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A New Composite Quadrature Rule

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Dedicated to Professor Graeme Fairweather on the occasion of his 70^{th} birthday.

Abstract. We present a new composite quadrature rule which is exact for polynomials of degree $2N+K-1$ with N abscissas at each subinterval and K boundary conditions. The corresponding orthogonal polynomials are introduced and the analytic formulae for abscissas and weight functions are presented. Numerical results show that the new quadrature rule is more efficient, compared with classical ones.

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Key words: Composite quadrature, orthogonal polynomial.

1 Introduction

The Gaussian quadrature rule is an interpolatory quadrature rule on zeros of certain orthogonal polynomials. A general form can be given by

$$\int_{-1}^{1} f(x)dx = \sum_{j=1}^{N} \omega_j f(x_j) + R_f$$

and its composite rule is

$$\int_a^b f(t)dt = \sum_{i=0}^{M-1} h\Big(\sum_{j=1}^{N} \omega_j f\Big(t_i + \frac{h}{2} + x_j \frac{h}{2}\Big)\Big) + \mathcal{O}(h^{2N}) \tag{1.1}$$

for a uniform partition $(h = t_{i+1} - t_i)$. The Gaussian quadrature rule is exact for polynomials of degree no larger than $2N-1$. The composite Gaussian quadrature rule is one of

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most efficient and commonly used methods for numerical integration [2,4,7,8,13], particularly for problems in which the integrand is defined piecewisely. The computational complexity of Gauss-type rules depends upon the number of points where the integrand is evaluated. The composite Gaussian quadrature rule needs to evaluate the integrand $f(x)$ at NM points.

Here we propose a new quadrature rule of the form

$$\int_{-1}^{1} f(x)dx \approx \sum_{j=1}^{N} \omega_j f(x_j) + \sum_{i=1}^{K} \beta_i (f^{(i-1)}(1) - f^{(i-1)}(-1)), \tag{1.2}$$

where ω_j, x_j and β_i are to be determined so that the formula is exact for any polynomials of degree no larger than $2N+K-1$. The corresponding composite rule is given by

$$\int_a^b f(t)dt = \sum_{i=0}^{M-1} \Big(\sum_{j=1}^{N} \frac{h}{2} \omega_j f\Big(t_i + \frac{h}{2} + x_j \frac{h}{2}\Big)\Big) + \sum_{i=1}^{K} \Big(\frac{h}{2}\Big)^i \beta_i (f^{(i-1)}(b) - f^{(i-1)}(a)) + R_f \tag{1.3}$$

with $R_f = \mathcal{O}(h^{2N+K})$ in general.

There are several interesting applications. For $K=1$, (1.3) becomes

$$\int_a^b f(x)dx = \sum_{i=0}^{M-1} \Big(\sum_{j=1}^{N} \frac{h}{2} \omega_j f\Big(t_i + \frac{h}{2} + x_j \frac{h}{2}\Big)\Big) + \frac{h}{2} \beta_1 (f(b) - f(a)) + \mathcal{O}(h^{2N+1}). \tag{1.4}$$

This composite quadrature rule only needs two extra evaluations of the integrand at endpoints. The error of this rule is one order higher than the classical one. The new rule is more significant for those small N, which are often used in practical computations. We shall show that the weight $\omega_j > 0$ and the new rule is stable although it is not a positive Gaussian quadrature.

When the integrand $f(t)$ satisfies some periodic conditions

$$f^{(k-1)}(b) = f^{(k-1)}(a), \quad k = 1, 2, \cdots, K, \tag{1.5}$$

we have

$$\int_a^b f(t)dt \approx \sum_{i=0}^{M-1} \Big(\sum_{j=1}^{N} \alpha_j f\Big(t_i + \frac{h}{2} + x_j \frac{h}{2}\Big)\Big) + \mathcal{O}(h^{2N+K}). \tag{1.6}$$

For more general case, one can use the classical sigmoidal transformation or IMI transformation [2,3], which changes the integrand $f(x)$ into one satisfying some periodic conditions in (1.5).

