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A ROBUST FINITE ELEMENT METHOD FOR A 3-D ELLIPTIC SINGULAR PERTURBATION PROBLEM*

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Abstract

This paper proposes a robust finite element method for a three-dimensional fourth-order elliptic singular perturbation problem. The method uses the three-dimensional Morley element and replaces the finite element functions in the part of bilinear form corresponding to the second-order differential operator by a suitable approximation. To give such an approximation, a convergent nonconforming element for the second-order problem is constructed. It is shown that the method converges uniformly in the perturbation parameter.

Mathematics subject classification: 65N30.
Key words: Finite element, Singular perturbation problem.

1. Introduction

Let Ω be a bounded polyhedral domain of R^n with $1 \le n \le 3$. Denote the boundary of Ω by $\partial\Omega$. For $f \in L^2(\Omega)$, we consider the following boundary value problem of the fourth-order elliptic singular perturbation equation:

$$\begin{cases} \varepsilon^2 \Delta^2 u - \Delta u = f, & \text{in } \Omega, \\ u|_{\partial\Omega} = \dfrac{\partial u}{\partial \nu}\Big|_{\partial\Omega} = 0, \end{cases} \tag{1.1}$$

where $\nu = (\nu_1, \cdots, \nu_n)^\top$ is the unit outer normal of $\partial\Omega$, Δ is the standard Laplacian operator and ε is a small parameter satisfying $0 < \varepsilon \le 1$. When $\varepsilon \to 0$ the differential equation formally degenerates to the Poisson equation.

In the two-dimensional case, the Morley element was proposed in [9] for the plate bending problem. The Morley element is convergent for fourth-order elliptic problems, but is divergent for second-order problems (see, e.g., [5, 8, 13]). The Morley element and an C^0 modified Morley element for problem (1.1) were discussed in [10]. It was shown that the modified Morley element is uniformly convergent with respect to ε while the Morley element does not converge when $\varepsilon \to 0$. Two non-C^0 nonconforming elements were proposed in [4] by the double set parameter technique. These two elements were also proved to be uniformly convergent. A modified Morley element method for problem (1.1) was proposed in [15]; it is convergent uniformly with respect to ε. This method also uses the Morley element (or the rectangle Morley element), but the linear approximation (or the bilinear approximation) of finite element functions is used in the part of the bilinear form corresponding to the second-order differential term.

In this paper, we consider the three-dimensional case. The three-dimensional Morley element can be found in [11] or in [14]. We will take a similar way used in [15] and propose a modified

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Morley element method for problem (1.1). We will use certain approximation of finite element functions in the part of the bilinear form corresponding to the second-order differential term. It will be shown that the modified method converges uniformly in the perturbation parameter ε. The three-dimensional Morley element uses the integral averages of the function over all edges as degrees of freedom instead of the function values at vertices. To given suitable approximation of the finite element function, we need to construct a convergent nonconforming finite element for the Poisson equation with the integral averages of the function over all edges as degrees of freedom.

Problem (1.1) is a boundary value problem of a stationary linearizing form of the Cahn-Hilliard equation. The modelling in material science makes use of the Cahn-Hilliard equations in three dimensions (see, e.g., [2, 3, 6]). Besides the theoretical interest, our new finite element method is expected to be useful in the computation of the Cahn-Hilliard equation.

The paper is organized as follows. The rest of this section lists some preliminaries. Section 2 describes a nonconforming finite element for the Poisson equation. Section 3 gives the detailed descriptions of the modified Morley element method. Section 4 shows the uniform convergence of the method.

