Journal of Computational Mathematics, Vol.25, No.1, 2007, 49–66.

ASYMPTOTIC BEHAVIOR OF THE TE AND TM APPROXIMATIONS TO SECOND HARMONIC GENERATION *

Naïma Aïssa
(*Faculté de Mathématiques, USTHB, BP32, El Alia, Bab Ezzouar, 16111, Alger;*
Centre de Mathématiques Appliquées, CNRS UMR 7641 & Ecole Polytechnique, Palaiseau 91128 Cedex, France
Email: aissa@cmapx.polytechnique.fr, n.aissa@yahoo.fr)

Abstract

We consider a Second Harmonic Generation (SHG) problem of an optical signal wave with an optical pump in a medium represented by a smooth bounded domain $\Omega \subset \mathbb{R}^d$, which is assumed to contain a heterogeneous material: a compactly imbedded subdomain $B^r \subset\subset \Omega$ in the shape of a small ball contains a nonlinear material, while $\Omega \setminus \overline{B^r}$ is filled with a linear material. We begin by proving existence and uniqueness of the solution to the TE approximation of SHG for arbitrary bounded susceptibilities, thus improving the result obtained by Bao and Dobson (Eur. J. Appl. Math. 6 (1995), 573-590) under small enough susceptibilities assuption. We then establish an existence and uniqueness result of a solution to the TM approximation problem. In both parts we study the asymptotic behavior of the system as the size of the nonlinear material vanishes: error estimates and asymptotic expansion of the solution are derived for both TE and TM approximations.

Mathematics subject classification: 35Q60, 35B40, 35C20.
Key words: Second harmonic generation, TE Approximation, TM Approximation.

1. Introduction

Franken *et al.* [12] in their second harmonic generation experiment developed a process for generating double frequency laser beams, thus marking the advent of the field of nonlinear optics. Though revolutionary, the theory that underpins the discovery is quite basic: a given medium is subjected to an intense beam of optical pump waves, causing the field in the medium to be polarized nonlinearly. The former and latter process are governed respectively by the constitutive equations and a linear system of Maxwell's equations. For further details, we refer the reader to the very interesting book by Shen [19].

In this paper, we consider a domain Ω which is filled with a heterogeneous material. Inside the domain, a ball-shaped subdomain $B^r \subset\subset \Omega$ of small size, with center x_0 and radius r contains a nonlinear material, while $\Omega \setminus \overline{B^r}$ is filled with a linear material.

We first use the two-dimensional space model introduced in [6] to deal with the TE approximation (*i.e* the diffracted electric fields are assumed to be directed in the vertical direction): we improve the existence and uniqueness result stated in [6] under small enough susceptibilities assumption. Then we study the TM approximation (*i.e.,* the diffracted magnetic and electric fields, with respectively the same and the double frequency as the incident wave, are assumed to be directed in the vertical direction) in the setting of the model proposed in [8] in the three-dimensional case, thereby bypassing the two-dimensional case which presents a technical difficulty for deriving the asymptotic expansion of the solution. More precisely, we prove existence and uniqueness of the solution, and furthermore establish the well-posedness of the problem.

* Received January 4, 2006; accepted February 14, 2006.

In both the TE and TM approximations, the behavior of the solution is considered as the size of the nonlinear material vanishes. Moreover, error estimates and asymptotic expansion are derived.

2. The TE Approximation

2.1. Model Problem

Consider the model set in [6]. Throughout the paper, we assume that the medium is nonmagnetic and has constant magnetic permeability. For the sake of convenience, the magnetic permeability parameter is set to 1. In addition, we also assume that no external charges nor current are present in the field.

The time-harmonic Maxwell equations which govern second harmonic generation (SHG) take the form

$$\begin{cases} \nabla \times \mathbf{E} = -\frac{i\omega}{c}\mathbf{H}, \quad \nabla \cdot \mathbf{H} = 0, \\ \nabla \times \mathbf{H} = \frac{i\omega}{c}\mathbf{D}, \quad \nabla \cdot \mathbf{D} = 0, \end{cases} \tag{1}$$

along with the constitutive equation

$$\mathbf{D} = \varepsilon\ (\mathbf{E} + 4\pi \mathbf{P}), \tag{2}$$

where $\mathbf{E}$ is the electric field, $\mathbf{H}$ the magnetic field, $\mathbf{D}$ the electric displacement, $\mathbf{P}$ the polarization field, ε the electric permittivity of the medium, c the speed of light and ω the angular frequency. The physics of SHG may be described as follows: when a plane wave with frequency $\omega = \omega_1$ is projected onto a nonlinear medium, it generates two diffracted waves with respective angular frequencies $\omega = \omega_1$ and $\omega = \omega_2 = 2\omega_1$ because of the interaction between the incident wave and the nonlinear medium. The presence of new frequency components is the most striking difference between nonlinear and linear optics. For most media however, the nonlinear optical effects are so negligible that they may be ignored. To observe nonlinear phenomena in the optical region, one needs high-intensity beams like high-intensity laser ones.

Let us consider the two wave fields $\mathbf{E}(x, \omega_1)$ and $\mathbf{E}(x, \omega_2 = \omega_1 + \omega_1)$. To simplify our notation, we denote $\mathbf{E}(x, \omega_i) = \mathbf{E}(\omega_i)$.

Since second harmonic generation can be considered as a special case of optical mixing [19], the polarization field at frequencies ω_1 and ω_2 respectively are given by [19, p. 68]

$$\begin{cases} \mathbf{P}(w_1) = \chi^{(1)}(\omega_1).\mathbf{E}(\omega_1) + \chi^{(2)}(x, \omega_1) : \mathbf{E}^\star(\omega_1)\mathbf{E}(\omega_2), \\ \mathbf{P}(w_2) = \chi^{(1)}(\omega_2).\mathbf{E}(\omega_2) + \chi^{(2)}(x, \omega_2) : \mathbf{E}(\omega_1)\mathbf{E}(\omega_1), \end{cases}$$

where $\chi^{(1)}$ is the linear susceptibility tensor of the medium, $\chi^{(2)}$ is the second-order nonlinear susceptibility tensor of third rank, that means that, $\chi^{(2)} : \mathbf{EE}$ is a vector whose jth component is $\sum_{k,l=1}^{3} \chi^{(2)}_{jkl}\mathbf{E}_k\mathbf{E}_l$, and $\mathbf{E}^\star$ is the complex conjugate of $\mathbf{E}$. Then the Maxwell equations (1)-(2) yield the following coupled system

$$\begin{cases} \left[\nabla \times (\nabla \times) - \frac{\omega_1^2 d_1}{c^2}\right]\mathbf{E}(\omega_1) = \frac{4\pi\omega_1^2\varepsilon}{c^2}\chi^{(2)}(\omega_1 = -\omega_1 + \omega_2) : \mathbf{E}^\star(\omega_1)\mathbf{E}(\omega_2), \\ \left[\nabla \times (\nabla \times) - \frac{\omega_2^2 d_2}{c^2}\right]\mathbf{E}(\omega_2) = \frac{4\pi\omega_2^2\varepsilon}{c^2}\chi^{(2)}(\omega_2 = \omega_1 + \omega_1) : \mathbf{E}(\omega_1)\mathbf{E}(\omega_1), \end{cases}$$

where $d_i = \varepsilon(1 + 4\pi\chi^{(1)}(\omega_i))$. The medium is said to be linear if $\mathbf{D} = \varepsilon(1 + \chi^{(1)}(\omega))\mathbf{E}$, *i.e.*, $\chi^{(2)}$ vanishes. We assume that all the fields are invariant in the vertical direction. Then the problem can be formulated in two dimensions. We shall also assume that the electric fields at ω_1 and ω_2 are **TE** polarized, which means that

$$\mathbf{E}(\omega_i) = e(\omega_i)\mathbf{u}_3, \quad i = 1, 2,$$

where $(\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3)$ is the canonical basis of $\mathbb{R}^3$.

Let Ω be a smooth domain in $\mathbb{R}^2$. We assume that Ω is occupied heterogeneously: a small ball $B^r \subset\subset \Omega$ is filled with a nonlinear material whereas $\Omega^r = \Omega \setminus \overline{B^r}$ contains a linear material. In the present case, the nonlinear material is represented by the ball B^r of center x_0 and radius r. The susceptibilities $\chi_1(\omega_1)$ and $\chi_2(\omega_2)$ are with support B^r so they depend on r likewise $e(\omega_1)$ and $e(\omega_2)$ also depend on r. In addition, as the material is assumed heterogeneous, the permittivity of the medium is a function ε of the space variable x.