A similar quadrature formula is the Gaussian-Lobatto rule, see [3] given in the form of

$$\int_{-1}^{1} f(x)dx = \sum_{j=1}^{N} \omega_j f(x_j) + \sum_{i=1}^{K} \beta_i \big(f^{(i-1)}(1) + (-1)^i f^{(i-1)}(-1)\big) + R_f.$$

No corresponding composite rule can be obtained due to some symmetry in the second summation.

A related work is the expansion of the error of classical quadrature rules in terms of the Euler-Machaurin summation. A well-known formula for trapezoidal rule is

$$\int_{-1}^{1} f(x)dx = f(1)+f(-1)-\sum_{i=1}^{m}\frac{2^{2i-1}}{(2i)!}B_{2i}\big(f^{(2i-1)}(1)-f^{(2i-1)}(-1)\big)+R_m(f),$$

where B_{2i} is the Bernoulli number. The corresponding composite formula is

$$\int_a^b f(t)dt=\sum_{j=1}^{N}hf(t_j)+\frac{h}{2}\big(f(a)+f(b)\big)-\sum_{i=1}^{m}\frac{B_{2i}}{(2i)!}h^{2i}\big(f^{(2i-1)}(b)-f^{(2i-1)}(a)\big)+\tilde{R}_m(f).$$

Formulae for more general Newton-Cotes quadrature rule were given in [10] and the formulae for some Gaussian quadrature rules were discussed in [2]. The quadrature rule developed in this paper is different from those in [2] in general since the weight β_i in our formulas is obtained so that the method is exact for higher-order polynomial. For $N=1$ and K being even, we can show that $x_1=0$, $\omega_1=2$ and $\beta_i=0$ for odd i and (1.2) becomes

$$\int_a^b f(t)dt=\sum_{i=0}^{M-1}hf(t_{i+1/2})+\sum_{i=1}^{K/2}\Big(\frac{h}{2}\Big)^{2i}\beta_{2i}\big(f^{(2i-1)}(b)-f^{(2i-1)}(a)\big)+\mathcal{O}(h^{K+2}), \tag{1.7}$$

which corresponds to the midpoint rule with the Euler-Machaurin summation.

2 Analysis

In this section, we show the existence and uniqueness of the abscissas and the weights for $K=1,2$. Let $\mathbf{P}_N$ be the space of polynomials of degree no larger than N, $P_N^{\alpha}:=P_N^{\alpha,\alpha}$ be the Jacobi polynomial with respect to the weight function $\omega(x)=(1-x^2)^{\alpha}$, which satisfies the recurrence relation [11]

$$(1-x^2)\frac{d}{dx}P_N^{\alpha}(x)=\frac{N+2\alpha+1}{2}(1-x^2)P_{N-1}^{\alpha}(x)=AP_{N-1}^{\alpha}(x)+CP_{N+1}^{\alpha}(x),$$

where

$$A=\frac{(N+\alpha)(N+2\alpha+1)}{2N+2\alpha+1},\qquad C=-\frac{2N(N++1)(N+2\alpha+1)}{(2N+2\alpha+1)(2N+2\alpha+2)},$$

and the following basic formulas

$$P_N^{\alpha}(1)=\begin{pmatrix} N+\alpha \\ N \end{pmatrix},\qquad P_N^{\alpha}(-1)=(-1)^N\begin{pmatrix} N+\alpha \\ N \end{pmatrix},$$

$$\int_{-1}^{1}P_N^1(x)dx=\frac{4\delta_N}{N+2},\qquad \int_{-1}^{1}xP_N^1(x)dx=\frac{4\delta_{N+1}}{N+2},$$

$$\int_{-1}^{1} P_N^2(x)dx = \frac{4(N+2)\delta_N}{N+4}, \qquad \int_{-1}^{1} xP_N^2(x)dx = \frac{4(N+2)\delta_{N+1}}{N+4},$$
$$\int_{-1}^{1}(1-x^2)P_N^2(x)dx = \frac{16\delta_N}{(N+3)(N+4)}, \qquad \int_{-1}^{1}(1-x^2)xP_N^2(x)dx = \frac{16\delta_{N+1}}{(N+3)(N+4)},$$
$$\frac{d}{dx}P_N^\alpha(x) = \frac{N+2\alpha+1}{2}P_{N-1}^{\alpha+1}(x),$$

where $\delta_N = (1+(-1)^N))/2$.

