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THE MAGNETIC POTENTIAL FOR THE ELLIPSOIDAL MEG PROBLEM *1)

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Abstract

In magnetoencephalography (MEG) a primary current is activated within a bounded conductive medium, *i.e.*, the head. The primary current excites an induction current and the total (primary plus induction) current generates a magnetic field which, outside the conductor, is irrotational and solenoidal. Consequently, the exterior magnetic field can be expressed as the gradient of a harmonic function, known as the magnetic potential. We show that for the case of a triaxial ellipsoidal conductor this potential is obtained by using integration along a specific path which is dictated by the geometrical characteristics of the ellipsoidal system as well as by utilizing special properties of ellipsoidal harmonics. The vector potential representation of the magnetic field is also obtained.

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1. Introduction

As Plonsey and Heppner [7] have demonstrated, in studying Bioelectromagnetic problems, the values of the physical parameters of the human body justify the replacement of Maxwell's equations with the equations of quasi-static theory of electromagnetism. This means that the time derivative terms of the magnetic induction and of the electric displacement fields in Maxwell's equations can be omitted. That renders the rotation of the magnetic field proportional to the current. Hence, in regions free of current the magnetic field becomes irrotational and since, due to the lack of magnetic monopoles, it is also solenoidal, it can be represented by the gradient of a harmonic function. This function was first obtained by Bronzan [1] via path integration in appropriate regions that avoid the support of the current. But the actual meaning of the scalar magnetic potential in MEG was demonstrated by Sarvas [8] in a celebrated paper where he showed that for a spherical conductor the exterior magnetic potential can be obtained from the radial component of the primary current alone. In particular, he obtained in closed form the potential and therefore the magnetic field as well, for the case of a homogeneous spherical conductor with a dipole source anywhere in its interior. His solution coincides with the one Bronzan gave for the general case. It is of interest to see though that this property of recovering the exterior magnetic field from the radial component of the primary current is not shared by any other geometry besides the spherical one. In other words, for non spherical conductors the geometry of the conductor influences directly the exterior magnetic field. The ellipsoidal geometry has the advantage of being a genuine three dimensional shape that can be well adjusted in any convex body, and in particular to the brain which anatomicaly is considered to be an ellipsoid with average semiaxes 6, 6.5 and 9 centimeters. On the other hand, it is exactly this freedom of adaptation to any convex body that makes the mathematics much more elaborate than the spherical (1-D) or even the spheroidal (2-D) geometries.

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Obviously, any attempt to calculate the magnetic potential for an ellipsoidal conductor has to incorporate the contribution from the surface distribution of dipoles as given by the Geselowitz formula [5]. Therefore, the direct electroencephalography (EEG) problem has to be solved as well, in order to determine the dipole density function on the boundary of the ellipsoidal conductor. The crucial part of the present work is to determine the path of integration that will allow the calculation of the line integral which provides the magnetic potential. We show that such a path is given by the non planar curve which is defined by the intersection of the one-sheet hyperboloid and the two-sheet hyperboloid that correspond to the “angular” ellipsoidal coordinates of the point where the potential is evaluated. It seams that this choice of integration path is the unique choice which allows for the integration of the ellipsoidal fields. It is the ellipsoidal analogue of the radial direction for the case of a sphere. Following this approach we were able to obtain the exterior magnetic field as a series solution in terms of multipole fields. The leading term of this series, which is the quadrupolic term, was obtained analytically by the author and Kariotou in [2].

We mention here that as far as the inverse MEG problem is concerned, it was shown by Fokas, Kurylev and Marinakis [4] for the sphere and by the author, Fokas and Kariotou [3] for any star-shape conductor, that from the three scalar functions needed to identify the current only one can be recovered, and this is true even when a complete knowledge of the magnetic potential outside the head is provided.

Section 2 states the direct problem of magnetoencephalography for a single dipole in ellipsoidal geometry and provides the solution to the corresponding problem of electroencephalography which concerns the electric potential. Section 3 elaborates a compact expresssion for the multipole expansion of the exterior magnetic field in dyadic form. The vector potential for the magnetic field is discussed in Section 4 while the corresponding scalar magnetic potential is obtained in Section 5.

2. The Ellipsoidal MEG Problem

Consider the ellipsoid

$$\frac{x_1^2}{\alpha_1^2} + \frac{x_2^2}{\alpha_2^2} + \frac{x_3^2}{\alpha_3^2} = 1, \tag{2.1}$$

where

$$0 < \alpha_3 < \alpha_2 < \alpha_1 < +\infty \tag{2.2}$$

are the three semiaxes and

$$\left.\begin{aligned} h_1 &= \sqrt{\alpha_2^2 - \alpha_3^2} \\ h_2 &= \sqrt{\alpha_1^2 - \alpha_3^2} \\ h_3 &= \sqrt{\alpha_1^2 - \alpha_2^2} \end{aligned}\right\} \tag{2.3}$$

are the three semifocal distances. Since

$$h_1^2 - h_2^2 + h_3^2 = 0, \tag{2.4}$$

only two out of the three semifocal distances are independent.

Introduce the ellipsoidal coordinates [6] (ρ,μ,ν) via

$$\left.\begin{aligned} x_1 &= \tfrac{1}{h_2 h_3}\rho\mu\nu \\ x_2 &= \tfrac{1}{h_1 h_3}\sqrt{\rho^2 - h_3^2}\sqrt{\mu^2 - h_3^2}\sqrt{h_3^2 - \nu^2} \\ x_3 &= \tfrac{1}{h_1 h_2}\sqrt{\rho^2 - h_2^2}\sqrt{h_2^2 - \mu^2}\sqrt{h_2^2 - \nu^2} \end{aligned}\right\} \tag{2.5}$$

where

$$0 < \nu^2 < h_3^2 < \mu^2 < h_2^2 < \rho^2 < +\infty. \tag{2.6}$$

The ρ-system forms a family of confocal ellipsoids where the value $\rho = \alpha_1$ corresponds to the ellipsoid (2.1). The μ-system forms a family of confocal hyperboloids of one-sheet and the ν-system forms a family of confocal hyperboloids of two-sheets. The three families form a confocal quadric system given by