Throughout this paper, we assume $n = 3$. For a nonnegative integer s, let $H^s(\Omega)$, $\|\cdot\|_{s,\Omega}$ and $|\cdot|_{s,\Omega}$ denote the usual Sobolev space, norm and semi-norm, respectively. Let $H_0^s(\Omega)$ be the closure of $C_0^\infty(\Omega)$ in $H^s(\Omega)$ with respect to the norm $\|\cdot\|_{s,\Omega}$ and $(\cdot,\cdot)$ denotes the inner product of $L^2(\Omega)$. Define

$$a(v,w) = \int_\Omega \sum_{i,j=1}^{3} \frac{\partial^2 v}{\partial x_i \partial x_j} \frac{\partial^2 w}{\partial x_i \partial x_j}, \quad \forall v, w \in H^2(\Omega), \tag{1.2}$$

$$b(v,w) = \int_\Omega \sum_{i=1}^{3} \frac{\partial v}{\partial x_i} \frac{\partial w}{\partial x_i}, \quad \forall v, w \in H^1(\Omega). \tag{1.3}$$

The weak form of problem (1.1) is: find $u \in H_0^2(\Omega)$ such that

$$\varepsilon^2 a(u,v) + b(u,v) = (f,v), \quad \forall v \in H_0^2(\Omega). \tag{1.4}$$

Let u^0 be the solution of following boundary value problem:

$$\begin{cases} -\Delta u^0 = f, & \text{in } \Omega, \\ u^0|_{\partial\Omega} = 0. \end{cases} \tag{1.5}$$

For a mesh size h, let $\mathcal{T}_h$ be a triangulation of Ω consisting of tetrahedra. For each $T \in \mathcal{T}_h$, let h_T be the diameter of the smallest ball containing T and ρ_T be the diameter of the largest ball contained in T. Let $\{\mathcal{T}_h\}$ be a family of triangulations with $h \to 0$. Throughout the paper, we assume that $h_T \le h \le \eta\rho_T$, $\forall T \in \mathcal{T}_h$, with η a positive constant independent of h.

2. A Nonconforming Element for the Poisson Equation

For a subset $B \subset R^3$ and a nonnegative integer r, let $P_r(B)$ be the space of all polynomials with degree not greater than r.

Given a tetrahedron T, its four vertices are denoted by a_j, $1 \le j \le 4$. The face of T opposite a_j is denoted by F_j, $1 \le j \le 4$. The edge with a_i and a_j as its vertices, is denoted by S_{ij},

$1 \le i < j \le 4$. Denote the measures of T, F_i and S_{ij} by $|T|$, $|F_i|$ and $|S_{ij}|$ respectively. Let $\lambda_1, \cdots, \lambda_4$ be the barycentric coordinates of T. Define

$$q_1 = (\lambda_1 - \lambda_3)(\lambda_2 - \lambda_4), \quad q_2 = (\lambda_1 - \lambda_2)(\lambda_4 - \lambda_3).$$

We define a nonconforming element (T, P_T^s, Φ_T^s) for the Poisson equation by

1) T is a tetrahedron.

2) $P_T^s = P_1(T) + \mathrm{span}\{q_1, q_2\}$.

3) For $v \in C^0(T)$,

$$\Phi_T^s(v) = (\phi_{12}(v), \phi_{13}(v), \phi_{14}(v), \phi_{23}(v), \phi_{24}(v), \phi_{34}(v))^\top$$

with

$$\phi_{ij}(v) = \frac{1}{|S_{ij}|}\int_{S_{ij}} v, \quad 1 \le i < j \le 4.$$

For $1 \le i < j \le 4$, let $1 \le k < l \le 4$ and $\{k, l\} \cap \{i, j\} = \emptyset$, and define

$$p_{ij} = \frac{2}{3}(\lambda_i + \lambda_j) - \frac{1}{3}(\lambda_k + \lambda_l) + 2\lambda_i\lambda_j + 2\lambda_k\lambda_l - \sum_{i_1=i,j}\sum_{i_2=k,l}\lambda_{i_1}\lambda_{i_2}. \tag{2.1}$$

Set

$$\tilde{p}_{ij} = \frac{2}{3}(\lambda_i + \lambda_j) - \frac{1}{3}(\lambda_k + \lambda_l).$$

Then the following identities can be verified:

$$\begin{cases} p_{12} = \tilde{p}_{12} + 2q_1 + q_2, \ p_{13} = \tilde{p}_{13} - q_1 - 2q_2, \ p_{14} = \tilde{p}_{14} - q_1 + q_2, \\ p_{23} = \tilde{p}_{23} - q_1 + q_2, \ p_{24} = \tilde{p}_{24} - q_1 - 2q_2, \ p_{34} = \tilde{p}_{34} + 2q_1 + q_2. \end{cases} \tag{2.2}$$