Theorem 2.1. *For* $K=1,2$, *there exist unique abscissas and weights* $\{x_j,\omega_j,\beta_1\}$ *with* $\omega_j>0$ *such that the quadrature rule* (1.2) *is exact for* $f(x)\in\mathbf{P}_{2N+K-1}$.

We prove the theorem below for $K=1$ and $K=2$, respectively.

2.1 $K=1$

Let

$$l(x) = c_0(x-x_1)(x-x_2)\cdots(x-x_N) = c_1P_{N-1}^1(x)+c_2P_N^1(x). \tag{2.1}$$

Comparing the leading terms in (2.1),

$$c_0 = \frac{1}{2^N}\begin{pmatrix} 2(N+2) \\ N \end{pmatrix}.$$

We assume that the quadrature rule (1.3) is exact for $f(x)\in\mathbf{P}_{2N}$. Taking $f(x) = l(x), xl(x)$, respectively, in (1.3), we obtain

$$\begin{aligned}
&\frac{4\delta_N}{N+1}c_1+\frac{4\delta_N}{N+2} = 2\beta_1\big(N\delta_{N+1}c_1+(N+1)\delta_{N+1}\big),\\
&\frac{4\delta_{N+1}}{N+1}c_1+\frac{4\delta_{N+1}}{N+2} = 2\beta_1\big(N\delta_Nc_1+(N+1)\delta_N\big).
\end{aligned}$$

Solving the above system gives

$$\beta_1 = \pm\frac{2}{(N+1)\sqrt{N(N+2)}} \tag{2.2}$$

and

$$c_1 = \pm\frac{N+1}{\sqrt{N(N+2)}}c_2. \tag{2.3}$$

Since

$$\begin{aligned}
&\mathrm{sign}\big(c_1P_{N-1}^1(\pm1)+P_N^1(\pm1)\big)\\
=&\mathrm{sign}\Big(\frac{(-1)^{N-1}N(N+1)}{\sqrt{N(N+2)}}\mathrm{sign}(c_1)+(-1)^N(N+1)\Big) = (\pm1)^N,
\end{aligned}$$

the polynomial $c_1P^1_{N-1}(x)+P^1_N(x)$ always has N distinct zeros in $(-1,1)$. We take zeros of this polynomial as the abscissas x_j. Taking $f(x)=l_j(x)(1-x^2)/(1-x_j^2)$, $j=1,2,\cdots,N$, respectively, where $l_j(x)=l(x)/l'(x_j)$, we obtain

$$\omega_j=\int_{-1}^{1} l_j(x)\frac{1-x^2}{1-x_j^2}dx, \quad j=1,2,\cdots,N. \tag{2.4}$$

Now we prove that the quadrature rule (1.3) with the abscissas and weights is exact for $f(x)\in \mathbf{P}_{2N}$. In fact, we have shown that the quadrature rule (1.3) is exact for $f(x)=l(x),xl(x),(1-x^2)l_j(x)$, $j=1,2,\cdots,N$, so it is exact for $f(x)\in \mathbf{P}_{N+1}$. For $f(x)\in \mathbf{P}_{2N}$, we always have

$$f(x)=l(x)(1-x^2)p_{N-2}+p_{N+1},$$

where $p_{N-2}\in \mathbf{P}_{N-2}$, $p_{N+1}\in \mathbf{P}_{N+1}$. We see that

$$f(x_j)=p_{N+1}(x_j)$$

and by noting the orthogonality of Jacobi polynomials,

$$\begin{aligned}\int_{-1}^{1} f(x)dx&=\int_{-1}^{1}(1-x^2)l(x)p_{N-2}dx+\int_{-1}^{1}p_{N+1}dx\\&=\sum_{j=0}^{N}\omega_j p_{N+1}(x_j)+\beta_1\big(p_{N+1}(1)-p_{N+1}(-1)\big)\\&=\sum_{j=0}^{N}\omega_j f(x_j)+\beta_1\big(f(1)-f(-1)\big),\end{aligned}$$