Define

$$\begin{cases} u_r = e(\omega_1),\ v_r = e(\omega_2), \\ \chi_1^r = -\frac{4\pi\omega_1^2}{c^2}\varepsilon(x)\chi_{333}^{(2)}(\omega_1 = -\omega_1 + \omega_2), \\ \chi_2^r = -\frac{4\pi\omega_2^2}{c^2}\varepsilon(x)\chi_{333}^{(2)}(\omega_2 = \omega_1 + \omega_1). \end{cases}$$

Writing the transmission conditions on the boundary ∂B^r and the Neumann boundary conditions on $\partial\Omega$, we get the following system

$$\begin{cases} \left(\Delta + \omega_1^2(1 + 4\pi\chi^{(1)}(\omega_1))\frac{\varepsilon(x)}{c^2}\right)u_r = \chi_1^r(\omega_1, x)u_r^\star v_r \quad in\ \Omega, \\ \left(\Delta + \omega_2^2(1 + 4\pi\chi^{(1)}(\omega_2)\frac{\varepsilon(x)}{c^2}\right)v_r = \chi_2^r(\omega_2, x)u_r^2 \quad in\ \Omega, \\ [u_r]_{|\partial B^r} = [v_r]_{|\partial B^r} = [\partial_n u_r]_{|\partial B^r} = [\partial_n v_r]_{|\partial B^r} = 0, \\ \partial_n u_r = f,\ \partial_n v_r = g \quad in\ \partial\Omega, \end{cases} \tag{3}$$

where

$$\varepsilon(x) = \begin{cases} \varepsilon_1 > 0 \quad in \quad B^r, \\ \varepsilon_2 > 0 \quad in \quad \Omega^r = \Omega \setminus \overline{B^r}. \end{cases} \tag{4}$$

Here $\varepsilon_1, \varepsilon_2$ are the permittivity of the medium occupied by B^r and Ω^r respectively. In the sequel, we set

$$k_j(x, \omega_j) = \begin{cases} \frac{\varepsilon_1(1+4\pi\chi^{(1)}(\omega_j))}{c^2} \quad in \quad B^r \\ \frac{\varepsilon_2(1+4\pi\chi^{(1)}(\omega_j))}{c^2} \quad in \quad \Omega^r = \Omega \setminus \overline{B^r} \end{cases} = \begin{cases} k_{j,1} \quad in \quad B^r, \\ k_{j,2} \quad in \quad \Omega^r = \Omega \setminus \overline{B^r}, \end{cases} \tag{5}$$

to rewrite (3) as follows

$$\begin{cases} \left(\Delta + \omega_1^2 k_1(x, \omega_1)\right)u_r = \chi_1^r(\omega_1, x)u_r^\star v_r \quad in\ \Omega, \\ \left(\Delta + \omega_2^2 k_2(x, \omega_2)\right)v_r = \chi_2^r(\omega_2, x)u_r^2 \quad in\ \Omega, \\ \partial_n u_r = f,\ \partial_n v_r = g \quad \text{on}\ \partial\Omega. \end{cases} \tag{6}$$

2.2. Existence and Uniqueness of the Solution and Uniform Estimates

Denote for $s > 0$, $H^s(\Omega)$ the classical Sobolev spaces of $\mathbb{C}$-values functions, $L^2(\Omega)$ the Lebesgue space of square integrable functions on Ω. In the sequel, the norm in $H^1(\Omega)$ will be denoted by $||.||$, while the norm in $L^2(\Omega)$ will be denoted by $|.|$.

Definition 2.1. *A weak solution to* (6) *is a pair* $(u_r, v_r) \in H^1(\Omega) \times H^1(\Omega)$ *that satisfies the weak formulation*

$$\begin{cases} \displaystyle\int_\Omega \nabla u_r \cdot \nabla\phi^\star\ dx - \omega_1^2 \int_\Omega k_1 u_r \phi^\star\ dx = \int_{\partial\Omega} f\phi^\star\ d\sigma - \int_{B^r} \chi_1^r u_r^\star v_r \phi^\star\ dx, \\ \displaystyle\int_\Omega \nabla v_r \cdot \nabla\psi^\star\ dx - \omega_2^2 \int_\Omega k_2 v_r \psi^\star\ dx = \int_{\partial\Omega} g\psi^\star\ d\sigma - \int_{B^r} \chi_2^r u_r^2 \psi^\star\ dx, \end{cases} \tag{7}$$

for any test function $(\phi, \psi) \in H^1(\Omega) \times H^1(\Omega)$. *Here* $\psi^\star$ *denotes the complex conjugate of* ψ.

Recall that as $\Omega \subset \mathbb{R}^2$, if $u_r \in H^1(\Omega)$ then $u_r \in L^p(\Omega)$ for any $1 \le p < \infty$. Proof of existence of solutions to (6) is given in [6] and [8] for χ_j^r small enough. We extend this result to arbitrary bounded susceptibilities χ_j^r by using the Schauder fixed point theorem. In order to prove the existence of solutions to the nonlinear problem, we first establish the following lemmas dealing with the linear problem.

Lemma 2.1. *Let* $\omega \in \mathbb{R}$ *and* $k(x,\omega)$ *be defined by* (5), *where* ω_j *is replaced by* ω. *Then, for all but a discrete set of frequencies* ω, *for any* $F \in L^2(\Omega)$ *and* $f \in L^2(\partial\Omega)$, *there exists a unique weak solution* $u \in H^1(\Omega)$ *to the linear problem*

$$\begin{cases} \Big(\Delta + \omega^2 k(x,\omega)\Big)u = F \text{ in } \Omega, \\ \partial_n u = f \text{ on } \partial\Omega. \end{cases} \tag{8}$$

In particular, if $f \in H^{\frac{1}{2}}(\partial\Omega)$ *then* $u \in H^2(\Omega)$ *is a strong solution to* (8).

Proof. We summarize the proof given in [6] or [8] as follows. The weak formulation of (8) writes

$$\int_\Omega \nabla u \cdot \nabla \phi^\star \, dx - \omega^2 \int_\Omega k(x,\omega) u \phi^\star \, dx = \int_{\partial\Omega} f \phi^\star \, d\sigma - \int_\Omega F \phi^\star \, dx, \ \forall \phi \in H^1(\Omega). \tag{9}$$

It can be written in the form $B(u,\phi) = L(\phi)$ where B and L are respectively the bilinear form and the linear form in $H^1(\Omega)$ given by the left-hand and the right-hand side of (9). We can write $B = B^1 + \omega^2 B^2$ where

$$\begin{cases} B^1(u,\phi) = \displaystyle\int_\Omega \nabla u \cdot \nabla \phi^\star \, dx + \omega^2 \int_\Omega u \phi^\star \, dx, \\ B^2(u,\phi) = -\displaystyle\int_\Omega k(x,\omega) u \phi^\star \, dx - \int_\Omega u \phi^\star \, dx. \end{cases}$$

Next, consider the operators $A(\omega,\omega) : H^1(\Omega) \to (H^1(\Omega))'$, $A^i(\omega) : H^1(\Omega) \to (H^1(\Omega))'$, $i = 1, 2$ defined by $(A(\omega,\omega)u,\phi) = B(u,\phi)$, $(A^i(\omega)u,\phi) = B^i(u,\phi)$. It is clear that $A(\omega,\omega) = A^1(\omega) + \omega^2 A^2(\omega)$. Our aim is to prove that the operator $A(\omega,\omega)$ is invertible for all but a discrete set of values of ω. By virtue of the Lax-Milgram theorem, the operator $A^1(\omega)$ is invertible and has bounded inverse. Moreover, the operator $A^2(\omega)$ is compact. Holding ω_1 fixed, consider the operator $A(\omega_1,\omega) = A^1(\omega_1) + \omega^2 A^2(\omega_1)$. We see that $A(\omega_1,\omega)^{-1}$ exists by Fredholm's theory for all $\omega \notin \mathcal{E}(\omega_1)$, where $\mathcal{E}(\omega_1)$ is some discrete set. Since

$$||A(\omega,\omega) - A(\omega_1,\omega)|| \to 0 \text{ as } \omega \to \omega_1,$$

it follows from the stability of the bounded invertibility that $A(\omega,\omega)^{-1}$ exists and is bounded for $|\omega - \omega_1|$ sufficiently small, $\omega \notin \mathcal{E}(\omega_1)$. Since ω_1 is an arbitrary real number, we have shown that $A(\omega,\omega)^{-1}$ exists for all but a discrete set of points and hence the equation $B(u,\phi) = L(\phi)$ admits a unique solution $u \in H^1(\Omega)$.

In the sequel, ω will denote a frequency for which the system (8) admits a unique solution u. Next, we will establish uniform estimates for the solution to the linear equation given by Lemma 2.1.

Lemma 2.2. *Assume that* $f \in L^2(\partial\Omega)$ *and* $F \in L^2(\Omega)$. *The solution* u *to* (8) *given by proposition* 2.1 *satisfies*

$$|u|^2 \le C(|f|^2 + |F|^2), \tag{10}$$

$$|\nabla u|^2 \le C(f|^2 + |F|^2), \tag{11}$$

for some constant C *independent of* r *and* u.

Proof. The proof of (10) is classical and can be obtained easily by a contradiction argument (see, e.g., [11, p. 306]). Next it stems from (9) that

$$|\nabla u|^2 - w^2|k^{\frac{1}{2}}u|^2 = \int_{\partial\Omega} f u^\star \, d\sigma - \int_\Omega F u^\star \, dx.$$

Using the following inequality for α small enough [14, p. 41]

$$\int_{\partial\Omega} |u|^2 \, d\sigma \leq \alpha |\nabla u|^2 + C_\alpha |u|^2,$$

we get the estimate $|\nabla u|^2 \leq C_1 |u|^2 + |f|^2 + |F|^2$. Hence estimate (11) is fulfilled thanks to (10). The proof of the lemma is then complete.