That is, $p_{ij} \in P_T^s$, $1 \le i < j \le 4$. Denote by δ_{ij} the Kronecker delta. By directly computations, we obtain

$$\frac{1}{|S_{kl}|}\int_{S_{kl}} p_{ij} = \delta_{ik}\delta_{jl}, \quad 1 \le i < j \le 4, \ 1 \le k < l \le 4. \tag{2.3}$$

Hence, p_{ij}, $1 \le i < j \le 4$, are the basis functions corresponding to the degrees of freedom. This indicates that Φ_T^s is P_T^s-unisolvent.

The interpolation operator Π_T^s corresponding to (T, P_T^s, Φ_T^s) is written as

$$\Pi_T^s v = \sum_{1 \le i < j \le 4} p_{ij}\phi_{ij}(v), \quad \forall v \in C^0(T). \tag{2.4}$$

For $v \in L^2(\Omega)$ and $v|_T \in C^0(T)$, $\forall T \in \mathcal{T}_h$, define $\Pi_h^s v$ by

$$\Pi_h^s v|_T = \Pi_T^s(v|_T), \quad \forall T \in \mathcal{T}_h. \tag{2.5}$$

By the interpolation theory (see, e.g., [5]) we obtain the following lemma.

Lemma 2.1. *There exists a constant C independent of h such that*

$$|v - \Pi_T^s v|_{m,T} \le Ch^{2-m}|v|_{2,T}, \quad 0 \le m \le 2, \ \forall v \in H^2(T), \tag{2.6}$$

is true for all $T \in \mathcal{T}_h$.

By a direct computation we have the following lemma.

Lemma 2.2. *Given a tetrahedron T, the following equality is true:*

$$\frac{1}{|F_i|}\int_{F_i} p = \frac{1}{9}\sum_{\substack{1\le j<k\le 4\\ j\ne i,k\ne i}} \frac{1}{|S_{jk}|}\int_{S_{jk}} p, \quad 1\le i\le 4,\ \forall p\in P_T^s. \tag{2.7}$$

By the above two lemmas and some relevant mathematical theories (see, e.g., [5, 8, 12]) we can verify that this element is convergent for the boundary value problem of the three-dimensional Poisson equation.

3. Modified Morley Element Method

The Morley element can be described by (T, P_T^M, Φ_T^M) with

1) T is a tetrahedron.

2) $P_T^M = P_2(T)$.

3) Φ_T^M is the vector of degrees of freedom whose components are:

$$\frac{1}{|S_{ij}|}\int_{S_{ij}} v,\ 1\le i<j\le 4; \quad \frac{1}{|F_j|}\int_{F_j}\frac{\partial v}{\partial \nu},\ \ 1\le j\le 4$$

for $v\in C^1(T)$.

For each $\mathcal{T}_h$, let V_h and V_{h0} be the corresponding finite element spaces associated with the Morley element for the discretization of $H^2(\Omega)$ and $H_0^2(\Omega)$, respectively. This defines two families of finite element spaces $\{V_h\}$ and $\{V_{h0}\}$. It is known that $V_h \not\subset H^2(\Omega)$ and $V_{h0}\not\subset H_0^2(\Omega)$. Let Π_h be the interpolation operator corresponding to the Morley element and $\mathcal{T}_h$.

We define, for $v, w\in L^2(\Omega)$ and $v|_T, w|_T\in H^2(T)$, $\forall T\in\mathcal{T}_h$,

$$a_h(v,w) = \sum_{T\in\mathcal{T}_h}\int_T\sum_{i,j=1}^{3}\frac{\partial^2 v}{\partial x_i\partial x_j}\frac{\partial^2 w}{\partial x_i\partial x_j}, \tag{3.1}$$

$$b_h(v,w) = \sum_{T\in\mathcal{T}_h}\int_T\sum_{i=1}^{3}\frac{\partial v}{\partial x_i}\frac{\partial w}{\partial x_i}. \tag{3.2}$$