$$\frac{x_1^2}{\rho^2} + \frac{x_2^2}{\rho^2 - h_3^2} + \frac{x_3^2}{\rho^2 - h_2^2} = 1, \qquad h_2^2 < \rho^2 < +\infty, \tag{2.7}$$

$$\frac{x_1^2}{\mu^2} + \frac{x_2^2}{\mu^2 - h_3^2} + \frac{x_3^2}{\mu^2 - h_2^2} = 1, \quad h_3^2 < \mu^2 < h_2^2, \tag{2.8}$$

and

$$\frac{x_1^2}{\nu^2} + \frac{x_2^2}{\nu^2 - h_3^2} + \frac{x_3^2}{\nu^2 - h_2^2} = 1, \quad 0 < \nu^2 < h_3^2, \tag{2.9}$$

which is also orthogonal. From each point (α_1, μ_0, ν_0) on the ellipsoid (2.1) springs a curve that is the particular intersection of (2.8) and (2.9) on which (α_1, μ_0, ν_0) lies. Parametrizing this curve by the ellipsoidal coordinate ρ we obtain its vectorial representation in the form

$$\mathbf{C}(\rho) = \frac{1}{h_1 h_2 h_3}\Bigg[h_1 \rho \mu_0 \nu_0 \hat{\mathbf{x}}_1 + h_2 \sqrt{\rho^2 - h_3^2}\sqrt{\mu_0^2 - h_3^2}\sqrt{h_3^2 - \nu_0^2}\hat{\mathbf{x}}_2 \\ + h_3 \sqrt{\rho^2 - h_2^2}\sqrt{h_2^2 - \mu_0^2}\sqrt{h_2^2 - \nu_0^2}\hat{\mathbf{x}}_3\Bigg], \tag{2.10}$$

where $\rho \in [\alpha_1, +\infty)$. From the orthogonality of the ellipsoidal system it follows that the tangent vector $\mathbf{r}'(\rho)$ of the curve (2.10) at any point $\rho \in [\alpha_1, +\infty)$ is normal to the ellipsoid (2.7). This is the key-property that will be used later to obtain the exterior magnetic potential. In the ellipsoidal system (ρ,μ,ν) the electric potential u that is due to an electric dipole at $\mathbf{r}_0 = (\rho_0, \mu_0, \nu_0)$ inside the ellipsoid, with moment $\mathbf{Q}$, solves the Neumann boundary value problem

$$\Delta u(\rho, \mu, \nu) = \frac{1}{\sigma}\mathbf{Q} \cdot \nabla \delta(\mathbf{r} - \mathbf{r}_0), \qquad h_2 \leq \rho < \alpha_1, \tag{2.11}$$

$$\frac{\partial}{\partial n} u(\rho, \mu, \nu) = 0, \qquad \rho = \alpha_1, \tag{2.12}$$

where σ is the conductivity of the ellipsoidal region. The solution of this problem has been obtained in [2] and it is equal to

$$u(\mathbf{r}) = \frac{1}{4\pi\sigma}\mathbf{Q} \cdot \frac{\mathbf{r} - \mathbf{r}_0}{|\mathbf{r} - \mathbf{r}_0|^3} - \frac{1}{\sigma}\sum_{n=1}^{\infty}\sum_{m=1}^{2n+1} \frac{\mathbf{Q} \cdot \nabla_{\mathbf{r}_0}\mathbb{E}_n^m(\mathbf{r}_0)}{(2n+1)\gamma_n^m} \frac{F_n^{m\,\prime}(\alpha_1)}{E_n^{m\,\prime}(\alpha_1)}\mathbb{E}_n^m(\mathbf{r}), \tag{2.13}$$

where $E_n^m(\rho)$ denotes the interior Lamé function of degree n and of order m,

$$F_n^m(\rho) = (2n+1)E_n^m(\rho) I_n^m(\rho) \tag{2.14}$$

with

$$I_n^m(\rho) = \int_\rho^{+\infty} \frac{dx}{(E_n^m(x))^2 \sqrt{x^2 - h_3^2}\sqrt{x^2 - h_2^2}} \tag{2.15}$$

denotes the corresponding exterior Lamé function and

$$\mathbb{E}_n^m(\mathbf{r}) = E_n^m(\rho) E_n^m(\mu) E_n^m(\nu), \tag{2.16}$$

and

$$\mathbb{F}_n^m(\mathbf{r}) = F_n^m(\rho)E_n^m(\mu)E_n^m(\nu) \tag{2.17}$$

are the interior and the exterior ellipsoidal harmonics, respectively. The ellipsoidal eigensolutions $\mathbb{E}_n^m(\mathbf{r})$ and $\mathbb{F}_n^m(\mathbf{r})$ are also known as Lamé products of the first and of the second kind respectively.

The primes in (2.13) stand for differentiation whith respect to ρ and

$$\gamma_n^m = \int_{\rho=\alpha_1} [E_n^m(\mu)E_n^m(\nu)]^2 \frac{ds}{\sqrt{\alpha_1^2-\mu^2}\sqrt{\alpha_1^2-\nu^2}} \tag{2.18}$$

are the ellipsoidal normalization constants, *i.e.*, the L^2-norms of the surface ellipsoidal harmonics with respect to the surface measure

$$l(\mu,\nu) = \frac{1}{\sqrt{\alpha_1^2-\mu^2}\sqrt{\alpha_1^2-\nu^2}}. \tag{2.19}$$

The Dirichlet values of the electric potential u on the ellipsoid $\rho = \alpha_1$ provide the exterior magnetic field via the Geselowitz integral representation [5]

$$\mathbf{B}(\mathbf{r}) = \frac{\mu_0}{4\pi}\mathbf{Q} \times \frac{\mathbf{r}-\mathbf{r}_0}{|\mathbf{r}-\mathbf{r}_0|^3} - \frac{\mu_0\sigma}{4\pi}\int_{\rho=\alpha_1} u(\mathbf{r}')\hat{\boldsymbol{\rho}}' \times \frac{\mathbf{r}-\mathbf{r}'}{|\mathbf{r}-\mathbf{r}'|^3} ds(\mathbf{r}'), \tag{2.20}$$

where

$$\hat{\boldsymbol{\rho}}' = \alpha_1\alpha_2\alpha_3 l(\mu',\nu')\sum_{i=1}^{3}\frac{x_i'}{\alpha_i^2}\bigg|_{\rho'=\alpha_1}\hat{\mathbf{x}}_i \tag{2.21}$$

is the outward unit normal on the ellipsoid $\rho = \alpha_1$ [2], and μ_0 is the magnetic permeability of the air.