which implies that the rule (1.3) is exact for $f(x)\in \mathbf{P}_{2N}$

Finally, taking $f(x)=(1-x^2)(l_j(x))^2$, $j=1,2,\cdots,N$, respectively, in (1.3) gives

$$\omega_j=\int_{-1}^{1}(1-x^2)(l_j(x))^2dx>0, \quad j=1,2,\cdots,N.$$

We complete the proof for $K=1$.

2.2 $K=2$

Here we take an analogous approach. Let

$$l(x)=c_0(x-x_1)(x-x_2)\cdots(x-x_N)=c_2P^2_{N-2}(x)+c_1P^2_{N-1}(x)+c_0P^2_N(x).$$

We assume that the quadrature rule (1.3) is exact for $f(x)\in\mathbf{P}_{2N+1}$. Taking $f(x)=xl(x)l(-x)$ in (1.3), we have

$$0=\int_{-1}^{1}xl(x)l(-x)dx=2\beta_1 l(1)l(-1),$$

which leads to

$$\beta_1=0, \tag{2.5}$$

if $l(1)l(-1)\neq 0$. Since

$$P_N^2(x)=(-1)^N P_N^2(-x),$$

taking $f(x)=l(x)(1-x^2)P_{N-1}^2, l(x)P_{N-1}^2$ in (1.3) results in

$$c_1\int_{-1}^{1}(1-x^2)(P_{N-1}^2(x))^2dx=-\beta_2\big(l(1)P_{N-1}^2(1)+l(-1)P_{N-1}^2(-1)\big)=-4\beta_2c_1(P_{N-1}^2(1))^2,$$
$$c_1\int_{-1}^{1}(P_{N-1}^2(x))^2dx=2\beta_2c_1P_{N-1}^2(1)P_{N-1}^{2'}(1),$$

where we have noted that $P_{N-1}^2(x)P_N^2(x)$, $P_{N+1}^2(x)P_N^2(x)$ are odd. It follows that

$$c_1=0$$

and therefore, the abscissas are symmetric, i.e.,

$$x_{N-j+1}=x_j.$$

For N being even, again taking $f(x)=l(x),(1-x^2)l(x)$, respectively, in (1.3), we have

$$c_2\int_{-1}^{1}P_{N-2}^2(x)dx+\int_{-1}^{1}P_N^2(x)dx=2\beta_2\big(c_2P_{N-2}^{2'}(1)+P_N^{2'}(1)\big),$$
$$c_2\int_{-1}^{1}(1-x^2)P_{N-2}^2(x)dx+\int_{-1}^{1}(1-x^2)P_N^2(x)dx=-4\beta_2\big(c_2P_{N-2}^2(1)+P_N^2(1)\big),$$

since $f(x)$ is even. By those classical formulas, the system becomes

$$c_2\frac{4N}{N+2}+\frac{4(N+2)}{N+4}=\beta_2\Big(c_2\frac{(N+3)N(N-1)(N-2)}{6}+\frac{(N-5)(N(N+1)(N+2))}{6}\Big), \tag{2.6a}$$
$$c_2\frac{16}{(N+1)(N+2)}+\frac{16}{(N+3)(N+4)}=-2\beta_2\big(c_2N(N-1)+(N+1)(N+2)\big), \tag{2.6b}$$

and moreover, c_2 satisfies the following quadratic equation

$$a_2c_2^2+a_1c_2+a_0=0, \tag{2.7}$$

where

$$a_2=\frac{2N(N-1)(2N^2+2N-3)}{3(N+1)(N+2)},$$
$$a_1=\frac{4N(2N^2+6N+7)}{3(N+4)},$$
$$a_2=\frac{2(N+1)(N+2)(2N^2+10N+9)}{3(N+3)(N+4)}.$$

