diffusion problems. To the Galerkin bilinear form

$$a_G(w,v) := \varepsilon\big(\nabla w, \nabla v\big) - \big(b \cdot \nabla w - cw, v\big)$$

the stabilization term

$$a_{stab}(w,v) := \sum_T \delta_T\big(-\varepsilon \Delta w - b\nabla w + cw,\ -b\cdot\nabla v\big)_T$$

is added.

Let $V_{0,h} \subset H_0^1(\Omega)$ be the subspace of linear or bilinear finite elements and let $u_h \in V_{0,h}$ be defined by

$$a_h(u_h,v_h) := a_G(u_h,v_h) + a_{stab}(u_h,v_h) = f_h(v_h) := (f,v_h) + \sum_T \delta_T\big(f, -b\cdot\nabla v_h\big)_T.$$

The streamline-diffusion parameters $\delta_T \geq 0$ are user chosen. The SDFEM is consistent, i.e.,

$$a_h(u,v_h) = f_h(v_h).$$

Therefore, one can use the orthogonality property $a_h(u-u_h,v_h)=0$ for all $v_h \in V_{0,h}$ in the error estimate. Introducing the streamline-diffusion norm

$$\|v\|_{SD}^2 := \|v\|_\varepsilon^2 + \sum_T \delta_T\big(b\cdot\nabla v, b\cdot\nabla v\big)_T,$$

we have for bounded δ_T coercivity of the bilinear form in the SD-norm and the error estimate

$$\|u-u_h\|_{SD} \leq C(\varepsilon^{1/2} + h^{1/2})\, h\, |u|_2;$$

see, e.g., [40].

Remark 2.1. In 1D one can choose δ for a problem with constant coefficients in such a way that the discrete solution is exact at the nodes. Two dimensions δ_T is usually determined by minimizing the factor $\varepsilon + \delta_T + \delta_T^{-1}h_T^2$ under constraints on δ_T guaranteeing coercivity. In [14] the authors propose the smart choice

$$\delta_T = \begin{cases} \dfrac{h_T^*}{2|b_T|}\left(1 - \dfrac{1}{P_T}\right) & \text{if} \quad P_T := \dfrac{|b_T| h_T^*}{2\varepsilon} > 1, \\ 0 & \text{if} \quad P_T \leq 1, \end{cases} \tag{2.1}$$

where $|b_T|$ is the Euclidean norm of the wind b at the element center and h_T^* the element length in the direction of the wind. Later we shall see that near characteristic layers this proposal is not optimal.

There are many variants of SDFEM, see for example the recent surveys [15] and [32]. Here we shall not discuss these variants and other methods closely related to SDFEM like residual free bubbles or variational multiscale methods, see [6, 20, 22, 23, 40].

The SDFEM solution is not free of oscillations. See Section 3.5.2 in [14] for the analysis of the related difference equations and its oscillation properties. Similar behaviour can be expected for all other stabilization techniques we are going to discuss, although we are not aware of a rigorous analysis for these methods analogously to that of SDFEM.

Oscillations can be suppressed by nonlinear modifications of the SDFEM [40]. Certain shock-capturing variants even satisfy a discrete maximum principle [7].

A careful analysis of the terms added in the SDFEM shows that only

$$\sum_T \delta_T \big(b \cdot \nabla w, b \cdot \nabla w\big)_T$$

is responsible for increased stability. However, if only this term is added not only consistency is lost, but also accuracy.

Local projection stabilization. An improved version of this idea uses two finite element spaces V_h and D_h and subtracts from $b \cdot \nabla u_h$ its L_2-projection π_h into D_h. In this way a stabilization term of the form

$$a_{LPS}(u_h, v_h) := \sum_M \tau_M \big(b \cdot \nabla u_h - \pi_h(b \cdot \nabla u_h), b \cdot \nabla v_h - \pi_h(b \cdot \nabla v_h)\big)_M \tag{2.2}$$

is added (assuming D_h lives on a triangulation with elements M).

The choice $D_h = V_h$ in [11] necessitates the computation of the global L_2-projection into the finite element space. In contrast the use of a possibly discontinuous space D_h on a macro mesh allows the local computation of $\pi_h(b \cdot \nabla u_h)$; see [4]. A general theory of these local projection stabilizations is given in [34]; see also [12].