Note that since

$$\mathbf{r}\cdot\hat{\boldsymbol{\rho}}(\mathbf{r}) = \alpha_1\alpha_2\alpha_3 l(\mu,\nu) \tag{2.22}$$

it follows that the product $\alpha_1\alpha_2\alpha_3 l(\mu,\nu)$ is the support function of the ellipsoid (2.1), *i.e.,* the projection of the position vector $\mathbf{r}$ of the surface on the Gaussian image $\hat{\boldsymbol{\rho}}(\mathbf{r})$ at the point $\mathbf{r}$.

The electric potential u, as a solution of a Neumann problem, is unique up to an additive constant. Hence, if u is a solution of (2.11)-(2.12) so is $u + c$ for every $c \in \mathbb{R}$. Nevertheless, since by Gausss theorem

$$\int_{\rho'=\alpha_1} u(\mathbf{r}')\hat{\boldsymbol{\rho}}' \times \frac{\mathbf{r}-\mathbf{r}'}{|\mathbf{r}-\mathbf{r}'|^3} ds(\mathbf{r}') = \int_V (\nabla u(\mathbf{r}')) \times \frac{\mathbf{r}-\mathbf{r}'}{|\mathbf{r}-\mathbf{r}'|^3} dv(\mathbf{r}'), \tag{2.23}$$

where V is the interior of the ellipsoid $\rho = \alpha_1$, it follows that $\mathbf{B}$ is independent of the additive constant c. Hence, we can always assume that $c = 0$.

3. The Multipole Expansion

Using the fundamental expansion

$$\frac{1}{|\mathbf{r}-\mathbf{r}_0|} = \sum_{n=0}^{\infty}\sum_{m=1}^{2n+1}\frac{4\pi}{2n+1}\frac{1}{\gamma_n^m}\mathbb{E}_n^m(\mathbf{r}_0)\mathbb{F}_n^m(\mathbf{r}) \tag{3.1}$$

which holds for $\rho > \rho_0$, and the identity

$$\nabla_{\mathbf{r}_0} \frac{1}{|\mathbf{r}-\mathbf{r}_0|} = \frac{\mathbf{r}-\mathbf{r}_0}{|\mathbf{r}-\mathbf{r}_0|^3}, \tag{3.2}$$

the solution (2.13), in a left neighborhood of the boundary $\rho = \alpha_1$ (i.e. for $\rho \in (\rho_0, \alpha_1)$), assumes the form

$$u(\rho,\mu,\nu) = \frac{1}{\sigma} \sum_{n=1}^{\infty} \sum_{m=1}^{2n+1} \frac{\mathbf{Q}\cdot\nabla_{\mathbf{r}_0}\mathbb{E}_n^m(\mathbf{r}_0)}{(2n+1)\gamma_n^m} \left[\frac{F_n^m(\rho)}{E_n^m(\rho)} - \frac{F_n^{m\,\prime}(\alpha_1)}{E_n^{m\,\prime}(\alpha_1)}\right] \mathbb{E}_n^m(\rho,\mu,\nu) \tag{3.3}$$

which holds for $\rho \in (\rho_0, \alpha_1]$. The satisfaction of the boundary condition (2.12) is obvious in the form (3.3) since

$$\frac{\partial}{\partial\rho}\left[F_n^m(\rho) - \frac{F_n^{m\,\prime}(\alpha_1)}{E_n^{m\,\prime}(\alpha_1)} E_n^m(\rho)\right]\Bigg|_{\rho=\alpha_1} = 0. \tag{3.4}$$

In particular, since

$$\frac{F_n^m(\alpha_1)}{E_n^m(\alpha_1)} - \frac{F_n^{m\,\prime}(\alpha_1)}{E_n^{m\,\prime}(\alpha_1)} = (2n+1)\frac{1}{E_n^{m\,\prime}(\alpha_1)E_n^m(\alpha_1)\alpha_2\alpha_3}, \tag{3.5}$$

it follows that on the surface of the ellipsoid the electric potential is written as

$$u(\alpha_1,\mu,\nu) = \sum_{n=1}^{\infty}\sum_{m=1}^{2n+1} A_n^m(\mathbf{r}_0) E_n^m(\mu) E_n^m(\nu), \tag{3.6}$$

where

$$A_n^m(\mathbf{r}_0) = \frac{\mathbf{Q}\cdot\nabla_{\mathbf{r}_0}\mathbb{E}_n^m(\mathbf{r}_0)}{\sigma\alpha_2\alpha_3\gamma_n^m E_n^{m\,\prime}(\alpha_1)}. \tag{3.7}$$

Next we expand the dipole field in the primed variable. For $\rho > \rho' = \alpha_1$

$$\begin{aligned}
\frac{\mathbf{r}-\mathbf{r}'}{|\mathbf{r}-\mathbf{r}'|^3}\Bigg|_{\rho'=\alpha_1} &= -\nabla_{\mathbf{r}} \frac{1}{|\mathbf{r}-\mathbf{r}'|}\Bigg|_{\rho'=\alpha_1} \\
&= -\sum_{n=0}^{\infty}\sum_{m=1}^{2n+1} \frac{4\pi}{2n+1}\frac{1}{\gamma_n^m}\left(\nabla_{\mathbf{r}}\mathbb{F}_n^m(\mathbf{r})\right)\mathbb{E}_n^m(\mathbf{r}')\Bigg|_{\rho'=\alpha_1} \\
&= -\sum_{n=0}^{\infty}\sum_{m=1}^{2n+1} \mathbf{B}_n^m(\mathbf{r}) E_n^m(\mu') E_n^m(\nu'),
\end{aligned} \tag{3.8}$$

where

$$\mathbf{B}_n^m(\mathbf{r}) = \frac{4\pi}{(2n+1)\gamma_n^m}(\nabla_{\mathbf{r}}\mathbb{F}_n^m(\mathbf{r}))E_n^m(\alpha_1). \tag{3.9}$$