A straightforward calculation shows that the equation has two negative roots and one is in $(-1,0)$ given by

$$c_2=\frac{-N(N+3)(2N^2+6N-7)+2(3+2N)\sqrt{3N(N^2+3N-1)(N+3)}}{(N+1)(N+2)(2N^2+10N+9)}. \tag{2.8}$$

Therefore,

$$\beta_2=-\frac{\frac{8c_2}{(N+1)(N+2)}+\frac{8}{(N+3)(N+4)}}{c_2N(N-1)+(N+1)(N+2)}>0. \tag{2.9}$$

Since $xl(x)$ and $x(1-x^2)l(x)$ are odd when N is even, the quadrature rule (1.3) is exact for $f(x)=xl(x),x(1-x^2)l(x)$.

For N being odd, we take $f(x)=xl(x),x(1-x^2)l(x)$, respectively, in (1.3) and we have

$$\begin{aligned}
&c_2\int_{-1}^{1}xP_{N-2}^2(x)dx+\int_{-1}^{1}xP_N^2(x)dx=2\beta_2\big(c_2(P_{N-2}^{2'}(1)+P_{N-2}^2(1))+P_N^{2'}(1)+P_N^2(1)\big),\\
&c_2\int_{-1}^{1}x(1-x^2)P_{N-2}^2(x)dx+\int_{-1}^{1}x(1-x^2)P_N^2(x)dx=-4\beta_2\big(c_2P_{N-2}^2(1))+P_N^2(1)\big),
\end{aligned}$$

or equivalently

$$\begin{aligned}
&c_2\frac{4(N-1)}{N+1}+\frac{4(N+1)}{N+3}=\beta_2\Big(c_2\frac{N^2(N-1)(N+1)}{6}+\frac{(N+2)^2(N+1)(N+3)}{6}\Big),\\
&c_2\frac{16}{(N+1)(N+2)}+\frac{16}{(N+3)(N+4)}=-2\beta_2\big(c_2N(N-1)+(N+1)(N+2)\big).
\end{aligned}$$

We can verify that c_2 satisfies the Eq. (2.7). Hence c_2 and β_2 are same as in the case of N being even and given in (2.8) and (2.9), respectively. Since $l(x)$ and $(1-x^2)l(x)$ are odd for N being odd, the quadrature rule (1.3) is exact for $f(x)=l(x),(1-x^2)l(x)$ in this case.

Moreover, since $-1<c_2<0$, the polynomial $c_2P_{N-2}^2(x)+P_N^2(x)$ always has N distinct zeros in $(-1,1)$ [1]. For $N>1$, taking $f(x)=l_j(x)(1-x^2)^2/(1-x_j^2)^2$, $j=1,2,\cdots,N$, respectively, in (1.3), we obtain

$$\omega_j=\int_{-1}^{1}l_j(x)\frac{(1-x^2)^2}{(1-x_j^2)^2}dx, \qquad j=1,2,\cdots,N. \tag{2.10}$$

For $N=1$, taking $f(x)=1$,

$$\omega_1=2. \tag{2.11}$$

We have shown that the quadrature rule (1.3) is exact for $f(x)=l(x),xl(x),(1-x^2)l(x)$, $(1-x^2)xl(x)$ and $f(x)=(1-x^2)^2l_j(x)$, $j=1,2,\cdots,N$, for $N>1$ and $f(x)=1$ for $N=1$, so it is exact for $f(x)\in\mathbf{P}_{N+3}$. For any $f(x)\in\mathbf{P}_{2N+1}$, we always have

$$f(x)=l(x)(1-x^2)^2p_{N-3}+p_{N+3},$$

where $p_{N-3} \in \mathbf{P}_{N_3}$ and $p_{N+3} \in \mathbf{P}_{N+3}$. We see that

$$f(x_j) = p_{N+3}(x_j), \qquad f'(\pm 1) = p'_{N+3}(\pm 1),$$

and by noting the orthogonality of the Jacobi polynomials,

$$\begin{aligned}\int_{-1}^{1} f(x)dx &= \int_{-1}^{1}(1-x^2)^2 l(x) p_{N-3} dx + \int_{-1}^{1} p_{N+3} dx \\ &= \sum_{j=0}^{N} \omega_j p_{N+3}(x_j) + \beta_1\big(p'_{N+3}(1) - p'_{N+3}(-1)\big) \sum_{j=0}^{N} \omega_j f(x_j) + \beta_2 (f'(1) - f'(-1)),\end{aligned}$$

which implies that the rule (1.3) is exact for $f(x) \in \mathbf{P}_{2N+1}$.