The spaces V_h and D_h can live on different meshes or on the same mesh. In the first case, for our linear or bilinear elements D_h consists of piecewise constant functions on a macro mesh $\mathcal{T}_{2h}$. The mesh $\mathcal{T}_h$ is the result of the decomposition of every triangle of $\mathcal{T}_{2h}$ into 3 subtriangles, alternatively every rectangle is decomposed into 4 subelements. If V_h and D_h live on the same mesh, standard choices are P_1^{bubble}/P_0^{disc} and Q_1^{bubble}/P_0^{disc}, where P_1^{bubble} is the space P_1 enriched by the standard cubic bubble.

For this local projection method we have the error bound [34]

$$\|u - u_h\|_{LPS} \le C\big(\varepsilon^{1/2} + h^{1/2}\big)\, h\, |u|_2, \tag{2.3}$$

with

$$\|v\|_{LPS}^2 := \|v\|_\varepsilon^2 + \sum_M \tau_M \big\|b \cdot \nabla v - \pi_h(b \cdot \nabla v)\big\|_{0,M}^2.$$

Remark 2.2. The proof of (2.3)—in particular the treatment of the convection term—employs the existence of a quasi-interpolant $j_h w \in V_h$ of w that possesses both standard approximation properties and the orthogonality property

$$(w - j_h w, q_h) = 0 \quad \text{for all} \quad q_h \in D_h. \tag{2.4}$$

The latter allows to expand

$$\big(u - j_h u, b \cdot \nabla v_h\big) = \big(u - j_h u, b \cdot \nabla v_h - \pi_h(b \cdot \nabla v_h)\big).$$

and, using Cauchy-Schwarz, this term can be estimated against the new term in the norm (and not against $\varepsilon^{1/2}|v|_1$, which would require the factor $\varepsilon^{1/2}$). This is similar to the analysis of SDFEM.

Edge stabilization or *continuous interior penalty* method CIP [10] adds certain jump terms on interior edges E to the Galerkin bilinear form:

$$J_h(u_h, v_h) := \sum_E \tau_E \big(b \cdot [\nabla u_h]_E, b \cdot [\nabla v_h]_E\big)_E. \tag{2.5}$$

It was observed numerically [41,43] that for edge stabilization it is advantageous to incorporate the boundary conditions in the weak sense. Let $V_h \subset H^1$ be the space of linear or bilinear finite element. The bilinear form corresponding to the new situation is given by

$$a_h(w, v) := a_G(w, v) + a_N(w, v) \tag{2.6}$$

with the Nitsche bilinear form

$$a_N(w, v) := -\varepsilon \left\langle \frac{\partial w}{\partial n}, v \right\rangle_\Gamma - \varepsilon \left\langle \frac{\partial v}{\partial n}, w \right\rangle_\Gamma + \langle |b \cdot n\, w|, v\rangle_\Gamma + \varepsilon \sum_{E \subset \Gamma} \frac{\gamma}{h_E} \langle w, v\rangle_E .$$

Then, edge stabilization with weakly incorporated boundary conditions is characterized by: Find $u_h \in V_h$ such that

$$a_h(u_h, v_h) + J_h(u_h, v_h) = (f, v_h) \quad \text{for all} \quad v_h \in V_h. \tag{2.7}$$

If on the given shape-regular mesh the L_2 projection is H_1-stable and $\tau = ch_E^2$, then [8,9]

$$\|u - u_h\|_{CIP} \le C\big(\varepsilon^{1/2} + h^{1/2}\big)\, h\, |u|_2 \tag{2.8}$$

with the norm

$$\|v\|_{CIP}^2 := \|v\|_\varepsilon^2 + J_h(v, v) + \big\||b \cdot n|^{1/2} v\big\|_{0,\Gamma}^2 + \sum_{E \subset \Gamma} \frac{\varepsilon}{h_E} \|v\|_{0,E}^2.$$

Remark 2.3. Instead of the standard nodal interpolant of u, the proof of (2.8) uses the L_2 projection $\pi u \in V_h$, the orthogonality property

$$(u - \pi u, b \cdot \nabla v_h) = (u - \pi u, b \cdot \nabla v_h - w_h) \quad \text{for arbitrary } w_h \in V_h$$

and the special choice $w_h = \tilde{\pi}_h(b \cdot \nabla v_h)$, where $\tilde{\pi}_h$ is the Oswald projection onto V_h. For this projection we have for both piecewise constant and piecewise linear b

$$h_K \|b \cdot \nabla v_h - \tilde{\pi}(b \cdot \nabla v_h)\|_{0,K}^2 \le C \sum_{E \subset \mathcal{E}(K)} \int_E h_E^2 \mid b \cdot [\nabla v_h]_E \mid^2 . \tag{2.9}$$

Thus, also from the theoretical point of view it is important to use V_h, but not $V_{0,h}$.