In view of (2.21), (3.6) and (3.8) the integral term assumes the expansion

$$-\int_{\rho'=\alpha_1} u(\mathbf{r}')\hat{\boldsymbol{\rho}}' \times \frac{\mathbf{r}-\mathbf{r}'}{|\mathbf{r}-\mathbf{r}'|^3} ds(\mathbf{r}') = \sum_{n=0}^{\infty}\sum_{m=1}^{2n+1}\sum_{k=1}^{\infty}\sum_{\lambda=1}^{2k+1} \mathrm{A}_\kappa^\lambda(\mathbf{r}_0)\mathbf{C}_{n\kappa}^{m\lambda} \times \mathbf{B}_n^m(\mathbf{r}) \tag{3.10}$$

where

$$\mathbf{C}_{n\kappa}^{m\lambda} = \int_{\rho'=\alpha_1} E_n^m(\mu')E_n^m(\nu')E_\kappa^\lambda(\mu')E_\kappa^\lambda(\nu')\hat{\boldsymbol{\rho}}(\mu',\nu')ds(\mu',\nu'). \tag{3.11}$$

Since

$$\hat{\boldsymbol{\rho}}(\mu',\nu')\,|_{\rho'=\alpha_1} = \frac{\alpha_1\alpha_2\alpha_3}{h_1h_2h_3}\sum_{i=1}^{3}\frac{h_i}{\alpha_i}E_1^i(\mu')E_1^i(\nu')l(\mu',\nu')\hat{\mathbf{x}}_i \tag{3.12}$$

the constants in (3.11) are also written as

$$\mathbf{C}_{n\kappa}^{m\lambda} = \frac{\alpha_1\alpha_2\alpha_3}{h_1h_2h_3}\sum_{i=1}^{3}\frac{h_i}{\alpha_i} \times \int\limits_{\rho'=\alpha_1} E_n^m(\mu')E_n^m(\nu')E_\kappa^\lambda(\mu')E_\kappa^\lambda(\nu')E_1^i(\mu')E_1^i(\nu')l(\mu',\nu')ds(\mu',\nu')\hat{\mathbf{x}}_i. \tag{3.13}$$

The integral in (3.13) represents the (nm)-coeffiecient of the expansion of the function $\mathrm{E}_\kappa^\lambda(\mu')$ $\mathrm{E}_\kappa^\lambda(\nu')$ $\mathrm{E}_1^i(\mu')$ $\mathrm{E}_1^i(\nu')$ in terms of surface ellipsoidal harmonics. Expansion (3.10) is useful, since it "separates" the $\mathbf{r}_0$-dependence of the source from the $\mathbf{r}$-dependence of the observation. Its importance is due to the fact that the $\mathbf{r}_0$-dependence is not explicit in the integral term but instead it is implicit within the electric potential u. In the expansion (3.10) this dependence enters clearly via the expressions $\mathbf{Q}\cdot\nabla_{\mathbf{r}_0}\mathbb{E}_\kappa^\lambda(\mathbf{r}_0)$.

If we do the same with the first term on the RHS of (2.20) we obtain

$$\mathbf{Q}\times\frac{\mathbf{r}-\mathbf{r}_0}{|\mathbf{r}-\mathbf{r}_0|^3} = -\mathbf{Q}\times\nabla_{\mathbf{r}}\frac{1}{|\mathbf{r}-\mathbf{r}_0|} = -\sum_{n=0}^{\infty}\sum_{m=1}^{2n+1}\frac{4\pi}{2n+1}\frac{1}{\gamma_n^m}\mathbb{E}_n^m(\mathbf{r}_0)\mathbf{Q}\times\nabla_{\mathbf{r}}\mathbb{F}_n^m(\mathbf{r}). \tag{3.14}$$

Inserting (3.14) and (3.10) into (2.20) and using expressions (3.7) and (3.9) we obtain the following separable (in $\mathbf{r}_0$ and $\mathbf{r}$) expression which holds true for $\mathbf{r}$ outside the ellipsoid

$$\mathbf{B}(\mathbf{r}) = \mu_0\sum_{n=1}^{\infty}\sum_{m=1}^{2n+1}\frac{\nabla_{\mathbf{r}}\mathbb{F}_n^m(\mathbf{r})}{(2n+1)\gamma_n^m}\times\Bigg[\mathbf{Q}\mathbb{E}_n^m(\mathbf{r}_0)$$

$$-\frac{1}{\alpha_2\alpha_3}\sum_{\kappa=1}^{\infty}\sum_{\lambda=1}^{2k+1}\mathbf{C}_{n\kappa}^{m\lambda}\frac{E_n^m(\alpha_1)}{E_\kappa^\lambda{}'(\alpha_1)}\,\frac{\mathbf{Q}\cdot\nabla_{\mathbf{r}_0}\mathbb{E}_\kappa^\lambda(\mathbf{r}_0)}{\gamma_\kappa^\lambda}\Bigg], \tag{3.15}$$

where we used the fact that the n = 0 term in (3.15) vanishes.

Indeed, for $n = 0$ we have the leading term

$$\mathbf{B}_0(\mathbf{r}) = \mu_0\frac{\nabla_{\mathbf{r}}I_0^1(\rho)}{\gamma_0^1}\times\left[\mathbf{Q}-\frac{1}{\alpha_2\alpha_3}\sum_{\kappa=1}^{\infty}\sum_{\lambda=1}^{2\kappa+1}\mathbf{C}_{0\kappa}^{1\lambda}\frac{1}{E_\kappa^\lambda{}'(\alpha_1)}\frac{\mathbf{Q}\cdot\nabla_{\mathbf{r}_0}\mathbb{E}_\kappa^\lambda(\mathbf{r}_0)}{\gamma_\kappa^\lambda}\right],$$

where

$$\mathbf{C}_{0\kappa}^{1\lambda} = \frac{\alpha_1\alpha_2\alpha_3}{h_1h_2h_3}\sum_{i=1}^{3}\frac{h_i}{\alpha_i}\hat{\mathbf{x}}_i \int\limits_{\rho'=\alpha_1} E_\kappa^\lambda(\mu')E_\kappa^\lambda(\nu')E_1^i(\mu')E_1^i(\nu')l(\mu',\nu')ds'$$