Also taking $f(x) = (1-x^2)(l_j(x))^2/(1-x_j^2)$, $j = 1,2,\cdots,N$, respectively, in (1.3) leads to

$$\omega_j = \int_{-1}^{1} \frac{(1-x^2)(l_j(x))^2}{1-x_j^2} dx + \frac{4\beta_2 l_j^2(1)}{1-x_j^2} > 0, \quad j = 1,2,\cdots,N.$$

We complete the proof of the theorem.

2.3 Some other cases

In Theorem 2.1, we have proved the existence and uniqueness of the weight and the abscissas for the quadrature rule (1.3) in the case of $K = 1,2$, where the abscissas are zeros of a quasi-orthogonal polynomial, a linear combination of classical Jacobi polynomials. Our numerical results show the existence in more general case although we cannot provide a theoretical analysis. For $K = 3,4$, it is possible to obtain a system of equations for c_i by an analogous approach with

$$l(x) = P_N^3 + c_1 P_{N-1}^3 + c_2 P_{N-2}^3 + c_3 P_{N_3}^3$$

and

$$l(x) = P_N^4 + c_1 P_{N-1}^4 + c_2 P_{N-2}^4 + c_3 P_{N_3}^4 + c_4, P_{N_4}^4,$$

respectively. The zeros of such quasi-orthogonal polynomials were studied by several authors [1, 15]. However, the existence of N real and distinct zeros in a certain interval for more general linear combination remains open.

Finally we consider a special case of $N = 1$. In this case, we always have $\omega_1 = 2$. For K being even, we take $x_1 = 0$ and $f(x) = x/2, x^3/2, \cdots, x^{K-1}/2$ in (1.3) and we obtain $\beta_k = 0$, $k = 1,3,\cdots,K-1$. The quadrature rule (1.3) holds also for $f(x) = x^{K+1}/2$. Again by taking $f(x) = x^2/2.x^4/2,\cdots,x^K/2$, respectively, in (1.3), we obtain $\beta_2 = 1/6$ and

$$\beta_{2j} = \frac{1}{(2j+1)!} \sum_{i=1}^{j-1} \frac{\beta_{2i}}{(2j-2i+1)!}, \quad j = 2,3,\cdots,K/2.$$

In this case, the quadrature rule (1.3) reduces to (1.7), the midpoint rule with Euler-Machaurin summation.

We list the abscissas and weights (x_j, ω_j, β_i) for $K = 1,2$ in Table 1.

Table 1: (x_j,ω_j,β_i) with $K=1,2$.

(N,K)	Abscissas	Weight ω_j; β_i
(1,1)	$-1/\sqrt{3}$	$2;\ 1/\sqrt{3}$
(2,1)	$\mp\frac{\sqrt{7}}{5}-\frac{\sqrt{2}}{5}$	$1\mp\frac{1}{3\sqrt{14}};\ \frac{\sqrt{2}}{6}$
(3,1)	-8.941766561414513D-01 -2.204556838379386e-01 5.613490048068953e-01	5.172041525280592D-01 8.033886116698080D-01 6.794072358021326D-01 1.290994448735810D-01
(4,1)	-9.322489257468869D-01 -4.767128611431370D-01 1.499209030642403D-01 7.147098298739979D-01	3.324811385435277D-01 5.753963247291207D-01 6.366909814459927D-01 4.554315552813591D-01 -8.1649658092769565-02
(5,1)	-9.529409172376568D-01 -6.271934369898662D-01 -1.400946289004881D-01 3.822706409793550D-01 8.001329073213428D-01	2.314519143323961D-01 4.235907382812989D-01 5.284695787860465D-01 4.923078787702703D-01 3.241798898299884D-01 -5.634361698189862D-02
(1,2)	0	$2;\ 0, 1/6$
(2,2)	$\mp\sqrt{1-\sqrt{\frac{8}{15}}}$	$1, 1;\ 0, \frac{1}{2}\sqrt{\frac{8}{15}}-\frac{1}{3}$
(3,2)	-7.114370355674900D-01 0.0 7.114370355674900D-01	6.171982912016719D-01 7.656034175966561D-01 6.171982912016719D-01 0.0 1.047147560344837D-02
(4,2)	-8.072338280399707D-01 -2.989538511730900D-01 2.989538511730900D-01 8.072338280399707D-01	4.180212114502936D-01 5.819787885497064D-01 5.819787885497064D-01 4.180212114502936D-01 0.0 4.463113967589422D-03