Finally, note the following estimate [9]

$$h_K \|b \cdot \nabla v_h - \tilde{\pi}(b \cdot \nabla v_h)\|_{0,K}^2 \ge C^* \sum_{E \subset \mathcal{E}(K)} \int_E h_E^2 \mid b \cdot [\nabla v_h]_E \mid^2,$$

which shows that CIP is equivalent to a special local projection method.

Discontinuous Galerkin. Last, but not least we like to mention stabilization by the discontinuous Galerkin method (dG). In the dG method the FE-space $V_h^{disc} \supset V_h$ consists of discontinuous functions. Therefore, similar to (2.6), the corresponding bilinear form contains many additional terms which arise from integration by parts. The stabilizing term in dG-methods is of the form

$$J_h^{disc}(u_h, v_h) := \sum_E \delta_E \big([u_h], [v_h]\big)_E$$

and consists of jumps of the function values across edges in contrast to jumps of derivatives in (2.5). Remark, there exists many variants of dG methods [1]; we sketch results for interior penalty methods based on the primal formulation.

For these dG methods we have in the associated norm [19]

$$\|u - u_h\|_{dG} \le C\big(\varepsilon^{1/2} + h^{1/2}\big)\, h|u|_2.$$

Remark it is widespread not to consider the upwind form of dG for the first order terms as a stabilization procedure. However in [5] it was shown that the upwind form of dG arises from the standard handling by adding a jump term for stabilization.

3. Solution Decomposition and Layer-adapted Meshes

In the simplest case of a one-dimensional problem of type (1.1) with an exponential boundary layer at $x = 0$ one can estimate

$$\big|u^{(i)}(x)\big| \le C\,\big(1 + \varepsilon^{-i}\exp(-\beta x/\varepsilon)\big) \quad \text{for} \quad i = 0, 1, \ldots, q, \tag{3.1}$$

where q depends on the regularity of the data of the problem. Surprisingly, (3.1) is equivalent to the existence of a decomposition of the solution $u = S + E$ into a smooth part S and a layer component E with

$$\big|S^{(i)}(x)\big| \le C, \quad \big|E^{(i)}(x)\big| \le C\varepsilon^{-i}\exp(-\beta x/\varepsilon) \quad \text{for} \quad i = 0, 1, ..., q,$$

and

$$LS = f \qquad \text{and} \qquad LE = 0.$$

We call such a decomposition S-decomposition because it was introduced by Shishkin to analyze upwind finite difference schemes.

Two dimensions sufficient conditions for the existence of such a decomposition are known in certain special cases and for small values of q only: for problems with exponential layers and for some problems with characteristic layers; see [24, 25, 31, 36].

The next two sections mainly discuss (1.1) with (1.5), i.e., problems with only exponential boundary layers. We shall assume the existence of a solution decomposition

$$u = S + E_1 + E_2 + E_{12} \tag{3.2a}$$

such that

$$\left|\frac{\partial^{i+j} S}{\partial x^i \partial y^j}(x,y)\right| \le C, \qquad \left|\frac{\partial^{i+j} E_1}{\partial x^i \partial y^j}(x,y)\right| \le C\varepsilon^{-i} e^{-\beta_1 x/\varepsilon} \tag{3.2b}$$

$$\left|\frac{\partial^{i+j} E}{\partial x^i \partial y^j}(x,y)\right| \le C\varepsilon^{-j} e^{-\beta_2 y/\varepsilon}, \quad \text{and} \quad \left|\frac{\partial^{i+j} E}{\partial x^i \partial y^j}(x,y)\right| \le C\varepsilon^{-(i+j)} e^{-(\beta_1 x + \beta_2 y)/\varepsilon} \tag{3.2c}$$

for all $(x,y) \in \bar\Omega$ and $0 \le i + j \le 3$.

The validity of (3.2) requires additional compatibility conditions of the data at the corners which are sometime unrealistic. Therefore, it is reasonable to search for analysis techniques of FEM on layer-adapted meshes that use the pointwise information of (3.2) for, say $0 \le i+j \le 2$ only, and weaker L_2 information for certain third-order derivatives. We are not going to discuss this issue in detail here and refer the reader for instance to [16, 39].