Next, in order to prove the existence of solutions to the nonlinear problem, consider $(U, V) \in H^1(\Omega) \times H^1(\Omega)$ and set $F = U^\star V$ and $G = U^2$. Then $F, G \in L^2(\Omega)$. Consequently, by Lemma 2.1, it follows that for all but a discrete set of frequencies ω_1, ω_2, there exists a unique weak solution $(u, v) \in H^1(\Omega) \times H^1(\Omega)$ to the linear problem

$$\begin{cases} (\Delta + \omega_1^2 k_1(x, \omega_1))u = \chi_1^r(\omega_1, x)F \quad \text{in } \Omega, \\ (\Delta + \omega_2^2 k_2(x, \omega_2))v = \chi_2^r(\omega_2, x)G \quad \text{in } \Omega, \\ \partial_n u = f, \;\; \partial_n v = g \quad \text{on } \partial\Omega. \end{cases} \tag{12}$$

Therefore, applying Lemma 2.2, we obtain

$$\begin{cases} |\nabla u|^2 \leq C\Big(|f|^2 + |\chi_1^r|_\infty \displaystyle\int_{B^r} |F|^2 \, dx\Big), \\ |\nabla v|^2 \leq C\Big(|g|^2 + |\chi_2^r|_\infty \displaystyle\int_{B^r} |G|^2 \, dx\Big). \end{cases}$$

As $H^1(\Omega)$ is continuously imbedded in $L^r(\Omega)$ for any $1 \leq r < \infty$, we have for all $q > 1$ and $p > 1$ satisfying $\frac{1}{p} + \frac{1}{q} = 1$

$$\begin{cases} \displaystyle\int_{B^r} |F|^2 dx \leq |B^r|^{\frac{1}{q}} \Big(\int_\Omega |U|^{2p}|V|^{2p}\Big)^{\frac{1}{p}} \leq C|B^r|^{\frac{1}{q}} ||U||^2 ||V||^2, \\ \displaystyle\int_{B^r} |G|^2 \, dx \leq |B^r|^{\frac{1}{q}} \Big(\int_\Omega |U|^{4p}\Big)^{\frac{1}{p}} \leq C|B^r|^{\frac{1}{q}} ||U||^4, \end{cases} \tag{13}$$

for some positive constant C independent of r. Then for all $q > 1$

$$\begin{cases} |\nabla u|^2 \leq C \; (|f|^2 + |\chi_1^r|_\infty |B^r|^{\frac{1}{q}} ||U||^2 ||V||^2), \\ |\nabla v|^2 \leq C \; (|v|^2 + |g|^2 + C \; |\chi_2^r|_\infty |B^r|^{\frac{1}{q}} ||U||^4). \end{cases} \tag{14}$$

Hence letting $q \to 1$ we get the following lemma.

Lemma 2.3. *Let $(U, V) \in H^1(\Omega) \times H^1(\Omega)$ and let (u, v) be the solution to* (12) *associated with $F = U^\star V$ and $G = U^2$. Then, there exists a constant $C > 0$ independent of r, (u, v) and the data F and g such that*

$$\begin{cases} |u|^2 \leq C(|f|^2 + |\chi_1^r|_\infty |B^r| ||U||^2 ||V||^2), \quad |v|^2 \leq C(|g|^2 + |\chi_2^r|_\infty |B^r| ||U||^4), \\ |\nabla u|^2 + |\Delta u|^2 \leq C(|f|^2 + |\chi_1^r|_\infty |B^r| ||U||^2 ||V||^2), \\ |\nabla v|^2 + |\Delta v|^2 \leq C(|g|^2 + \; |\chi_2^r|_\infty |B^r| ||U||^4). \end{cases}$$

Let $R > 0$ be a fixed positive number and denote by $Q_R = \{u \in H^1(\Omega), ||u|| \leq R\}$. We are now in the position to prove the following result.

Theorem 2.1. *Assume that χ_j^r is uniformly bounded in $L^\infty(\Omega)$ and that $(f,g) \in L^2(\partial\Omega) \times L^2(\partial\Omega)$. Then for all but a discrete set of frequencies ω_1, ω_2, there exists a constant C independent of r such that for all $R > C\max\{|f|^2, |g|^2\}$ and $r > 0$ small enough,* (6) *admits a unique weak solution $(u_r, v_r) \in Q_R \times Q_R$.*

Moreover, if $(f,g) \in H^{\frac{1}{2}}(\partial\Omega) \times H^{\frac{1}{2}}(\partial\Omega)$ then $(u_r, v_r) \in (Q_R \times Q_R) \cap (H^2(\Omega) \times H^2(\Omega))$ and satisfies

$$\begin{cases} |u_r|^2_{H^2(\Omega)} \le C\Big(|f|^2_{H^{\frac{1}{2}}(\partial\Omega)} + |\chi_1^r|_\infty |B^r| R^4\Big), \\ |v_r|^2_{H^2(\Omega)} \le C\Big(|g|^2_{H^{\frac{1}{2}}(\partial\Omega)} + |\chi_2^r|_\infty |B^r| R^4\Big), \end{cases} \tag{15}$$

for some positive constant C which is independent of r, u_r, v_r and the data.

Proof. We will apply Schauder's fixed point theorem to prove existence of solutions for appropriate R. In what follows, C denotes a generic positive constant which is independent of r. Let $X = L^4(\Omega) \times L^4(\Omega)$ and define the operator $\mathcal{R}$: $X \to X$ by $\mathcal{R}(U,V) = (u,v)$ where $(u,v) \in H^1(\Omega \times H^1(\Omega)$ is the solution to (12) associated with $F = U^\star V$ and $G = U^2$. For $R > 0$, set $K_R = Q_R \times Q_R$. It is clear that K_R is a convex and a compact subspace of X. Next, we prove that there exists $R > 0$ such that $\mathcal{R}$ maps K_R into itself. Indeed, let $(U,V) \in K_R$, then thanks to Lemma 2.3, we get

$$||u||^2 \le C(|f|^2 + |\chi_1^r|_\infty |B^r| R^4), \quad ||v||^2 \le C(|g|^2 + |\chi_2^r|_\infty |B^r| R^4).$$

Assuming that $|\chi_j^r|_\infty \le C_j$ with C_j independent of r, it is enough to choose $R > 0$ such that

$$C(|f|^2 + C_1 |B^r| R^4) \le R^2, \quad C(|g|^2 + C_2 |B^r| R^4) \le R^2.$$

We have $1 - 4C^2 C_1 |B^r||f|^2 > 0$ and $1 - 4C^2 C_1 |B^r||g|^2 > 0$ for r small enough then set

$$A_1^r = \frac{1 + \sqrt{1 - 4C^2 C_1 |B^r||f|^2}}{2CC_1|B^r|}, \quad A_2^r = \frac{1 + \sqrt{1 - 4C^2 C_2 |B^r||g|^2}}{2CC_2|B^r|},$$

$$D_1^r = \frac{1 - \sqrt{1 - 4C^2 C_1 |B^r||f|^2}}{2CC_1|B^r|}, \quad D_2^r = \frac{1 - \sqrt{1 - 4C^2 C_2 |B^r||g|^2}}{2CC_2|B^r|}.$$

Since $A_j^r \to \infty$ as $r \to 0$, for all arbitrary fixed $R > 0$ we have $R \le A_j^r$ for r small enough. Moreover, $D_1^r \to C|f|^2$ and $D_2^r \to C|g|^2$. Then assuming $R > C\max\{|f|^2, |g|^2\}$ we get $D_j^r \le R$ for r small enough. Consequently, if $R > C\max\{|f|^2, |g|^2\}$ and r is small enough we have $(u,v) \in Q_R \times Q_R$ for all $(U,V) \in Q_R \times Q_R$ and $\mathcal{R}$ maps K_R into itself. We complete the proof by proving that $\mathcal{R}$ is a contraction on K_R. Let $(u_j, v_j) \in K_R$ be two solutions to (12) associated with $(U_j, V_j) \in K_R$, $j = 1, 2$. Set $(u,v) = (u_1 - u_2, v_1 - v_2)$ and $(U,V) = (U_1 - U_2, V_1 - V_2)$. Then

$$\begin{cases} (\Delta + \omega_1^2 k_1(x,\omega_1))u = \chi_1^r(U_1^\star V + U^\star V_2), \ \partial_n u = 0, \\ (\Delta + \omega_2^2 k_2(x,\omega_2))v = \chi_2^r(U_1 + U_2)U, \ \partial_n v = 0. \end{cases}$$

Using Lemma 2.2, we get

$$||u||^2 \le C|\chi_1^r|^2_\infty \int_{B^r} |U_1^\star V + U^\star V_2|^2 \, dx, \quad ||v||^2 \le C|\chi_2^r|^2_\infty \int_{B^r} |(U_1 + U_2)U|^2 \, dx.$$

Next, using the same arguments as in (13) we get for all $q > 1$

$$\begin{cases} ||u||^2 \le C|\chi_1^r|^2_\infty |B^r|^{\frac{1}{q}}(||U_1||^2||V||^2 + ||V_2||^2||U||^2), \\ ||v||^2 \le C|\chi_2^r|^2_\infty |B^r|^{\frac{1}{q}}(||U_1||^2 + ||U_2||^2)||U||^2. \end{cases}$$

Since $|\chi_j^r|^2_\infty \le C_j$ and U_j, V_j belong to Q_R, we get

$$||u||^2 \le CC_1|B^r|^{\frac{1}{q}} R^2(||V||^2 + ||U||^2), \quad ||v||^2 \le CC_2|B^r|^{\frac{1}{q}} R^2||U||^2.$$

Hence, for all $q > 1$,

$$||u||^2 + ||v||^2 \leq C \max\{C_1, C_2\} |B^r|^{\frac{1}{q}} (||U||^2 + ||V||^2).$$

Since $|B^r| \to 0$ as $r \to 0$, for r small enough $C \max\{C_1, C_2\}|B^r|^{\frac{1}{q}} < 1$ and then the map $\mathcal{R}$ is a contraction on K_R. This completes the proof of existence and uniqueness of a weak solution to (6). Furthermore, if the data $f, g \in H^{\frac{1}{2}}(\partial\Omega)$ then the weak solution $(u, v) \in H^2(\Omega) \times H^2(\Omega)$ is a strong solution and satisfies (15) by virtue of Lemma 2.3.

2.3. Error Estimates and Asymptotic Expansion

We assume in this section that the nonlinear material occupies a ball $B^r \subset\subset \Omega$ with center x_0 and radius r and that the data $f, g \in H^{\frac{1}{2}}(\partial\Omega)$. We have

Proposition 2.1. *Let (u_r, v_r) be the solution to (6). Then for subsequences, we have $(u_r, v_r) \to (u, v)$ in $H^1(\Omega)$ and $(u_r, v_r) \rightharpoonup (u, v)$ in $H^2(\Omega)$. The limit (u, v) satisfies the problem*

$$\begin{cases} (\Delta + \omega_1^2 k_{1,2})u = 0, \ (\Delta + \omega_2^2 k_{2,2})v = 0 \quad in \ \Omega, \\ \partial_n u = f, \ \partial_n v = g \quad on \ \partial\Omega. \end{cases} \tag{16}$$

Proof. Thanks to Theorem 2.1, (u_r, v_r) is uniformly bounded in $H^2(\Omega)$ so it converges strongly in $H^1(\Omega)$ and weakly in $H^2(\Omega)$ to a limit (u, v). Since χ_j^r are bounded functions with compact support in B^r, we have $\chi_j^r(x) \to 0$ *a.e.* $x \in \Omega$. Then $\chi_1^r u_r^\star v_r \to 0$ and $\chi_2^r u_r^2 \to 0$ in $L^1(\Omega)$. Hence, we get (16) by passing to limit in the weak formulation (9).