3 Numerical examples

In this section, we present our numerical results to confirm the advantage of the proposed quadrature rule. We test the proposed rule against five examples, E21, E35, E29, E41 and E42, of 50 functions given in [2,7], compared with classical Gauss-Legendre rules.

The integrands in the first two examples are smooth, the last two have a singularity

Table 2: Numerical results and comparison with classical Gaussian quadrature $(N=1,2,3,\ K=1,2)$.

$\int_a^b f(x)dx$	methods	$M=3$	$M=6$	$M=12$	h^α
$\int_0^1 e^{-x}dx$ (E21)	G-L $(N=1)$	2.917D-3	7.310D-4	1.829D-4	$h^{2.00}$
	new (1,1)	1.890D-4	2.356D-5	2.940D-6	$h^{3.00}$
	new (1,2)	9.456D-6	5.923D-7	3.704D-8	$h^{4.00}$
	G-L $(N=2)$	1.800D-6	1.128D-7	7.055D-9	$h^{4.00}$
	new (2,1)	5.114D-8	1.599D-9	4.994D-11	$h^{5.00}$
	new (2,2)	1.269D-9	1.987D-11	3.109D-13	$h^{6.00}$
	G-L $(N=3)$	4.285D-10	6.714D-12	1.050D-13	$h^{6.00}$
	new (3,1)	7.545D-12	5.909D-14	5.121D-16	$h^{6.85}$
	new (3,2)	1.223D-13	2.220D-16	7.930D-19	$h^{8.13}$
$\int_0^1 \frac{1}{1+x}dx$ (E35)	G-L $(N=1)$	3.392D-3	8.628D-4	2.167D-4	$h^{1.99}$
	new (1,1)	5.170D-4	6.537D-5	8.165D-6	$h^{3.00}$
	new (1,2)	7.973D-5	5.196D-6	3.284D-7	$h^{3.98}$
	G-L $(N=2)$	1.501D-5	9.866D-7	6.250D-8	$h^{3.98}$
	new (2,1)	1.740D-6	5.786D-8	1.833D-9	$h^{4.98}$
	new (2,2)	2.080D-7	3.584D-9	5.754D-11	$h^{5.96}$
	G-L $(N=3)$	6.964D-8	1.208D-9	1.943D-11	$h^{5.96}$
	new (3,1)	7.202D-9	6.392D-11	5.166D-13	$h^{6.95}$
	new (3,2)	7.701D-10	3.569D-12	1.464D-14	$h^{7.93}$
$\int_a^b f(x)dx$	methods	$M=60$	$M=120$	$M=240$	h^α
$\int_0^{2\pi} x\sin(30x)cos(x)dx$ (E29)	G-L $(N=1)$	1.198D-1	2.320D-2	5.482D-3	$h^{2.08}$
	new (1,1)	6.768D-3	3.212D-4	1.888D-5	$h^{4.09}$
	new (1,2)	3.364D-2	1.664D-3	9.879D-5	$h^{4.07}$
	G-L $(N=2)$	6.768D-3	3.212D-4	1.888D-5	$h^{4.09}$
	new (2,1)	7.766D-5	9.148D-7	1.342D-8	$h^{6.09}$
	new (2,2)	4.282D-4	5.068D-6	7.445D-8	$h^{6.09}$
	G-L $(N=3)$	1.479D-4	1.722D-6	2.519D-8	$h^{6.10}$
	new (3,1)	6.022D-7	1.734D-9	6.370D-12	$h^{8.09}$
	new (3,2)	3.8222D-6	1.107D-8	4.045D-11	$h^{8.10}$
		$M=30$	$M=60$	$M=120$	h^α
$\int_0^1 \lvert x^2-0.25\rvert^{1/2}dx$ (E40)	G-L $(N=1)$	6.877D-4	2.487D-4	8.930D-5	$h^{1.48}$
	new (1,1)	8.541D-5	3.077D-5	1.099D-5	$h^{1.49}$
	new (1,2)	6.681D-4	2.438D-4	8.807D-5	$h^{1.47}$
	G-L $(N=2)$	8.876D-5	3.138D-5	1.110D-5	$h^{1.50}$
	new (2,1)	2.401D-5	8.530D-6	3.027D-6	$h^{1.49}$
	new (2,2)	2.078D-4	7.493D-5	2.685D-5	$h^{1.48}$
	G-L $(N=3)$	3.061D-5	1.083D-5	3.833D-6	$h^{1.50}$
	new (3,1)	9.912D-6	3.527D-6	1.250D-6	$h^{1.50}$
	new (3,2)	9.295D-5	3.334D-5	1.191D-5	$h^{1.49}$
$\int_0^1 \lvert x^2-0.25\rvert^{3/2}dx$ (E42)	G-L $(N=1)$	1.136D-4	2.889D-5	7.309D-6	$h^{1.98}$
	new (1,1)	1.563D-6	1.736D-7	1.735D-8	$h^{3.32}$
	new (1,2)	8.817D-5	2.253D-5	5.720D-6	$h^{1.98}$
	G-L $(N=2)$	5.310D-7	9.370D-8	1.655D-8	$h^{2.50}$
	new (2,1)	4.710D-8	8.840D-9	1.608D-9	$h^{2.46}$
	new (2,2)	1.734D-5	4.390D-6	1.108D-6	$h^{1.99}$
	G-L $(N=3)$	7.631D-8	1.349D-8	2.384D-9	$h^{2.50}$
	new (3,1)	1.112D-8	2.011D-9	3.594D-10	$h^{2.48}$
	new (3,2)	5.773D-6	1.457D-6	3.666D-7	$h^{1.99}$