Based on the above solution decomposition we construct layer-adapted meshes. Let us assume that for a one-dimensionsl problem on the interval $[0,1]$ a layer of type $\exp(-\beta x/\varepsilon)$ is located at $x=0$. As early as 1969, Bakhvalov [3] proposed a special mesh with mesh points x_i near $x=0$ given by

$$q\left(1-\exp\left(-\frac{\beta x_i}{\sigma\varepsilon}\right)\right)=\xi_i:=\frac{i}{N}.$$

The parameter $q \in (0,1)$ determines how many mesh points are used to resolve the layer, while $\sigma > 0$ controls the spacing within the layer region. Outside the layer an equidistant mesh is used. To be precise, Bakhvalov's mesh is specified by $x_i = \varphi(i/N)$, $i = 0, 1, \ldots, N$, where

$$\varphi(\xi)=\begin{cases}\chi(\xi):=-\dfrac{\sigma\varepsilon}{\beta}\ln\dfrac{q-\xi}{q} & \text{for} \quad \xi\in[0,\tau],\\ \chi(\tau)+\dfrac{\xi-\tau}{1-\tau}\big(1-\chi(\tau)\big) & \text{for} \quad \xi\in[\tau,1].\end{cases}$$

Here τ is a transition point between the fine and coarse submeshes. Originally Bakhvalov chose τ to ensure that the mesh generating function φ lay in $C^1(0,1)$ with $\varphi(1)=1$. However the explicit definition

$$\tau=\frac{\gamma\varepsilon}{\beta}|\ln\varepsilon| \qquad (e^{-\beta\tau/\varepsilon}=\varepsilon^{\gamma}\,!)$$

is also possible and gives a mesh we shall refer to as a *B-type mesh*.

From the numerical point of view when choosing the transition point, it seems better to replace the smallness of the layer term with respect to ε by smallness with respect to the discretization error. Assume the formal order of the method is σ. Then imposing

$$\exp\left(-\frac{\beta\tau}{\varepsilon}\right)=N^{-\sigma}$$

yields the choice $\tau = \big((\sigma\varepsilon)/\beta\big)\ln N$ for the transition point. We call a mesh an *S-type mesh* if it is generated by

$$\varphi(\xi)=\begin{cases}\dfrac{\sigma\varepsilon}{\beta}\hat\varphi(\xi) \quad \text{with} \quad \hat\varphi(1/2)=\ln N & \text{for } \xi\in[0,1/2],\\ 1-2\Big(1-\dfrac{\sigma\varepsilon}{\beta}\ln N\Big)(1-\xi) & \text{for } \xi\in[1/2,1].\end{cases}$$

In particular, when $\hat\varphi(\xi) = 2(\ln N)\xi$, the mesh generated is piecewise equidistant. This *S-mesh* was introduced by Shishkin in 1988. For surveys concerning layer adapted meshes, see [28, 29].

The analysis of certain difference methods for one-dimensional problems in [29] shows: If the pointwise error of a particular method on an S-mesh is proportional to $(N^{-1}\ln N)^{\sigma}$, then on a B-type mesh (and on S-type meshes with certain optimality properties) the error is of $\mathcal{O}(N^{-\sigma})$.

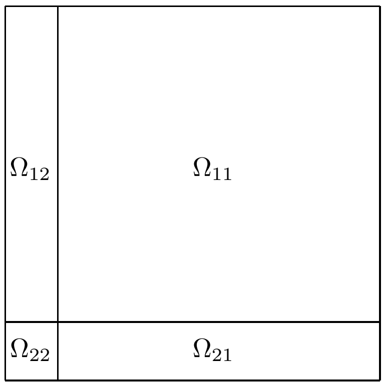
For finite element methods the situation is different. So far, except for [37], there are no optimal error estimates for B-type meshes. If a transition point in the sense of Bakhvalov is used and a piecewise constant or locally uniform meshes, then the error weakly depends on ε (see [2]). In some papers, for instance in [47], the different impact of the choice of the transition point in the sense of Bakhvalov or Shishkin is discussed. On a piecewise constant or polynomial graded mesh (i.e., not on the original Bakhvalov mesh) the authors of [47] demonstrate numerical results which indicate the different error behavior on these two types of meshes.