$$= \frac{\alpha_1\alpha_2\alpha_3}{h_1h_2h_3}\sum_{i=1}^{3}\frac{h_i}{\alpha_i}\hat{\mathbf{x}}_i\gamma_1^i\delta_{\kappa 1}\delta_{i\lambda}$$

$$= \frac{4\pi}{3}\alpha_1\alpha_2\alpha_3h_1h_2h_3\sum_{i=1}^{3}\frac{\hat{\mathbf{x}}_i}{\alpha_ih_i}\delta_{\kappa 1}\delta_{i\lambda}$$

$$= \frac{4\pi}{3}\alpha_1\alpha_2\alpha_3h_1h_2h_3\frac{\hat{\mathbf{x}}_\lambda}{\alpha_\lambda h_\lambda}\delta_{\kappa 1}. \tag{3.16}$$

Inserting (3.16) in the expression for the $n=0$ term of the expansion (3.15) we obtain

$$\mathbf{B}_0(\mathbf{r}) = \frac{\mu_0}{4\pi}\nabla_{\mathbf{r}}I_0^1(\rho)\times\left[\mathbf{Q}-\sum_{\lambda=1}^{3}\hat{\mathbf{x}}_\lambda\frac{h_\lambda}{h_1h_2h_3}\mathbf{Q}\cdot\nabla_{\mathbf{r}_0}\left(\frac{h_1h_2h_3}{h_\lambda}x_{0\lambda}\right)\right]$$

$$= \frac{\mu_0}{4\pi}\nabla_{\mathbf{r}}I_0^1(\rho)\times\left[\mathbf{Q}-\sum_{\lambda=1}^{3}\mathbf{Q}\cdot\hat{\mathbf{x}}_\lambda\hat{\mathbf{x}}_\lambda\right] = \mathbf{0}. \tag{3.17}$$

The above calculations furnish a proof of the fact that the dipole contribution from the source at $\mathbf{r}_0$ and that from the interface at $\rho=\alpha_1$ cancel each other. So, the exterior magnetic field starts with the quadrupolic contribution

$$\nabla_{\mathbf{r}}F_1^i(\mathbf{r}) = O\left(\frac{1}{r^3}\right). \tag{3.18}$$

The explicit form of this leading term has been obtained in [2]. Obviously, $\mathbf{C}_{1\kappa}^{m\lambda}$ with $\kappa\geq 3$ should vanish by orthogonality. Note that the surface ellipsoidal harmonic

$$E_n^m(\mu)E_n^m(\nu)E_1^i(\mu)E_1^i(\nu)$$

lives in the subspace generated by the surface harmonics of degree less or equal to $n+1$. So, by orthgonality

$$\mathbf{C}_{n\kappa}^{m\lambda} = \mathbf{0},\qquad \kappa\geq n+2, \tag{3.19}$$

because any surface harmonic $E_\kappa^\lambda(\mu)E_\kappa^\lambda(\nu)$ with $\kappa\geq n+2$ lives in the orthogonal complement of the above subspace. Hence (3.15) is written as

$$\mathbf{B}(\mathbf{r}) = \mu_0\sum_{n=1}^{\infty}\sum_{m=1}^{2n+1}\frac{\nabla_{\mathbf{r}}\mathbb{F}_n^m(\mathbf{r})}{(2n+1)\gamma_n^m}\times\Bigg[\mathbf{Q}\mathbb{E}_n^m(\mathbf{r}_0)$$

$$-\frac{1}{\alpha_2\alpha_3}\sum_{\kappa=1}^{n+1}\sum_{\lambda=1}^{2\kappa+1}\mathbf{C}_{n\kappa}^{m\lambda}\frac{E_n^m(\alpha_1)}{E_\kappa^{\lambda}{}'(\alpha_1)}\frac{\mathbf{Q}\cdot\nabla_{\mathbf{r}_0}\mathbb{E}_\kappa^\lambda(\mathbf{r}_0)}{\gamma_\kappa^\lambda}\Bigg], \tag{3.20}$$

which can also be represented by the compact dyadic form

$$\mathbf{B}(\mathbf{r}) = \mu_0\sum_{n=1}^{\infty}\sum_{m=1}^{2n+1}\mathbf{Q}\cdot\widetilde{\mathbb{D}}_n^m(\mathbf{r}_0)\times\nabla\mathit{I\!F}_n^m(\mathbf{r}), \tag{3.21}$$

where the source dependent dyadic is given by

$$\tilde{\mathbb{D}}{}_n^m(\mathbf{r}_0) = \frac{1}{(2n+1)\gamma_n^m}\Bigg[-\mathbb{E}_n^m(\mathbf{r}_0)\,\tilde{\mathbb{I}}$$

$$+\frac{1}{\alpha_2\alpha_3}\sum_{\kappa=1}^{n+1}\sum_{\lambda=1}^{2k+1}\frac{1}{\gamma_\kappa^\lambda}\frac{E_\kappa^\lambda(\alpha_1)}{\mathrm{E}_\kappa^\lambda{}'(\alpha_1)}\left(\nabla_{\mathbf{r}_0}\mathbb{E}_\kappa^\lambda(\mathbf{r}_0)\right)\otimes\mathbf{C}_{n\kappa}^{m\lambda}\Bigg]. \tag{3.22}$$

Expression (3.21) provides the multipole expansion of $\mathbf{B}$ outside the ellipsoid.