Proposition 2.2. *Let (u_r, v_r) be the weak solution to (6) and (u, v) be the solution to (16). Then there exists $C > 0$ independent of r, depending only on $|f|_{H^{\frac{1}{2}}(\partial\Omega)}$, $|g|_{H^{\frac{1}{2}}(\partial\Omega)}$, $|\chi_i^r|_{L^\infty}$, ω_i, $k_{i,j}$ such that*

$$||u_r - u||^2 + ||v_r - v||^2 \leq C r^2.$$

Proof. In what follows, C are various constants independent of r. We have

$$\begin{cases} (\Delta + \omega_1^2 k_1)(u_r - u) = F_r \quad in \ \Omega, \\ (\Delta + \omega_2^2 k_2)(v_r - v) = G_r \quad in \ \Omega, \\ \partial_n(u_r - u) = 0, \ \partial_n(v_r - v) = 0 \quad on \ \partial\Omega, \end{cases} \tag{17}$$

where

$$F_r = \chi_1^r u_r^\star v_r + 1_{B^r}\omega_1^2(k_{1,2} - k_{1,1})u, \quad G_r = \chi_2^r u_r^2 + 1_{B^r}\omega_2^2(k_{2,2} - k_{2,1})v. \tag{18}$$

Since u_r, v_r are uniformly bounded in $H^2(\Omega)$, they are uniformly bounded in $\mathcal{C}^0(\Omega)$. Consequently

$$|F_r|^2 \leq C|B^r|, \quad |G_r|^2 \leq C|B^r|.$$

Then, by virtue of Lemma 2.2, we get

$$||u_r - u||^2 + ||v_r - v||^2 \leq C|B^r|,$$

and the proof of the proposition is complete.

Corollary 2.1. *Let (u_r, v_r) be the weak solution to (6) and (u, v) be the solution to (16). Then there exists an $C > 0$ independent on r such that*

$$||u_r - u||^2_{H^2(\Omega)} + ||v_r - v||^2_{H^2(\Omega)} \leq C r^2.$$

Proof. Set $W_r = \nabla(u_r - u)$. Then $W_r \in L^2(\Omega)$, $\mathrm{curl} W_r = 0 \in L^2(\Omega)$, $W_r \cdot n = 0 \in H^{\frac{1}{2}}(\partial\Omega)$. Moreover, thanks to (17) , we have $\mathrm{div} W_r = \Delta(u_r - u) = F_r - \omega_1^2 k_{1,2}(u_r - u) \in L^2(\Omega)$. It follows that $W_r \in H^1(\Omega)$ and there exists C depending only on Ω such that [10]

$$||W_r||_{H^1(\Omega)} \leq C(||u_r - u||_{H^1(\Omega)} + |F_r|_{L^2(B^r)}),$$

and the corollary follows from the previous proposition.

Next, as $H^2(\Omega) \subset \mathcal{C}^0(\Omega)$, then u, v are both continuous functions. Hence $u(x_0), v(x_0)$ are well defined. The following asymptotic expansion holds.

Theorem 2.2. *Assume that $\chi_j^r(x) = \chi_{j,x_0} + \epsilon^r(x)$ in $L^\infty(B^r)$ with $|\epsilon^r|_{L^\infty(B^r)} \to 0$ as $r \to 0$ and χ_{j,x_0} are some constants, j=1,2. Let $E_j(x, y)$ be the Green function to the Neumann problem*

$$(\Delta_x + \omega_j^2 k_{j,2})E_j(x, y) = \delta_y(x), \quad in \ \Omega, \quad \partial_n E_j = 0, \quad on \ \partial\Omega \quad j = 1, 2,$$

where $k_{j,2}$ are defined by (5). Then for any compact $K \subset \Omega$ containing x_0 such that $dist(x_0, \Omega \setminus K) > 0$, the solution (u_r, v_r) to (6) has the following pointwise asymptotic expansion for $x \in \Omega \setminus K$

$$\begin{cases} u_r(x) = u(x) + |B^r|[\omega_1^2(k_{1,1} - k_{1,2})u(x_0) + \chi_{1,x_0} u(x_0)^\star v(x_0)]E_1(x, x_0) + o(|B^r|)(x), \\ v_r(x) = v(x) + |B^r|[\omega_2^2(k_{2,1} - k_{2,2})v(x_0) + \chi_{2,x_0} u(x_0)^2]E_2(x, x_0) + o(|B^r|)(x), \end{cases}$$

where (u, v) is the solution to (16) and the remainder satisfies

$$|o(|B^r|)|_{L^\infty(\Omega\setminus K)} \leq C \ |B^r| \max(r, |\varepsilon^r|_{L^\infty(B^r)}),$$

for some constant C independent of r, but depending on $dist(x_0, \Omega \setminus K)$, $|f|_{H^{\frac{1}{2}}(\partial\Omega)}$, $|g|_{H^{\frac{1}{2}}(\partial\Omega)}$, the diameter of Ω and the constants $\omega_i, k_{i,j}$.

Proof. Since $u_r - u$ satisfies the following problem

$$\begin{cases} \Delta(u_r - u) + \omega_1^2 k_{1,2}(u_r - u) = M_r, \ \text{ in } \ \Omega, \ \partial_n(u_r - u) = 0 \text{ on } \ \partial\Omega, \\ M_r = \chi_1^r u_r^\star v_r + \omega_1^2(k_{1,1} - k_{1,2})u_r 1_{B^r}, \end{cases}$$

and $M_r \in L^\infty(\Omega)$, we use the Green's representation formula to get [10, p. 727]

$$u_r(x) - u(x) = \int_{B^r} E_1(x, y)M_r(y)dy.$$

Set

$$U_r(x) = u_r(x) - u(x) - |B^r|[\omega_1^2(k_{1,1} - k_{1,2})u(x_0) + \chi_{1,x_0} u(x_0)^\star v(x_0)]E_1(x, x_0).$$

We then have

$$\frac{U_r(x)}{|B^r|} = \int_{B^r} E_1(x, y)\frac{M_r(y)}{|B^r|}dy - \Big(\omega_1^2(k_{1,1} - k_{1,2})u(x_0) + \chi_{1,x_0} u(x_0)^\star v(x_0)\Big)E_1(x, x_0). \tag{19}$$

Let K be a compact subset of Ω containing x_0 such that $dist(x_0, \Omega \setminus K) > 0$ and assume $r < \frac{1}{2} dist(x_0, \Omega \setminus K)$. Then $(\Omega \setminus K) \cap B^r = \emptyset$. Moreover,

$$E_1(x, y) = -\tfrac{1}{2\pi} \log(|x - y|) + H(x, y),$$

where $H(x, y)$ is a smooth function. Hence, there exists $C_{K,x_0} > 0$ independent of r, but depending on the diameter of Ω, $dist(x_0, \Omega \setminus K)$ and H such that [18, p.227] or [3, p.761]

$$|E_1(x, y)| + |\nabla_y E_1(x, y)| \leq C_{K,x_0}, \ \ \forall x \in \Omega \setminus K, \ \forall y \in B^r. \tag{20}$$

Moreover, we have for any $x \in \Omega \setminus K$

$$\frac{1}{|B^r|}\int_{B^r} \chi_1^r(y) u_r^\star v_r(y) E_1(x,y) dy = \frac{1}{|B^r|}\int_{B^r} \chi_1^r(y)(u_r^\star - u^\star) v_r(y) E_1(x,y) dy$$

$$+\frac{1}{|B^r|}\int_{B^r} \chi_{1,x_0} u^\star(y) v(y) E_1(x,y) dy + \frac{1}{|B^r|}\int_{B^r} [\chi_1^r(y) - \chi_{1,x_0}] u^\star(y) v(y) E_1(x,y) dy$$

$$+\frac{1}{|B^r|}\int_{B^r} \chi_1^r(y) u^\star(y)(v_r - v)(y) E_1(x,y) dy = I_1 + I_2 + I_3 + I_4.$$

Using (20), (15) and Corollary 2.1, gives

$$|I_1|_{L^\infty(\Omega\setminus K)} \leq C|u_r - u|_{L^\infty(\Omega)} \leq C|B^r|^{\frac{1}{2}}, \tag{21}$$

$$|I_4|_{L^\infty(\Omega\setminus K)} \leq C|v_r - v|_{L^\infty(\Omega)} \leq C|B^r|^{\frac{1}{2}}, \tag{22}$$

$$|I_3|_{L^\infty(\Omega\setminus K)} \leq C\ |\varepsilon^r|_{L^\infty(B^r)}, \tag{23}$$

for some positive constant independent of r, but depending only on $|f|_{H^{\frac{1}{2}}(\partial\Omega)}$, $|g|_{H^{\frac{1}{2}}(\partial\Omega)}$, $|\chi_i^r|_{L^\infty(\Omega)}$ and the constant C_{K,x_0} given in (20).