The simple structure of S-type meshes allows error estimates for many stabilization methods as we will see in Section 4. For simplicity, we restrict ourselves to S-meshes in this paper—in most cases a generalization of the results to S-type meshes is possible.

For the two-dimensional problem (1.1) in $\Omega = (0,1)^2$ with exponential boundary layers let us define

$$\lambda_x := \min\left\{q, \frac{\sigma\varepsilon}{\beta_1}\ln N\right\} \quad \text{and} \quad \lambda_y = \min\left\{q, \frac{\sigma\varepsilon}{\beta_2}\ln N\right\}.$$

This definition allows to include the non-singularly perturbed case. Divide the domain Ω as in Fig. 3.1.



$$\begin{aligned}
\Omega_{11} &:= [\lambda_x, 1] \times [\lambda_y, 1] \\
\Omega_{12} &:= [0, \lambda_x] \times [\lambda_y, 1] \\
\Omega_{21} &:= [\lambda_x, 1] \times [0, \lambda_y] \\
\Omega_{22} &:= [0, \lambda_x] \times [0, \lambda_y]
\end{aligned}$$

Fig. 3.1. Subregions of Ω with exponential layers.

The nodes of our rectangular mesh are obtained from the tensor product of a set of N_x points in x-direction and N_y points in y-direction. A one-dimensional Shishkin mesh is characterized by an equidistant mesh size h in $[0, \lambda_x]$ and H in $[\lambda_x, 1]$, in the transition point λ_x the mesh switches from coarse to fine. For simplicity, let us assume

$$\beta_1 = \beta_2 \quad \text{and} \quad \lambda_x = \frac{\sigma\varepsilon}{\beta_1}\ln N.$$

Thus

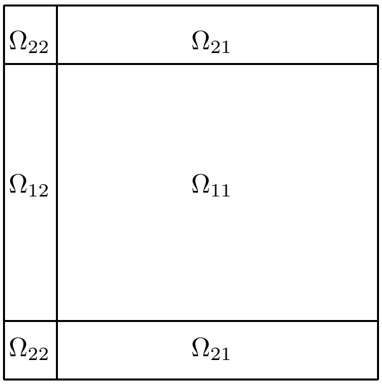
$$h = 2\lambda_x/N \quad \text{and} \quad H = 2(1-\lambda_x)/N$$

and

$$\begin{aligned}
x_i = y_i &= ih && \text{for} \quad i = 0, 1, \cdots, N/2 \\
x_i = y_i &= (1-\lambda_x) + H\Big(i - \frac{N}{2}\Big) && \text{for} \quad i = N/2+1, \cdots, N.
\end{aligned}$$

Remark 3.1. In the case $b_1 > \beta_1 > 0$, $b_2 \equiv 0$ the parabolic boundary layers at $y = 0$ and $y = 1$ are of width $\mathcal{O}(\varepsilon^{1/2}|\ln\varepsilon|)$. Therefore, they require a different choice of transition point in y-direction: $\tau_y = \mathcal{O}(\varepsilon^{1/2}\ln N)$, while $\tau_x = \mathcal{O}(\varepsilon\ln N)$ remains unchanged; see Fig. 3.2.

For more complicated domains the construction of layer-adapted meshes is of course much more involved; see, for instance, [35] for a description of the generation of such meshes.



$$\begin{aligned}
\Omega_{11} &:= [\lambda_x, 1] \times [\lambda_y, 1-\lambda_y] \\
\Omega_{12} &:= [0, \lambda_x] \times [\lambda_y, 1-\lambda_y] \\
\Omega_{21} &:= [\lambda_x, 1] \times ([0, \lambda_y] \cup [1-\lambda_y, 1]) \\
\Omega_{22} &:= [0, \lambda_x] \times ([0, \lambda_y] \cup [1-\lambda_y, 1])
\end{aligned}$$

Fig. 3.2. Subregions of Ω with exponential and parabolic layers.

4. Stabilization on S-meshes and Recovery

In this section we consider problem (1.1) assuming both (1.5) and the existence of the solution decomposition (3.2). We discretize the problem on an S-mesh by linear/bilinear finite elements and various stabilization techniques.

It was first observed numerically in [30] that both for a Galerkin method and for streamline diffusion the convergence rates in L_∞ for linear and bilinear elements on $\Omega \setminus \Omega_{11}$ different significantly: the rates for bilinears are twice the rates for linears! This fact can be explained with superconvergence phenomena for bilinears and is the reason for us to prefer bilinears in layer regions.