4. The Vector Potential

The induction field $\mathbf{B}$, being solenoidal, is written as

$$\mathbf{B}(\mathbf{r}) = \nabla\times\mathbf{A}(\mathbf{r}) \tag{4.1}$$

where $\mathbf{A}(\mathbf{r})$ is the vector potential of $\mathbf{B}(\mathbf{r})$. In fact, it is straightforward to see that (2.20) is written as

$$\mathbf{B}(\mathbf{r}) = \nabla_{\mathbf{r}}\times\left[\frac{\mu_0}{4\pi}\frac{\mathbf{Q}}{|\mathbf{r}-\mathbf{r}_0|} - \frac{\mu_0\sigma}{4\pi}\int\limits_{\rho'=\alpha_1} u(\mathbf{r}')\frac{\hat{\boldsymbol{\rho}}'}{|\mathbf{r}-\mathbf{r}'|}ds(\mathbf{r}')\right], \tag{4.2}$$

so that the vector potential is given by

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi}\frac{\mathbf{Q}}{|\mathbf{r}-\mathbf{r}_0|} - \frac{\mu_0\sigma}{4\pi}\int\limits_{\rho'=\alpha_1} u(\mathbf{r}')\frac{\hat{\boldsymbol{\rho}}'}{|\mathbf{r}-\mathbf{r}'|}ds(\mathbf{r}'), \tag{4.3}$$

or, in separable ellipsoidal coordinates,

$$\mathbf{A}(\mathbf{r}) = -\mu_0\sum_{n=1}^{\infty}\sum_{m=1}^{2n+1}\left(\mathbf{Q}\cdot\tilde{\mathbb{D}}_n^m(\mathbf{r}_0)\right)\mathbb{F}_n^m(\mathbf{r}). \tag{4.4}$$

Note that $\mathbf{A}$ satisfies the Coulomb gauge

$$\nabla\cdot\mathbf{A}(\mathbf{r}) = -\mu_0\sum_{n=1}^{\infty}\sum_{m=1}^{2n+1}\left(\mathbf{Q}\cdot\tilde{\mathbb{D}}_n^m(\mathbf{r}_0)\right)\cdot\nabla\mathbb{F}_n^m(\mathbf{r}) = 0. \tag{4.5}$$

Indeed, since $\nabla\times\mathbf{B}=\mathbf{0}$ outside the ellipsoid and since $|\mathbf{r}-\mathbf{r}'|^{-1}$ is harmonic for $\mathbf{r}\neq\mathbf{r}'$ the integral representation (2.20) implies that

$$\nabla_{\mathbf{r}}\nabla_{\mathbf{r}}\cdot\left[\frac{\mathbf{Q}}{|\mathbf{r}-\mathbf{r}_0|} - \sigma\int\limits_{\rho'=\alpha_1}\frac{u(\mathbf{r}')\hat{\boldsymbol{\rho}}'}{|\mathbf{r}-\mathbf{r}'|}ds(\mathbf{r}')\right] = \mathbf{0}. \tag{4.6}$$

Equation (4.6) implies that

$$\mathbf{Q}\cdot\frac{\mathbf{r}-\mathbf{r}_0}{|\mathbf{r}-\mathbf{r}_0|^3} = c + \sigma\int\limits_{\rho'=\alpha_1} u(\mathbf{r}')\hat{\boldsymbol{\rho}}'\cdot\frac{\mathbf{r}-\mathbf{r}'}{|\mathbf{r}-\mathbf{r}'|^3}ds(\mathbf{r}') \tag{4.7}$$

and taking $r\to\infty$ we obtain $c=0$. Consequently, (4.5) is justified.

Define the dyadics

$$\tilde{\mathbf{S}}(\mathbf{r}) = \mathbf{Q} \otimes \frac{\mathbf{r} - \mathbf{r}_0}{|\mathbf{r} - \mathbf{r}_0|^3} \tag{4.8}$$

for the source and

$$\tilde{\mathbf{C}}(\mathbf{r}) = \sigma \int\limits_{\rho'=\alpha_1} u(\mathbf{r}')\hat{\boldsymbol{\rho}}' \otimes \frac{\mathbf{r} - \mathbf{r}'}{|\mathbf{r} - \mathbf{r}'|^3} ds(\mathbf{r}') \tag{4.9}$$

for the conductive medium. Let

$$S_S(\mathbf{r}) = \mathbf{Q} \cdot \frac{\mathbf{r} - \mathbf{r}_0}{|\mathbf{r} - \mathbf{r}_0|^3}, \quad \mathbf{S}_V(\mathbf{r}) = \mathbf{Q} \times \frac{\mathbf{r} - \mathbf{r}_0}{|\mathbf{r} - \mathbf{r}_0|^3} \tag{4.10}$$

be the scalar and the vector invariants of $\tilde{\mathbf{S}}$ respectively, and

$$C_S(\mathbf{r}) = \sigma \int\limits_{\rho'=\alpha_1} u(\mathbf{r}')\hat{\boldsymbol{\rho}}' \cdot \frac{\mathbf{r} - \mathbf{r}'}{|\mathbf{r} - \mathbf{r}'|^3} ds(\mathbf{r}'), \tag{4.11}$$

$$\mathbf{C}_V(\mathbf{r}) = \sigma \int\limits_{\rho'=\alpha_1} u(\mathbf{r}')\hat{\boldsymbol{\rho}}' \times \frac{\mathbf{r} - \mathbf{r}'}{|\mathbf{r} - \mathbf{r}'|^3} ds(\mathbf{r}') \tag{4.12}$$

be the scalar and the vector invariants of $\tilde{\mathbf{C}}$, respectively. Then

$$S_S(\mathbf{r}) = C_S(\mathbf{r}), \quad \frac{4\pi}{\mu_0}\mathbf{B}(\mathbf{r}) = \mathbf{S}_V(\mathbf{r}) - \mathbf{C}_V(\mathbf{r}). \tag{4.13}$$

In other words, the induction field $\mathbf{B}$ outside the conductive medium is the difference between the vector invariant of the source dyadic and the vector invariant of the conductivity dyadic.

5. The Scalar Magnetic Potential

Since $\mathbf{B}$ is irrotational in the current free region outside the ellipsoid it follows that

$$\mathbf{B}(\mathbf{r}) = \mu_0 \nabla_{\mathbf{r}} U(\mathbf{r}), \tag{5.1}$$

where $U(\mathbf{r})$ is the magnetic potential, which has to vanish at infinity.

In order to evaluate this magnetic potential U for the case of the sphere, Sarvas [8] integrated along a ray, in the direction of $\hat{\mathbf{r}}$, from the position $\mathbf{r}$ where the potential is evaluated all the way to infinity where the potential vanishes. In doing so, he actually used only the radial component of $\mathbf{B}$ and since $\hat{\mathbf{r}}$ was constantly tangent to the path of integrations all the necessary calculations were possible.