Using the Lebesgue differentiation theorem, we get for any $x \in \Omega \setminus K$

$$I_2(x) \to \chi_{1,x_0} u^\star(x_0) v(x_0) E_1(x,x_0) \ \text{ as } r \to 0.$$

Moreover, the convergence is uniform with respect to $x \in \Omega \setminus K$. Indeed, if $w = u$ or $w = v$ and $B^r \subset\subset K' \subset\subset \Omega$ then

$$|w(y) - w(x_0)|_{L^\infty(B^r)} \leq Cr|\nabla w|_{L^\infty(K')} \leq Cr|w|_{H^3(K')} \leq Cr||w||_{H^1(\Omega)}, \tag{24}$$

for some positive constant independent on r. We deduce from this last estimate and (20) that there exists a positive constant independent of r such that

$$|I_2(x) - \chi_{1,x_0} u^\star(x_0) v(x_0) E_1(x,x_0)|_{L^\infty(\Omega\setminus K)} \leq C\ r. \tag{25}$$

Furthermore, using the arguments above and the Lebesgue differentiation theorem, we get

$$\Big|\frac{1}{|B^r|}\int_{B^r} u_r(y) E_1(x,y) dy - u(x_0) E_1(x,x_0)\Big|_{L^\infty(\Omega\setminus K)} \leq C\ r. \tag{26}$$

Finally, we deduce from (21)-(23) and (25)-(26) that for $x \in \Omega \setminus K$

$$\frac{1}{|B^r|}\int_{B^r} M_r(y) E_1(x,y)\ dy \to [\omega_1^2(k_{1,1} - k_{1,2}) u(x_0) + \chi_{1,x_0} u^\star(x_0) v(x_0)] E_1(x,x_0),$$

and

$$|U_r|_{L^\infty(\Omega\setminus K)} \leq C\ |B^r| \max(r, |\varepsilon^r|_{L^\infty(B^r)}),$$

for some positive constant C independent of r, but depending on $|f|_{H^{\frac{1}{2}}(\partial\Omega)}$, $|g|_{H^{\frac{1}{2}}(\partial\Omega)}$, $|\chi_i^r|_{L^\infty(\Omega)}$, ω_i, $k_{i,j}$ and the constant C_{K,x_0} given in (20). The asymptotic expansion of u_r is then proved. We can proceed similarly to get the asymptotic expansion of v_r.

3. The TM Approximation

3.1. The Model Problem

In this section, we will use the same notations as in Subsection 2.1 except that the fields are not assumed to be independent of the vertical variable. Then we will deal with the three dimensional case.

Consider a plane wave of frequency $\omega = \omega_1$ incident on Ω. Because of the presence of the nonlinear material consisting of a ball $B^r \subset\subset \Omega$ with center x_0 and radius r, the nonlinear optical interaction gives rise to diffracted waves of frequencies $\omega = \omega_1$ and $\omega = \omega_2 = 2\omega_1$. This process represents the simplest situation in SHG. Using the well-known undepleted-pump approximation in the literature (see, e.g., [16] and [17]), Eq (2) for frequencies $\omega = \omega_1$ and $\omega = \omega_2$ respectively may be written as [7, p.324]

$$\begin{cases} \mathbf{D}(x,\omega_1) = 4\pi\varepsilon(1+\chi^{(1)}(\omega_1))\mathbf{E}(x,\omega_1), \\ \mathbf{D}(x,\omega_2) = 4\pi\varepsilon\Big((1+\chi^{(1)}(\omega_2))\mathbf{E}(x,\omega_2) + 4\pi\chi^{(2)}(x,\omega_2) : \mathbf{E}(x,\omega_1)\mathbf{E}(x,\omega_1)\Big), \end{cases} \tag{27}$$

where $\chi^{(2)}(x,\omega_2)$ is assumed in this section to be with compact support in the ball $B^{\alpha r} \subset\subset \Omega$ with center x_0 and radius αr, $0 < \alpha < 1$. We assume further that the electromagnetic fields are **TM** polarized at frequency ω_1 and **TE** polarized at frequency ω_2. Therefore,

$$\mathbf{H}(x,\omega_1) = u^r(x_1,x_2,x_3,\omega_1)\mathbf{e}_3, \quad \mathbf{E}(x,\omega_2) = v^r(x_1,x_2,x_3,\omega_2)\mathbf{e}_3,$$

where $(\mathbf{e}_1,\mathbf{e}_2,\mathbf{e}_3)$ is the canonical basis of $\mathbb{R}^3$. Since

$$\mathbf{E}(x,\omega_1) = \frac{c}{i\omega_1\varepsilon(1+\chi^{(1)}(\omega_1))}\nabla\times\mathbf{H}(x,\omega_1),$$

Eqs. (1) and (27) can be simplified to

$$\begin{cases} \nabla\cdot(\frac{1}{\omega_1^2 k_1(x,\omega_1)}\nabla u^r) + u^r = 0, \\ (\Delta + \omega_2^2 k_2(x,\omega_2))v^r = -\dfrac{4\pi\omega_2^2\,\varepsilon}{c^2}\displaystyle\sum_{j,l=1}^{3}\chi^r_{jl}\;\partial_{x_j}u^r\partial_{x_l}u^r. \end{cases}$$

Then writing the transmission conditions on ∂B^r and the Neumann conditions on $\partial\Omega$ we get

$$\begin{cases} (\Delta + \omega_2^2 k_2(x,\omega_2))v^r = \chi^r(\nabla u^r)\otimes\nabla u^r \quad in\ \Omega, \\ \nabla\cdot(\frac{1}{k_1(x,\omega_1)}\nabla u^r) + \omega_1^2 u^r = 0 \quad in\ \Omega, \\ [u^r]_{|\partial B^r} = [\frac{1}{k_1}\partial_n u^r]_{|\partial B^r} = 0, \\ \partial_n v^r = g,\ \frac{1}{k_{1,2}}\partial_n u^r = f \quad \text{on } \partial\Omega, \end{cases} \tag{28}$$

where

$$\chi^r(\nabla u^r)\otimes\nabla u^r = \sum_{1\le j,l\le 3}\chi^r_{j,l}\partial_j u^r\partial_l u^r.$$

In the sequel, we omit to write the sum for j,l.

3.2. Existence, Uniqueness and Uniform Bounds

Our aim is to establish the following theorem.

Theorem 3.1. *Assume that $f,g\in L^2(\partial\Omega)$ and that $\chi^r\in L^\infty$. Then, for all but a discrete set of frequencies ω_1,ω_2, (28) admits a unique weak solution $(u^r,v^r)\in H^1(\Omega)\times H^1(\Omega)$. Moreover, there exists $C>0$ independent of r such that*

$$||u^r||^2 \le C|f|^2,\ |\chi^r(\nabla u^r)\otimes\nabla u^r|^2 \le Cr|f|^2,\ ||v^r||^2 \le C(r|f|^2+|g|^2).$$

Furthermore, if $g\in H^{\frac12}(\partial\Omega)$ then $v^r\in H^2(\Omega)$ and there exists an $C>0$ independent of r such that

$$||v^r||^2_{H^2(\Omega)} \le C(r|f|^2 + |g|^2_{H^{\frac12}(\partial\Omega)}).$$

Proof. Assume that $f, g \in L^2(\partial\Omega)$. We first deal with the following generalized Helmholtz problem satisfied by u^r

$$\begin{cases} \nabla \cdot (\frac{1}{k_1(x,\omega_1)} \nabla u^r) + \omega_1^2 u^r = 0 \quad in\ \Omega, \\ \frac{1}{k_{1,2}} \partial_n u^r = f \quad on\ \partial\Omega, \end{cases} \tag{29}$$

whose weak solution is defined by $u^r \in H^1(\Omega)$ satisfying

$$\int_\Omega \frac{1}{k_1(x,\omega_1)} \nabla u^r \cdot \nabla \phi^\star \ dx - \omega_1^2 \int_\Omega u^r \phi^\star \ dx = -\int_{\partial\Omega} f \phi^\star \ d\sigma,\ \forall \phi \in H^1(\Omega). \tag{30}$$

Using the arguments of Lemmas 2.1 and 2.2, we get, for all but a discrete set of frequencies ω_1, existence and uniqueness of a weak solution $u^r \in H^1(\Omega)$ to (29). Moreover, the following uniform bounds hold

$$|u^r| \leq C|f|, \quad |\nabla u^r| \leq C|f|, \tag{31}$$

for some constant C independent of r. Next, we set

$$k_0^2 = k_{1,2}\omega_1^2, \quad k_\star^2 = k_{1,1}\omega_1^2. \tag{32}$$

We deduce from (30) and (31) that the solution u^r to (29) satisfies (for a subsequence)

$$u^r \to u \text{ in } L^2(\Omega), \quad u^r \rightharpoonup u \text{ in } H^1(\Omega),$$

where u is the weak solution to

$$\Delta u + k_0^2\ u = 0 \text{ in } \Omega, \quad \frac{1}{k_{1,2}} \partial_n u = f \text{ in } \partial\Omega. \tag{33}$$

Next we have the following error estimate.

Lemma 3.1. *Let u^r be the solution to* (29) *and let u be defined by* (33)*. Then there exists an $C > 0$ independent of r such that*

$$||u^r - u||^2 \leq C\ |B^r|.$$

Proof. Throughout this proof, C will denote a positive constant independent of r. We have for all $\varphi \in H^1(\Omega)$,

$$\int_\Omega -\frac{1}{k_1} \nabla(u^r - u) \cdot \nabla\varphi^\star + \omega_1^2 (u^r - u)\varphi^\star \ dx = (\frac{1}{k_{1,1}} - \frac{1}{k_{1,2}}) \int_{B^r} \nabla u \cdot \nabla \varphi^\star \ dx. \tag{34}$$

In particular, for $\varphi = u^r - u$, using the Cauchy's inequality with a coefficient α, we get

$$|\nabla(u^r - u)|^2 \leq C(|u^r - u|^2 + |\nabla u|^2_{B^r}). \tag{35}$$

Moreover, proceeding by contradiction argument as in the previous section, we get

$$|u^r - u|^2 \leq C|\nabla u|^2_{B^r}. \tag{36}$$

Hence

$$||u^r - u||^2 \leq C|\nabla u|^2_{B^r}. \tag{37}$$

Since $\nabla u \in L^2(\Omega)$, there exists an $C > 0$ independent of r such that (cf. Appendix)

$$|\nabla u|^2_{L^2(B^r)} \leq C|B^r|.$$

Consequently,

$$||u^r - u||^2 \leq C\ |B^r|,$$

and the lemma is proved.