In [45] Stynes and Tobiska analyzed the SDFEM for bilinears on an S-mesh. The SD-parameter is chosen by assuming throughout that $\varepsilon \le CN^{-1}$

$$\delta_K = \begin{cases} N^{-1} & \text{if } K \in \Omega_{11}, \\ 0 & \text{otherwise.} \end{cases} \tag{4.1}$$

A detailed analysis shows, for instance, on Ω_{12} the stabilization parameter δ_{12} should satisfy $\delta_{12} \le C\varepsilon N^{-2}$. This value is much smaller than the natural diffusion ε and therefore, switching of the stabilization by setting $\delta_{12} = 0$ is reasonable.

Using the so called Lin identities for bilinears, see e.g. [19], one gets the supercloseness result

$$\left\| u^I - u^N \right\|_{SD} \le C\big(\varepsilon N^{-3/2} + N^{-2} \ln^2 N\big). \tag{4.2}$$

However, if bilinear elements are used in the layer region on $\Omega \setminus \Omega_{11}$, but linear elements on the coarse-mesh region Ω_{11}, then only the standard ingredients of the SDFEM analysis are available to estimate the error contribution on Ω_{11}. Fortunately, on Ω_{11} the layer components are small. One obtains for the method on a hybrid mesh consisting of rectangles and triangles

$$\left\| u^I - u^N \right\|_{SD} \le C\big(\varepsilon^{1/2} N^{-1} + N^{-3/2}\big).$$

Remark 4.1. For the problem with characteristic boundary layers it is more difficult to tune the SD-parameter. In the region Ω_{21} (see Fig. 3.2) the recommendation (2.1) gives $\delta_{21} = \mathcal{O}(N^{-1})$, but this choice is not appropriate [26]. For bilinears it was shown in [16] that $\delta_{21} \le C\varepsilon^{-1/4} N^{-2}$ should be satisfied.

Based on (4.2) and the standard postprocessing approach of [27] one can construct a local postprocessing operator P such that Pu^N approximates u better than u^N with respect to $\|\cdot\|_\varepsilon$.

Consider a family of S-meshes $\mathcal{T}_N$ where we require $N/2$ to be even. Then we can build a coarser mesh composed of disjoint macro rectangles M, each comprising four mesh rectangles from $\mathcal{T}_N$, where M belongs to only one of the four domains $\Omega_{11}, \Omega_{12}, \Omega_{21}, \Omega_{22}$. Associate with each macro rectangle M an interpolation operator $P_M : C(\overline{M}) \to Q_2(M)$ defined by standard biquadratic interpolation. As usual, P_M can be extended to a continuous global interpolation operator $P : C(\overline{\Omega}) \to W^N$, where W^N is the space of piecewise quadratic elements.

Then properties (7.2), (7.3) and (7.4) of [27]—approximation property, stability of P and consistency of P—give

$$\left\|u - Pu^N\right\|_\varepsilon \le C\big(\varepsilon N^{-3/2} + N^{-2}\ln^2 N\big).$$

The proof starts from the consistency of P:

$$u - Pu^N = u - Pu + P(u^I - u^N).$$

Then the stability of P is applied in order to give

$$\left\|u - Pu^N\right\|_\varepsilon \le \|u - Pu\|_\varepsilon + \left\|P(u^I - u^N)\right\|_\varepsilon \le \|u - Pu\|_\varepsilon + C\,\left\|u^I - u^N\right\|_\varepsilon . \tag{4.3}$$

The terms on the right-hand side will be bounded using the approximation property of P and the supercloseness of the method.

However, when linear elements on triangles are used in Ω_{11}, supercloseness results are known only for special triangulations. Moreover, only for isosceles and Friedrichs-Keller triangulations postprocessing procedures are presented in [27].

Now, let us consider edge stabilization and local projection. Corresponding to the choice (4.1) of the SD-parameter we shall stabilize on Ω_{11} only, while on $\Omega\backslash\Omega_{11}$ we shall use unstabilized Galerkin based on bilinear elements.