For the case of the ellipsoid though, the radial direction specified by the linear path of integration is not connected to anyone of the ellipsoidal directions $\hat{\boldsymbol{\rho}}, \hat{\boldsymbol{\mu}}, \hat{\boldsymbol{\nu}}$ and that makes the calculations impossible. This difficulty can be avoided if we choose an appropriate path of integration which is dictated by the geometry of the ellipsoidal system itself.

To this end, we consider the ellipsoidal representation (ρ,μ,ν) of the point $\mathbf{r}$ where the magnetic field U is to be evaluated. From the point (ρ,μ,ν) there passes an ellipsoid specified by the value of ρ, a hyperboloid of one sheet specified by the value of μ and a hyperboloid of two sheets specified by the value of ν. If we fix the values of μ and ν and we let the ellipsoidal coordinate vary from ρ to infinity we obtain the path (2.10) that is generated from the intersection of the two hyperboloids corresponding to the constant values of μ and ν. This path is a coordinate curve of the ellipsoidal system and its tangent at any point coincides with the ellipsoidal direction $\hat{\boldsymbol{\rho}}$ at the particular point. Since the system is orthogonal, the tangent $\hat{\boldsymbol{\rho}}$ remains normal to $\hat{\boldsymbol{\mu}}$ and $\hat{\boldsymbol{\nu}}$ as we travel along the path $\mathbf{C}$. Hence, integration along this path for the case of the ellipsoid, corresponds to integration along the ray for the spherical case.

Consequently, we evaluate the value of U at the point $\mathbf{r} = (\rho, \mu, \nu)$ by integrating along the ellipsoidal coordinate curve

$$\mathbf{C}(\rho') = (x_1(\rho'),\ x_2(\rho'),\ x_3(\rho')), \qquad \rho' \in [\rho, \infty) \tag{5.2}$$

given in (2.10). Indeed, since

$$\nabla = \frac{\hat{\boldsymbol{\rho}}}{h_\rho}\partial_\rho + \frac{\hat{\boldsymbol{\mu}}}{h_\mu}\partial_\mu + \frac{\hat{\boldsymbol{\nu}}}{h_\nu}\partial_\nu, \tag{5.3}$$

where $\hat{\boldsymbol{\rho}}, \hat{\boldsymbol{\mu}}, \hat{\boldsymbol{\nu}}$ are the orthogonal unit base vectors and h_ρ, h_μ, h_ν are the corresponding Lamé coefficients of the ellipsoidal system, it follows that

$$\frac{\partial}{\partial \rho} = h_\rho \hat{\boldsymbol{\rho}} \cdot \nabla \tag{5.4}$$

and we can represent U as follows

$$\begin{aligned} U(\mathbf{r}) &= U(\rho,\mu,\nu) = -\int_\rho^\infty \frac{\partial}{\partial \rho'} U(\rho',\mu,\nu) d\rho' \\ &= -\frac{1}{\mu_0}\int_\rho^\infty h_{\rho'}\hat{\boldsymbol{\rho}}' \cdot \nabla(\mu_0 U(\rho',\mu,\nu)) d\rho' = -\frac{1}{\mu_0}\int_\rho^\infty h_{\rho'}\hat{\boldsymbol{\rho}}' \cdot \mathbf{B}(\rho',\mu,\nu) d\rho', \end{aligned} \tag{5.5}$$

where μ_0 stands for the magnetic permeability. Using the identity

$$\begin{aligned} \nabla \mathbb{F}_n^m(\rho,\mu,\nu) &= [\nabla F_n^m(\rho)] E_n^m(\mu) E_n^m(\nu) + F_n^m(\rho)[\nabla E_n^m(\mu) E_n^m(\nu)] \\ &= \frac{\hat{\boldsymbol{\rho}}}{h_\rho}\left(\frac{\partial}{\partial \rho} F_n^m(\rho)\right) E_n^m(\mu) E_n^m(\nu) + F_n^m(\rho)\left[\frac{\hat{\boldsymbol{\mu}}}{h_\mu}\frac{\partial}{\partial \mu} + \frac{\hat{\boldsymbol{\nu}}}{h_\nu}\frac{\partial}{\partial \nu}\right] E_n^m(\mu) E_n^m(\nu) \end{aligned} \tag{5.6}$$

and (3.21) we obtain

$$\begin{aligned} \hat{\boldsymbol{\rho}}' \cdot (\mathbf{Q} \cdot \widetilde{\mathbb{D}}{}_n^m) \times \nabla \mathbb{F}_n^m(\rho',\mu,\nu) &= \hat{\boldsymbol{\rho}}' \cdot (\mathbf{Q} \cdot \widetilde{\mathbb{D}}{}_n^m) \times \frac{\hat{\boldsymbol{\rho}}'}{h_{\rho'}}\left(\frac{\partial}{\partial_{\rho'}} F_n^m(\rho')\right) E_n^m(\mu) E_n^m(\nu) \\ & -F_n^m(\rho')\hat{\boldsymbol{\rho}}' \cdot \left[\left(\frac{\hat{\boldsymbol{\mu}}}{h_\mu}\frac{\partial}{\partial \mu} + \frac{\hat{\boldsymbol{\nu}}}{h_\nu}\frac{\partial}{\partial \nu}\right) E_n^m(\mu) E_n^m(\nu)\right] \times (\mathbf{Q} \cdot \widetilde{\mathbb{D}}{}_n^m) \\ &= -F_n^m(\rho')\hat{\boldsymbol{\rho}}' \times \left[\left(\frac{\hat{\boldsymbol{\mu}}}{h_\mu}\frac{\partial}{\partial \mu} + \frac{\hat{\boldsymbol{\nu}}}{h_\nu}\frac{\partial}{\partial \nu}\right) E_n^m(\mu) E_n^m(\nu)\right] \cdot (\mathbf{Q} \cdot \widetilde{\mathbb{D}}{}_n^m) \\ &= F_n^m(\rho')\left[\left(\frac{\hat{\boldsymbol{\nu}}}{h_\mu}\frac{\partial}{\partial \mu} - \frac{\hat{\boldsymbol{\mu}}}{h_\nu}\frac{\partial}{\partial \nu}\right) E_n^m(\mu) E_n^m(\nu)\right] \cdot (\mathbf{Q} \cdot \widetilde{\mathbb{D}}{}_n^m) \end{aligned} \tag{5.7}$$

since the order of the ellipsoidal base is $(\hat{\boldsymbol{\rho}}, \hat{\boldsymbol{\nu}}, \hat{\boldsymbol{\mu}})$.