Next, we study the Helmholtz equation satisfied by v^r with a source term

$$\begin{cases} (\Delta + \omega_2^2 k_2(x,\omega_1))v^r = S^r \quad in\ \Omega, \\ \partial_n v^r = g \quad on\ \partial\Omega, \end{cases} \tag{38}$$

where $S^r = \chi^r(\nabla u^r) \otimes \nabla u^r$. As $u^r \in H^1(B^r)$ and satisfies

$$\Delta u^r + \omega_1^2 k_{1,1}\ u^r = 0 \quad in\ \mathcal{D}'(B^r), \tag{39}$$

we have $u^r \in H^3(B^{\alpha r}) \subset L^\infty(B^{\alpha r})$. Consequently, $\nabla u^r \in L^4(B^{\alpha r})$. So having in mind that χ^r is with support $B^{\alpha r} \subset\subset B^r$, we get $S^r \in L^2(\Omega)$. Then, thanks to Lemma 2.1, for all but a discrete set of frequencies ω_2, there exists a unique weak solution $v^r \in H^1(\Omega)$ solving (38). Moreover, by Lemma 2.2

$$|v^r| \le C(|g| + |S^r|), \quad |\nabla v^r| \le C(|g| + |S^r|), \tag{40}$$

for some constant $C > 0$ independent of r. Furthermore, if $g \in H^{\frac{1}{2}}(\partial\Omega)$ then $v^r \in H^2(\Omega)$ and $||v^r||_{H^2(\Omega)} \le C(|g|_{H^{\frac{1}{2}}(\partial\Omega)} + |S^r|)$. It remains to give uniform estimates on S^r which will be the consequence of following uniform estimate on $|\nabla u^r|_{L^\infty(B^{\alpha r})}$.

Lemma 3.2. *Let u^r be the solution to* (29)*. Then there exists $C > 0$ independent of r such that*

$$|\nabla u^r|_{L^\infty(B^{\alpha r})} \le \frac{C}{r}|f|_{L^2(\partial\Omega)}. \tag{41}$$

Proof. The proof relies on the integral representation formula of u^r. Let us first recall some notations involving in this latter formula.

For $k > 0$, $\phi_k(x)$ denotes the fundamental solution to $\Delta + k^2$. In the present case

$$\phi_k(x) = \frac{e^{ik|x|}}{4\pi|x|}. \tag{42}$$

Next, for a bounded regular domain D in $\mathbb{R}^3$, let $\mathcal{S}_D^k$ and $\mathcal{D}_D^k$ be respectively the single and double layer potentials defined by $\phi_k(x)$, that is, for a potential $\eta \in L^2(\partial D)$

$$\mathcal{S}_D^k \eta(x) = \int_{\partial D} \phi_k(x-y)\eta(y)dy,\ x \in \mathbb{R}^d,$$
$$\mathcal{D}_D^k \eta(x) = \int_{\partial D} \frac{\partial \phi_k(x-y)}{\partial \nu(y)}\eta(y)dy,\ x \in \mathbb{R}^d \setminus \partial D.$$

Moreover, for $k_0 > 0$ given by (32), we define

$$H^r(x) = \mathcal{D}_\Omega^{k_0}(u^r_{|\partial\Omega}) - \mathcal{S}_\Omega^{k_0}(f), \quad x \in \mathbb{R}^d \setminus \partial\Omega.$$

Assume r small enough ($r < \frac{1}{4}dist(x_0, \partial\Omega)$ for example). Then using (42), we conclude that for every $n \in \mathbb{N}$, there exists a positive constant $C_n > 0$ depending only on n and $dist(x_0, \partial\Omega)$ such that (see also [4, Proposition 3.2])

$$|H^r|_{C^n(\overline{B^r})} \le C_n |f|_{L^2(\partial\Omega)}. \tag{43}$$

Finally, for $k_0,\ k_\star > 0$ given by (32), there exists $(\varphi, \psi) \in L^2(\partial B^r) \times L^2(\partial B^r)$ as the unique solution to the following integral equation [4, Theorem 2.1]

$$\begin{cases} \mathcal{S}_{B^r}^{k_\star}\varphi - \mathcal{S}_{B^r}^{k_0}\psi = H^r, \\ \frac{1}{k_{1,1}}\frac{\partial(\mathcal{S}_{B^r}^{k_\star}\varphi)}{\partial\nu}|_- - \frac{1}{k_{1,2}}\frac{\partial(\mathcal{S}_{B^r}^{k_0}\varphi)}{\partial\nu}|_+ = \frac{1}{k_{1,2}}\frac{\partial H^r}{\partial\nu} \end{cases} on\ \partial B^r.$$

Moreover, there exists r_0 such that for any $r \le r_0$ we have [4, Proposition 4.1]

$$|\varphi|_{L^2(\partial B^r)} + |\psi|_{L^2(\partial B^r)} \le C(r^{-1}|H^r|_{L^2(\partial B^r)} + |\nabla H^r|_{L^2(\partial B^r)}), \tag{44}$$

where C is a constant independent of r. We finally come to the representation formula of the solution to (29) [4, Theorem 3.1]

$$u^r(x) = \mathcal{S}_{B^r}^{k_\star}\varphi(x), \quad x \in B^r,$$

which implies that

$$\partial_i u^r(x) = \int_{\partial B^r} \partial_i \phi_{k_\star}(x-y)\varphi(y)dy, \;\; x \in B^{\alpha r}.$$

Since there exists an $C>0$ depending only on $k_\star$ such that

$$|\partial_i \phi_{k_\star}(x-y)| \leq \frac{c}{|x-y|^2}, \quad x \neq y,$$

there exists some $C>0$ depending only on $k_\star$ and α such that

$$|\partial_i u^r(x)| \leq \frac{C}{r^2}\int_{\partial B^r} |\varphi(y)|dy, \quad x \in B^{\alpha r}.$$

Hence

$$\sup_{B^{\alpha r}} |\partial_i u^r(x)| \leq \frac{C}{r^2}|\varphi|_{L^2(\partial B^r)}|\partial B^r|^{\frac{1}{2}}.$$

Hence, it stems from (43) and (44) that

$$\sup_{B^{\alpha r}} |\partial_i u^r(x)| \leq \frac{C}{r^2}|\partial B^r|^{\frac{1}{2}}|f|_{L^2(\partial\Omega)} \leq \frac{C}{r}|f|_{L^2(\partial\Omega)},$$

for some constant $C>0$ independent of r, but depending only on $k_\star$ and α. The lemma is then proved. We are now ready to establish uniform estimates on the source term as follows

Corollary 3.1. *There exists an $C>0$ independent of r such that $|S^r|^2_{L^2(\Omega)} \leq C\ r\ |f|^2_{L^2(\partial\Omega)}$.*

Proof. According to Lemma 3.2

$$|S^r|^2_{L^2(\Omega)} = \int_{B^{\alpha r}} (\chi^r_{i,j})^2(y)(\partial_i u^r(y))^2(\partial_j u^r(y))^2 dy \leq |\chi^r|^2_{L^\infty}\frac{C}{r^2}|f|^2_{L^2}\int_{B^{\alpha r}} (\partial_j u^r(y))^2 dy.$$

Then we deducte from Lemmas 3.1 and 4.1 that there exists an $C>0$ independent of r such that

$$|S^r|^2_{L^2(\Omega)} \leq C\ \frac{r^3}{r^2}|f|^2_{L^2(\partial\Omega)} = C\ r|f|^2_{L^2(\partial\Omega)}.$$

This completes the proof of Theorem 3.1.

3.3. Convergence and Asymptotic Analysis

The following result derives directly from Theorem 3.1.

Proposition 3.1. *Let (u^r, v^r) be the solution to (28) given by Theorem 3.1, associated with $(f,g) \in L^2(\partial\Omega)\times H^{\frac{1}{2}}(\partial\Omega)$. Then*

$$u^r \to u \text{ in } L^2(\Omega), \quad v^r \to v \text{ in } H^1(\Omega),$$

where u, v are respectively the weak and strong solution to

$$\begin{cases} \Delta u + \omega_1^2 k_{1,2} u = 0 \text{ in } \Omega, & \frac{1}{k_{1,2}}\partial_n u = f \text{ in } \partial\Omega, \\ \Delta v + \omega_2^2 k_{2,2} v = 0 \text{ in } \Omega, & \partial_n v = g \text{ in } \partial\Omega. \end{cases} \tag{45}$$

Next we have the following error estimates.

Proposition 3.2. *Let (u^r, v^r) be the solution to (28) given by Theorem 3.1, associated with $(f,g) \in L^2(\partial\Omega)\times H^{\frac{1}{2}}(\partial\Omega)$, and let u, v be defined by (45). Then there exists an $C>0$ independent of r such that*

$$||u^r - u||^2 \leq C|B^r|, \quad ||v^r - v||^2 \leq Cr, \quad ||v^r - v||^2_{H^2(\Omega)} \leq Cr.$$

Proof. The estimate on $u^r - u$ has already been established in Lemma 3.1. Observe that $v^r - v$ satisfies

$$\begin{cases} \Delta(v^r - v) + k_2\omega_2^2(v^r - v) = \chi^r(\nabla u^r) \otimes \nabla u^r + \omega_2^2(k_{2,2} - k_{2,1})v1_{B^r} \text{ in } \Omega, \\ \partial_n(v^r - v) = 0 \text{ on } \partial\Omega. \end{cases}$$

Then using Lemma 2.2 and Corollary 3.1, we get the desired estimates.