First, let us discuss the consistent edge stabilization method. If we use bilinear elements everywhere we can estimate the Galerkin part of the error by

$$\left|a_G\left(u - u^I, u^N\right)\right| \le C\big(\varepsilon N^{-3/2} + N^{-2}\ln^2 N\big)\|u^N\|_\varepsilon.$$

Hence, from a theoretical point of view it is not necessary to incorporate the boundary conditions in a weak sense for applying the technique described in Section 2 for bounding the convective error term. Likewise, a triangular mesh on Ω_{11} that allows for the estimate

$$\left|\int_{\Omega_{11}} (S - S^I)_x\, w^N\right| \le CN^{-3/2}\left\|w^N\right\|_{0,\Omega_{11}},$$

enables an error analysis of edge stabilization on Ω_{11} without the use of (2.9); see [38]. For example, a Friedrichs-Keller triangulation possesses this property.

However, when an arbitrary triangulation is used on Ω_{11} it seems necessary to apply (2.9) and therefore to incorporate the boundary conditions on $\partial\Omega_{11} \cap \Gamma = \Gamma_{11}$ weakly. Introducing

$$V_h = \Big\{v_h \text{ with } v_h|_{\Gamma\backslash\Gamma_{11}} = 0,\ v_h|_K \in Q_1 \text{ if } K \subset \Omega \setminus \Omega_{11} \text{ and } v_h|_K \in P_1 \text{ if } K \in \Omega_{11}\Big\},$$

we define

$$a_h(w,v) := a_G(w,v) - \varepsilon\left\langle \frac{\partial w}{\partial n}, v\right\rangle_{\Gamma_{11}} - \varepsilon\left\langle w, \frac{\partial v}{\partial n}\right\rangle_{\Gamma_{11}} + \langle b\cdot n\, w, v\rangle_{\Gamma_{11}} + \sum_{E\in\Gamma_{11}} \frac{\varepsilon\gamma}{h_E}\langle w,v\rangle_E\,.$$

Note that $-b \cdot n < 0$ on Γ_{11} and $h_E = cN^{-1}$. As we like to stabilize on Ω_{11} only, we penalize jumps of the streamline derivative across all interior edges of Ω_{11} by adding

$$J_h(u_h, v_h) := \sum_{E \subset \Omega_{11}} \tau N^{-2} \big(b \cdot [\nabla u_h]_E \,, b \cdot [\nabla v_h]_E\big)_E$$

to the bilinear form.

Let us define a generalized interpolation operator $\pi w \in V_h$ for $w \in H^2(\Omega)$ by combining standard interpolation and L_2 projection:

$$\pi w := \begin{cases} w^I \text{ on } \bar{\Omega} \setminus \big((x_{N/2-1}, 1) \times (y_{N/2-1}, 1)\big), \\ L_2\text{-projection onto piecewise linears on } \Omega_{11}. \end{cases}$$

On the missing strip πw is uniquely since $\pi w \in V_h$.

Combing the results for standard Galerkin using bilinear elements in the layer region with the technique described in Section 2, we obtain

$$\|u^N - \pi u\|_{CIP} \leq C(N^{-3/2} + N^{-2} \ln^2 N);$$

see [17] for details of the analysis.

Next we consider the local-projection stabilization on the coarse mesh region of our S-mesh. We consider the version of LPS-schemes characterized by enrichment of the original finite element space and the use of a single mesh rather than the version based on macro meshes. The stabilization term is given by

$$a_{LPS}(u_h, v_h) := \sum_{T \in \Omega 11} \tau N^{-1} \big(b \cdot \nabla u_h - \pi(b \cdot \nabla u_h), b \cdot \nabla v_h - \pi(b \cdot \nabla v_h)\big).$$

For the LPS-scheme let $V_h \subset H_0^1(\Omega)$ be finite element space consisting of bilinears on $\Omega \setminus \Omega_{11}$ and linear elements enriched by a single bubble per element on Ω_{11}. Furthermore, let π project onto the space of piecewise constants on the triangulation of Ω_{11}.

For the analysis of the method a special interpolant $j_h w \in V_h$ of a given function w is used. On a triangle K it is defined by

$$j_h w(P_i) = w(P_i) \ \text{ for all three vertices } P_i \text{ of } K$$

and

$$(w - j_h w, q)_K = 0 \ \text{ for arbitrary constants } q.$$

Note, $j_h w$ used on $\bar{\Omega}_{11}$ and the standard bilinear interpolant w^I used in $\Omega \setminus \Omega_{11}$ match continuously. We call its composite π^*.