In order to isolate the ρ' dependence in (5.7) we use the identities

$$\hat{\boldsymbol{\nu}} = \frac{1}{h_\nu}\sum_{i=1}^{3}\frac{\nu}{\nu^2-\alpha_1^2+\alpha_i^2}x_i\hat{\mathbf{x}}_i, \qquad \hat{\boldsymbol{\mu}} = \frac{1}{h_\mu}\sum_{i=1}^{3}\frac{\mu}{\mu^2-\alpha_1^2+\alpha_i^2}x_i\hat{\mathbf{x}}_i$$

$$h_\nu^2 = \frac{(\mu^2-\nu^2)(\rho^2-\nu^2)}{(h_2^2-\nu^2)(h_3^2-\nu^2)}, \qquad h_\mu^2 = \frac{(\mu^2-\nu^2)(\rho^2-\mu^2)}{(h_2^2-\mu^2)(\mu^2-h_3^2)} \tag{5.8}$$

to obtain

$$\begin{aligned}
&\left(\frac{\hat{\boldsymbol{\nu}}}{h_\mu}\frac{\partial}{\partial\mu} - \frac{\hat{\boldsymbol{\mu}}}{h_\nu}\frac{\partial}{\partial\nu}\right)E_n^m(\mu)E_n^m(\nu)\\
&= \frac{1}{h_\nu h_\mu}\sum_{i=1}^{3}x_i\hat{\mathbf{x}}_i\left[\frac{\nu E_n^m{}'(\mu)E_n^m(\nu)}{\nu^2-\alpha_1^2+\alpha_i^2} - \frac{\mu E_n^m(\mu)E_n^m{}'(\nu)}{\mu^2-\alpha_1^2+\alpha_i^2}\right]\\
&= \frac{\sqrt{\mu^2-h_3^2}\sqrt{h_3^2-\nu^2}\sqrt{h_2^2-\mu^2}\sqrt{h_2^2-\nu^2}}{h_1h_2h_3(\mu^2-\nu^2)}\sum_{i=1}^{3}h_i\mathbb{E}_1^i(\rho',\mu,\nu)\\
&\quad\frac{1}{\sqrt{\rho'^2-\mu^2}\sqrt{\rho'^2-\nu^2}}\left[\frac{\nu E_n^m{}'(\mu)E_n^m(\nu)}{\nu^2-\alpha_1^2+\alpha_i^2} - \frac{\mu E_n^m(\mu)E_n^m{}'(\nu)}{\mu^2-\alpha_1^2+\alpha_i^2}\right]\hat{\mathbf{x}}_i\\
&= \sum_{i=1}^{3}\frac{E_1^i(\rho')}{\sqrt{\rho'^2-\mu^2}\sqrt{\rho'^2-\nu^2}}f_{ni}^m(\mu,\nu)\hat{\mathbf{x}}_i,
\end{aligned} \tag{5.9}$$

where

$$\begin{aligned}
f_{ni}^m(\mu,\nu) = {}& \frac{E_1^2(\mu)E_1^2(\nu)E_1^3(\mu)E_1^3(\nu)}{h_1h_2h_3(\mu^2-\nu^2)}h_iE_1^i(\mu)E_1^i(\nu)\times\\
&\times\left[\frac{\nu E_n^m{}'(\mu)E_n^m(\nu)}{\nu^2-\alpha_1^2+\alpha_i^2} - \frac{\mu E_n^m(\mu)E_n^m{}'(\nu)}{\mu^2-\alpha_1^2+\alpha_i^2}\right].
\end{aligned} \tag{5.10}$$

Substituting (3.21), (5.7) and (5.9) in (5.5) we obtain the magnetic potential

$$U(\mathbf{r}) = -\sum_{n=1}^{\infty}\sum_{m=1}^{2n+1}\sum_{i=1}^{3}f_{ni}^m(\mu,\nu)\mathbf{Q}\cdot\widetilde{\mathbb{D}}{}_n^m(\mathbf{r}_0)\cdot\hat{\mathbf{x}}_i\int_\rho^{+\infty}\frac{F_n^m(\rho')E_1^i(\rho')}{\sqrt{\rho'^2-h_3^2}\sqrt{\rho'^2-h_2^2}}d\rho'. \tag{5.11}$$

Note that

$$\frac{\nu E_1^i(\nu)}{\nu^2-\alpha_1^2+\alpha_i^2} = E_1^i{}'(\nu), \qquad \frac{\mu E_1^i(\mu)}{\mu^2-\alpha_1^2+\alpha_i^2} = E_1^i{}'(\mu), \qquad i=1,2,3 \tag{5.12}$$

so that (5.10) is also written as

$$f_{ni}^m(\mu,\nu) = \frac{h_iE_2^5(\mu)E_2^5(\nu)}{h_1h_2h_3(\mu^2-\nu^2)}\left[E_1^i(\mu)E_1^i{}'(\nu)E_n^m{}'(\mu)E_n^m(\nu) - E_1^i{}'(\mu)E_1^i(\nu)E_n^m(\mu)E_1^i{}'(\nu)\right]. \tag{5.13}$$

In the expression (5.11), $f_{ni}^m(\mu,\nu)$ depends only on the "orientation" (μ,ν) of the point $\mathbf{r}$ while the dependence on the "distance" ρ enters via the integral factor. The quantities $\mathbf{Q}\cdot\widetilde{\mathbb{D}}{}_n^m(\mathbf{r}_0)\cdot\hat{\mathbf{x}}_i$ are dependent solely on the source. Hence, (5.11) provides a separable expansion for the magnetic field in terms of "orientation" (μ,ν) and "distance" (ρ) variables.

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