Before giving the asymptotic expansion of u^r and v^r, we first define the following functions. We denote $G(x, y)$ the following Green function

$$\Delta_x G(x, y) + \omega_2^2 k_{2,2} G(x, y) = \delta_y \text{ on } \Omega, \quad \partial_n G(x, y) = 0 \text{ in } \partial\Omega. \tag{46}$$

Let B be the ball with center x_0 and radius 1. For $k_0, k_\star$ given by (32), we set

$$\mu_0 :\equiv \frac{1}{k_0^2}, \ \ \mu_\star :\equiv \frac{1}{k_\star^2}, \quad \mu(\xi) = \mu_0 \ \text{ in } \ \mathbb{R}^3 \setminus \overline{B}, \ \ \mu(\xi) = \mu_\star \ \text{ in } \ B.$$

Then if $A^{\mu,i}$ and $B^{\mu,i,j}$ are respectively unique solution to the following problems

$$\nabla \cdot (\frac{1}{\mu}\nabla A^{\mu,i})(\xi) = 0 \text{ in } \mathbb{R}^3, \ \ A^{\mu,i}(\xi) - \xi_i \to 0 \ \text{ as } \ |\xi| \to +\infty, \ \ 1 \le i \le 3, \tag{47}$$

$$\nabla \cdot (\frac{1}{\mu}\nabla B^{\mu,i,j})(\xi) = \frac{1}{\mu_0}\delta_{i,j} \text{ in } \mathbb{R}^3, \ B^{\mu,i,j}(\xi) - \frac{1}{2}\xi_i\xi_j \to 0 \ \text{ as } |\xi| \to +\infty, \ 1 \le i, j \le 3, \tag{48}$$

we define

$$\begin{aligned} \mathcal{E}_1(\xi) &= \sum_{i=1}^{3}(A^{\mu,i}(\xi) - \xi_i)\partial_i u(x_0), \\ \mathcal{E}_2(\xi) &= \sum_{1\le i,j\le 3}(B^{\mu,i,j}(\xi) - \frac{1}{2}\xi_i\xi_j)\partial_{i,j}^2 u(x_0). \end{aligned} \tag{49}$$

Finally, we have the following asymptotic expansion.

Theorem 3.2. *Let (u^r, v^r), (u, v) be the solutions to (28) and (45), respectively and $\mathcal{E}_i$ be defined by (49). Then we have*

$$u^r(x) = u(x) + r\mathcal{E}_1(\frac{x - x_0}{r}) + r^2\mathcal{E}_2(\frac{x - x_0}{r}) + O(r^3)(x) \text{ in } H^1(\Omega), \tag{50}$$

where $||O(r^3)||_{H^1(\Omega)} \le Cr^3|f|$ for some positive constant C independent of r. Moreover, if we assume that the susceptibility tensor writes

$$\chi_{i,j}^r(x) = \chi_{i,j} + \epsilon^r(x) \ \text{ in } \ L^\infty(B^r), \ \ 1 \le i, j \le 3,$$

where $\chi_{i,j} \in \mathbb{R}$ and $|\epsilon^r(x)|_{L^\infty(B^r)} \to 0$ as $r \to 0$, then, for any compact subset K containing x_0 such that $dist(x_0, \Omega \setminus K) > 0$, we have for $x \in \Omega \setminus K$ the pointwise asymptotic

$$v^r(x) = v(x) + \Big(\omega_2^2(k_{2,2} - k_{2,1})|B^r|v(x_0) + r^3M^\mu(\nabla u) \otimes \nabla u(x_0)\Big)G(x, x_0) + o(r^3)(x), \tag{51}$$

where G is the Green function defined by (46) and the polarization tensor M^μ is given by

$$M_{i,j}^\mu = \sum_{1\le m,n\le 3} \chi_{m,n} \int_{B^\alpha} \partial_m A^{\mu,i}(z)\partial_n A^{\mu,j}(z)\, dz, \ \ 1 \le i, j \le 3, \tag{52}$$

with $A^{\mu,i}$ defined by (47). The remainder in (51) satisfies

$$|o(r^3)|_{L^\infty(\Omega\setminus K)} \le Cr^3 \max\Big(|\epsilon^r|_{L^\infty(B^r)}, r^{\frac{1}{2}}\Big),$$

for some positive constant C independent of r.

Proof. The asymptotic expansion (50) is due to [3] and the asymptotic formula (51) will be deduced from (50) as follows. Since

$$\begin{cases} \Delta(v^r - v) + \omega_2^2 k_{2,2}(v^r - v) = \omega_2^2(k_{2,2} - k_{2,1})v^r 1_{B^r} + \chi^r(\nabla u^r) \otimes \nabla u^r, \\ \partial_n(v^r - v) = 0, \end{cases}$$

we have

$$v^r(x) - v(x) = \omega_2^2(k_{2,2} - k_{2,1}) \int_{B^r} G(x,y)v^r(y)dy + \int_{B^r} G(x,y)\chi^r(\nabla u^r) \otimes \nabla u^r(y)dy.$$

The proof follows from the following lemmas.

Lemma 3.3. *The following pointwise asymptotic expansion holds for* $x \in \Omega \setminus K$

$$\int_{B^r} G(x,y)v^r(y)dy = |B^r|G(x,x_0)v(x_0) + o(r^{\frac{7}{2}})(x),$$

with $|o(r^{\frac{7}{2}})|_{L^\infty(\Omega\setminus K)} \leq Cr^{\frac{7}{2}}$ *for some positive constant* C *independent of* r.

Proof. Indeed

$$\int_{B^r} G(x,y)v^r(y)dy = \int_{B^r} G(x,y)(v^r - v)(y)dy + \int_{B^r} G(x,y)v(y)dy.$$

Note that there exists a constant $C > 0$ independent of r, but depending on $dist(x_0, \Omega \setminus K)$ such that

$$|G|_{L^\infty((\Omega\setminus K)\times B^r)} + |\nabla_y G|_{L^\infty((\Omega\setminus K)\times B^r)} \leq C. \tag{53}$$

Then, by virtue of Proposition 3.2 and the above estimate, we have

$$\begin{aligned} |\int_{B^r} G(x,y)(v^r - v)(y)dy| &\leq C|v^r - v|_{L^\infty(B^r)}|B^r|, \\ &\leq C\ r^{\frac{1}{2}}|B^r|, \quad \forall x \in \Omega \setminus K. \end{aligned}$$

Moreover, arguing as in the proof of Theorem 2.2, we get

$$\int_{B^r} G(x,y)v(y)dy = |B^r|G(x,x_0)v(x_0) + O(r|B^r|).$$

This completes the proof of the lemma.

Lemma 3.4. *The following pointwise asymptotic expansion holds for* $x \in \Omega \setminus K$

$$\int_{B^r} G(x,y)\chi^r(\nabla u^r) \otimes \nabla u^r(y)dy = r^3 G(x,x_0)M^\mu(\nabla u) \otimes \nabla u(x_0) + o(r^3)(x),$$

where the remainder satisfies $|o(r^3)|_{L^\infty(\Omega\setminus K)} \leq C\ r^3 \max(|\epsilon^r|_{L^\infty(B^r)}, r)$ *for some positive constant* C *independent of* r*. The polarization tensor is given by* (52).

Proof. By virtue of (50), we have

$$
\begin{aligned}
&\int_{B^r} G(x,y)\chi^r(\nabla u^r)\otimes\nabla u^r(y)dy\\
&=\sum_{i,j}\sum_{m,n}\Big(\int_{B^{\alpha r}}\chi^r_{i,j}(y)G(x,y)\partial_i A^{\mu,m}(\frac{y-x_0}{r})\partial_j A^{\mu,n}(\frac{y-x_0}{r})dy\Big)\partial_m u(x_0)\partial_n u(x_0)\\
&+\sum_{i,j}\int_{B^{\alpha r}}\chi^r_{i,j}(y)G(x,y)[\partial_i u(y)(\partial_j u(y)-\partial_j u(x_0))+\partial_i u(x_0)(\partial_j u(x_0)-\partial_j u(y))]dy\\
&+\sum_{i,j}\int_{B^{\alpha r}}\chi^r_{i,j}(y)G(x,y)[\sum_m \partial_j A^{\mu,m}(\frac{y-x_0}{r})\partial_m u(x_0)](\partial_i u(y)-\partial_i u(x_0))dy\\
&+\sum_{i,j}\int_{B^{\alpha r}}\chi^r_{i,j}(y)G(x,y)[\sum_m \partial_i A^{\mu,m}(\frac{y-x_0}{r})\partial_m u(x_0)](\partial_j u(y)-\partial_j u(x_0))dy\\
&+r\sum_{i,j}\int_{B^{\alpha r}}\chi^r_{i,j}(y)G(x,y)\sum_{l,m,n}\partial_i B^{\mu,l,n}(\frac{y-x_0}{r}))\partial_j A^{\mu,m}(\frac{y-x_0}{r}))\partial^2_{l,n}u(x_0)\partial_m u(x_0)dy\\
&+r\sum_{i,j}\int_{B^{\alpha r}}\chi^r_{i,j}(y)G(x,y)\sum_{l,m,n}\partial_j B^{\mu,l,n}(\frac{y-x_0}{r}))\partial_i A^{\mu,m}(\frac{y-x_0}{r}))\partial^2_{l,n}u(x_0)\partial_m u(x_0)dy\\
&+r\sum_{i,j}\int_{B^{\alpha r}}\chi^r_{i,j}(y)G(x,y)\sum_{m,n}\partial_j B^{\mu,m,n}(\frac{y-x_0}{r})\partial^2_{m,n}u(x_0)(\partial_i u(y)-\partial_i u(x_0))dy\\
&+r\sum_{i,j}\int_{B^{\alpha r}}\chi^r_{i,j}(y)G(x,y)\sum_{n,l}\partial_i B^{\mu,n,l}(\frac{y-x_0}{r})\partial^2_{n,l}u(x_0)(\partial_j u(y)-\partial_j u(x_0))\ dy\\
&+r\sum_{i,j}\int_{B^{\alpha r}}\chi^r_{i,j}(y)G(x,y)\sum_{l}\frac{y_l-x_{0,l}}{r}\partial^2_{j,l}u(x_0)(\partial_i u(x_0)-\partial_i u(y))dy\\
&+r\sum_{i,j}\int_{B^{\alpha r}}\chi^r_{i,j}(y)G(x,y)\sum_{l}\frac{y_l-x_{0,l}}{r}\partial^2_{i,l}u(x_0)(\partial_j u(x_0)-\partial_j u(y))dy\\
&-2r\sum_{i,j}\sum_{l,m}\Big(\int_{B^{\alpha r}}\chi^r_{i,j}(y)G(x,y)\partial_i A^{\mu,l}(\frac{y-x_0}{r})\frac{y_m-x_{0,m}}{r}dy\Big)\partial^2_{j,m}u(x_0)\partial_l u(x_0)\\
&+r^2\sum_{i,j}\int_{B^{\alpha r}}\chi^r_{i,j}(y)G(x,y)\sum_{l,m,n,p}\partial_i B^{\mu,l,p}(\frac{y-x_0}{r})\partial_j B^{\mu,m,n}(\frac{y-x_0}{r})\partial^2_{l,p}u(x_0)\partial^2_{m,n}u(x_0)dy\\
&-r^2\sum_{i,j}\int_{B^{\alpha r}}\chi^r_{i,j}(y)G(x,y)\sum_{m,n,p}\partial_j B^{\mu,m,n}(\frac{y-x_0}{r})(\frac{y_p-x_{0,p}}{r})\partial^2_{m,n}u(x_0)\partial^2_{i,p}u(x_0)dy\\
&-r^2\sum_{i,j}\int_{B^{\alpha r}}\chi^r_{i,j}(y)G(x,y)\sum_{m,n,p}\partial_i B^{\mu,m,n}(\frac{y-x_0}{r})(\frac{y_p-x_{0,p}}{r})\partial^2_{m,n}u(x_0)\partial^2_{j,p}u(x_0)dy\\
&+r^2\sum_{i,j}\int_{B^{\alpha r}}\chi^r_{i,j}(y)G(x,y)\sum_{l,p}(\frac{y_l-x_{0,l}}{r})(\frac{y_p-x_{0,p}}{r})\partial^2_{i,l}u(x_0)\partial^2_{j,p}u(x_0)dy+o(r^3).
\end{aligned}
$$