The well known results for the bilinear Galerkin method in the layer region and the technique of [34] described in Section 2 give

$$\|u^N - \pi^* u\|_{LPS} \leq C(N^{-3/2} + N^{-2} \ln^2 N).$$

In [33] the author considers Q_r-elements for $r \geq 2$ enriched by six additional functions (such that the element contains P_{r+1}) and local projection stabilization on Ω_{11} with $\tau = \mathcal{O}(N^{-2})$. The resulting error estimates are however not optimal; it seems better not to enrich the space on Ω/Ω_{11} as in [34].

Finally, let us mention that it is also possible to combine the Galerkin finite element method with bilinears in the layer region with discontinuous Galerkin in Ω_{11} as a stabilization technique. There are many variants of discontinuous Galerkin, see the survey in [1]. In [42] the nonsymmetric version with interior penalties (NIPG, c.f., [21]) was considered, but similar results hold for its symmetric version SIPG too. For the so-called local discontinuous Galerkin method LDG see [46,47], the authors present numerical studies which indicate superconvergence phenomena as well for the L_2 norm of the solution as for the one side flux.

For NIPG in the associated dG-norm the supercloseness result

$$\|u^N - \pi u\|_{dG} \le C\big(\varepsilon^{1/2}N^{-1} + N^{-3/2}\big)$$

was proved in [42]. Here πu denotes the L_2-projection onto $P_1^{disc}(\Omega_{11})$ and $Q_1^{disc}(\Omega_{11})$, respectively, on Ω_{11}, but the standard bilinear interpolant on $\Omega \setminus \Omega_{11}$.

For edge stabilization and LPS-stabilization with bilinears one can use the same recovery technique as for SDFEM to get improved approximations with respect to $\|\cdot\|_\varepsilon$. This is because the finite element spaces used are continuous. However, for dGFEM recovery on Ω_{11} requires new ingredients. Let us sketch the basic idea from [18]. We restrict ourselves to Q_1^{disc}. In [18] the general case Q_n^{disc} is considered.

Again macroelements belonging to only one of the four subdomains Ω_{11}, Ω_{12}, Ω_{21} and Ω_{22} are used. Any macro consists of four mesh rectangles of the given mesh. We describe the recovery procedure for $Q_1^{disc}(\Omega_{11})$ on $K_1 := [0,1]^2, K_2 := [-1,0] \times [0,1], K_3 := [-1,0]^2$ and $K_4 := [0,1] \times [-1,0]$, and the macroelement $M := [-1,1]^2$. Let

$$R_{i,t}(v) := \int_0^1 \eta_i(t)v(t)dt; \quad L_{i,t}(v) := \int_{-1}^0 \eta_i(t+1)v(t)dt,$$

where the η_i are the Legendre polynomials on $[0,1]$. The tensor product structure allows to define the following nine degrees of freedom:

$$\begin{aligned}
N^{i,j} &:= (R_{i,x} + L_{i,x})(R_{j,y} + L_{j,y}) \quad \text{for} \quad i,j = 0,1,\\
N^{2,j} &:= (R_{1,x} - L_{1,x})(R_{j,y} + L_{j,y}) \quad \text{for} \quad j = 0,1,\\
N^{i,2} &:= (R_{i,x} + L_{i,x})(R_{1,y} - L_{1,y}) \quad \text{for} \quad i = 0,1,\\
N^{2,2} &:= (R_{1,x} - L_{1,x})(R_{1,y} - L_{1,y}).
\end{aligned} \tag{4.4}$$

This definition fixes nine combinations of the 16 values

$$\int_{K_i} \eta_l(x)\eta_m(y)\, v \quad \text{for} \quad l = 0,1,\ m = 0,1, \quad i = 1,2,3,4.$$

Next, $Pv \in Q_2(M)$ is defined by

$$N^{i,j}(Pv) = N^{i,j}(v)$$

for the nine degrees of freedom due to (4.4).

In [18] it is shown that P is consistent and stable. It is interesting to note that P is different from the L_2-projection onto $Q_2(M)$ which is moreover not H^1-stable.

Consistency, stability and the supercloseness property allow to prove

$$|||Pu - u|||_\varepsilon \le C\big(\varepsilon^{1/2}N^{-1} + N^{-3/2}\big),$$

where $|||\cdot|||_\varepsilon$ is the piecewise ε-weighted H^1-norm because Pu is discontinuous on Ω_{11}.

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