at $x=0.5$, and the integrand in E29 is highly oscillatory. We present our numerical results in Table 2, compared with the results obtained by the corresponding Gauss-Legendre quadrature rule (G-L). Some observation can be made. For smooth integrands (E21 and E35), we take M=3,6,12, respectively. Numerical results show that the accuracy of Gauss-Legendre quadrature rule and the new quadrature rule is $\mathcal{O}(h^{2N})$ and $\mathcal{O}(h^{2N+K})$ $(K=1,2)$, respectively, which is in good agreement with theoretical analysis and the new one is K-order higher than the classical one. For the highly oscillatory example E29, more subintervals are needed for both Gauss-Legendre rule and the new quadrature rule. Here we present numerical results with $M=60$, $M=120$ and $M=240$, respectively. One can see that new rule has better performance. In particular, the new rule with $K=1$ shows its superconvergence for this example. For these two singular problems, the new quadrature rule does not give much improvement as given in the first three examples. It is known that the accuracy of a quadrature rule depends upon the regularity of integrand and high-order methods cannot give a high-order accuracy when the integrand is non-smooth. Clearly our numerical results illustrate that the new composite rule is more efficient than classical Gauss quadrature rules in general.

4 Conclusions

We have presented a class of quadrature rules for numerical integration in a finite interval and proved the existence and uniqueness of the weight and the abscissas for $K=1,2$. The rule proposed in the paper is of higher order accuracy than classical ones when the integrand is smooth, and has almost the same computational complexity, which have been confirmed by our numerical results.

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