Making a change of variables in the above expansion, writing a Taylor formula and using (53) and (24), we check that all the terms on the right-hand side of the above expansion, except the first one, are $O(r^4)$. Then, the last equality becomes

$$
\begin{aligned}
&\int_{B^{\alpha r}} G(x,y)\chi^r(\nabla u^r)\otimes\nabla u^r(y)dy\\
&=\sum_{i,j}\sum_{m,n}\Big(\int_{B^{\alpha r}}\chi^r_{i,j}(y)G(x,y)\partial_i A^{\mu,m}(\frac{y-x_0}{r})\partial_j A^{\mu,n}(\frac{y-x_0}{r})dy\Big)\partial_m u(x_0)\partial_n u(x_0)+o(r^3)(x).
\end{aligned}
$$

Using again the change of variables $y=x_0+rz$ and writing the Taylor's formula, we obtain that

$$\sum_{i,j}\sum_{m,n}\Big(\int_{B^{\alpha r}}\chi^r_{i,j}(y)G(x,y)\partial_i A^{\mu,m}(\frac{y-x_0}{r})\partial_j A^{\mu,n}(\frac{y-x_0}{r})dy\Big)\partial_m u(x_0)\partial_n u(x_0)$$
$$= r^3\sum_{i,j}\sum_{m,n}\Big(\int_{B^{\alpha}}\chi_{i,j}G(x,x_0)\partial_i A^{\mu,m}(z)\partial_j A^{\mu,n}(z)dz\Big)\partial_m u(x_0)\partial_n u(x_0)$$
$$+r^3\sum_{i,j}\sum_{m,n}\Big(\int_{B^{\alpha}}\epsilon^r(x_0+rz)G(x,x_0+rz)\partial_i A^{\mu,m}(z)\partial_j A^{\mu,n}(z)dz\Big)\partial_m u(x_0)\partial_n u(x_0)$$
$$+r^4\sum_{i,j}\sum_{m,n}\Big(\int_{B^{\alpha}}z\nabla_y G(x,x_0+rz)\partial_i A^{\mu,m}(z)\partial_j A^{\mu,n}(z)dz\Big)\partial_m u(x_0)\partial_n u(x_0).$$

Consequently, using (53) we get

$$\int_{B^{\alpha r}}G(x,y)\chi^r(\nabla u^r)\otimes\nabla u^r(y)dy = r^3M^\mu(\nabla u)\otimes\nabla u(x_0)G(x,x_0)+O(r^3|\epsilon^r|_\infty)+O(r^4),$$

where M^μ is the polarization tensor given by (52). This completes the proof of the lemma and Theorem 3.2 is then proved.

4. Appendix

Lemma 4.1. *Let $f\in L^2(\mathbb{R}^n)$, $x_0\in\mathbb{R}^n$, $r>0$ and $B(x_0,r)$ be the ball of center x_0 and radius r. Then there exists a positive constant C independent of r, but depending on f, x_0 such that*

$$\int_{B(x_0,r)}f^2(x)dx\le Cr^n. \tag{54}$$

Proof. By virtue of [11, p.649] , for a.e. $y_0\in\mathbb{R}^n$,

$$\frac{1}{|B(y_0,r)|}\int_{B(y_0,r)}f^2(x)\ dx\to f^2(y_0)\ \text{ as }\ r\to 0.$$

A point y_0 of $\mathbb{R}^n$ for which the above convergence holds is called a Lebesgue's point of f^2. If x_0 is a Lebesgue's point of f^2, it is clear that the estimate (54) holds for some positive constant C independent of r, but depending on x_0 and f. If x_0 is not a Lebesgue's point of f^2 then $B(x_0,r)$ containts at least a Lebesgue's point x_1 of f^2 and the estimate

$$\int_{B(x_1,2r)}f^2(x)dx\le Cr^n$$

holds with C a positive constant independent of r, but depending on x_1,f. As $B(x_0,r)\subset B(x_1,2r)$, then (54) holds. Consequently the lemma is proved.

Remark 4.1. We can find also in [3] an asymptotic expansion of u^r in the two-dimensional case. However, it is not exploitable to deduce the asymptotic expansion of v^r because the products of the functions involving in this asymptotic formula are not integrable.

Acknowledgements. I am very grateful to Professors Habib Ammari and Kamel Hamdache for their valuable remaks and for their full support during this work.

References

[1] H. Ammari, G. Bao, K. Hamdache, The effect of thin coatings on second harmonic generation, *J. Diff. Eq.,* **36** (1999), 1-13.

[2] H. Ammari, H. Kang, Reconstruction of Small Inhomogeneities from Boundary Measurements, Lectures Notes in Mathematics, Vol 186, Springer, 2004.

[3] H. Ammari, A. Khelifi, Electromagnetic scattering by small dielectric inhomogeneities, *J. Math. Pures. Appl.,* **82** (2003), 749-842.

[4] H. Ammari, H. Kang, Boundary layer techniques for solving the Helmholtz equation in the presence of small inhomogeneities, *J. Math. Anal. Appl.,* **296** (2004), 190-208.

[5] H. Ammari, G. Bao, K. Hamdache, Analysis of thin coatings of nonlinear diffration gratings, *Nonlinear Anal. Real World Appl.,* **3**:4 (2002), 487–502.

[6] G. Bao, D.C. Dobson, Diffractive optics in nonlinear media with periodic structure, *Euro. J. Appl. Math.,* **6** (1995), 573-590.

[7] G. Bao, Y. Chen, A nonlinear grating problem in diffractive optics, *SIAM J. Math. Anal.,* **28** (1997), 322-337.

[8] G. Bao, D.C. Dobson, Second harmonic generation in nonlinear optical films, *J. Math. Phys.,* **35**:4 (1994), 1622-1633.

[9] H. Brezis, Analyse Fonctionnelle, Théorie et Aplications, Masson, 1987.

[10] R. Dautrey, J.L. Lions, Analyse Mathématique et Calcul Numérique, Tome 2, Masson, 1987.

[11] L.C. Evans, Partial Differential Equations, Graduate Studies in Maths, Vol 19, AMS, Providence, 1998.

[12] P.A. Franken, A.E. Hill, C.W. Peters, G. Weinreich, Generation of optical harmonics, *Phys. Rev. Lett,* **7**, (1961), p.118.

[13] A. Friedman, Mathematics in Industrial Problems, Part 7, Springer-Verlag, Heindelberg, 1994.

[14] P. Grisvard, Elliptic Boundary Value Problems in Nonsmooth Domains, Pitman, Boston, 1985.

[15] H. Reinhard, Equations aux Dérivées Partielles, Fondements et Applications, Dunod Université, 1987.

[16] R. Reinisch, M. Nevière, Electromagnetic theory of diffraction in nonlinear optics and surface-enhanced nonlinear optical effects, second-harmonic generation, *Phys. Rev. B,* **28** (1983), 1870-1885.

[17] R. Reinisch, M. Nevière, H. Akhouayri, J. Coutaz, D. Maystre, E. Pic, Grating enhanced second-harmonic generation through electromagnetic resonances, *Opt. Engrg.,* **27** (1988), 961-971.

[18] G.F. Roach, Green Functions: Introductory Theory with Applications, London, New York, Toronto, Van Nostrand Reinhold, 1970.

[19] Y. R. Shen, The Principles of Nonlinear Optics, John Wiley, New York, 1984.

[20] S. Soussi, Second harmonic generation in the undepleted pump approximation, *Multiscale. Model. Simul,* **4** (2005), 115-148.