The standard finite element method for problem (1.4) corresponding to the Morley element is: find $u_h\in V_{h0}$ such that

$$\varepsilon^2 a_h(u_h,v_h) + b_h(u_h,v_h) = (f,v_h), \quad \forall v_h\in V_{h0}. \tag{3.3}$$

We consider the following modified Morley element method: find $u_h\in V_{h0}$ such that

$$\varepsilon^2 a_h(u_h,v_h) + b_h(\Pi_h^s u_h,\Pi_h^s v_h) = (f,\Pi_h^s v_h), \quad \forall v_h\in V_{h0}. \tag{3.4}$$

Problem (3.4) has a unique solution when $\varepsilon>0$. When $\varepsilon=0$, the problem degenerates to

$$b_h(\Pi_h^s u_h,\Pi_h^s v_h) = (f,\Pi_h^s v_h), \quad \forall v_h\in V_{h0}. \tag{3.5}$$

Although the solution of problem (3.5) is not unique yet, $\Pi_h^s u_h$ is uniquely determined. Actually, $\Pi_h^s u_h$ is the exact finite element solution for problem (1.5) given in the previous section.

Now we consider two examples. Let $\Omega = [-1,1]^3$ and

$$u_1(x) = (1-x_1^2)^2(1-x_2^2)^2(1-x_3^2)^2,$$

$$u_2(x) = (1+\cos\pi x_1)(1+\cos\pi x_2)(1+\cos\pi x_3).$$

Let $i \in \{1,2\}$. For $\varepsilon \geq 0$, set $f = \varepsilon^2\Delta^2 u_i - \Delta u_i$. Then u_i is the solution of problem (1.1) when $\varepsilon > 0$, and is the solution of problem (1.5) when $\varepsilon = 0$.

We first divide Ω into 12 tetrahedral elements with $h = 2$ as shown in Fig. 3.1. Then we use the global regular refinement strategy provided in [1] to get the mesh sequence.

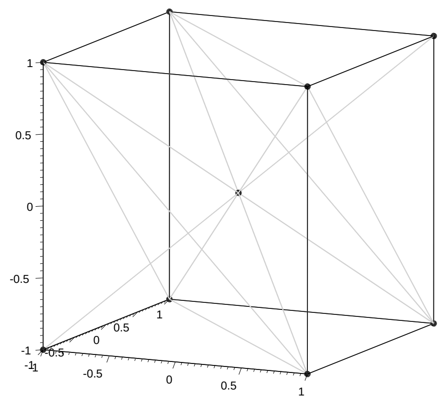

Fig. 3.1. The initial mesh.

Define

$$|||v_h|||_{\varepsilon,h} = \left(\varepsilon^2 a_h(v_h,v_h) + b_h(\Pi_h^s v_h, \Pi_h^s v_h)\right)^{1/2}, \quad \forall v_h \in V_{h0}.$$

Different values of ε and h are chosen to demonstrate the behaviors of the following relative error of the modified Morley element method,

$$E_{\varepsilon,h} = \frac{|||\Pi_h u - u_h|||_{\varepsilon,h}}{|||\Pi_h u|||_{\varepsilon,h}}, \tag{3.6}$$

where u_h is the solution of problem (3.4).

Let $g = \Delta^2 u_i$. Then u_i is the solution of the following boundary value problem of biharmonic equation,

$$\begin{cases} \Delta^2 u = g, & \text{in } \Omega, \\ u|_{\partial\Omega} = \dfrac{\partial u}{\partial \nu}\Big|_{\partial\Omega} = 0. \end{cases} \tag{3.7}$$

For comparison, we also consider the error of the finite element solution to problem (3.7). Let $\tilde{u}_h \in V_{h0}$ be the solution of the following problem,

$$a_h(\tilde{u}_h, v_h) = (g, \Pi_h^s v_h), \quad \forall v_h \in V_{h0}. \tag{3.8}$$

In this situation, the relative error $\tilde{E}_h$ is represented by

$$\tilde{E}_h^2 = \frac{a_h(\Pi_h u - \tilde{u}_h, \Pi_h u - \tilde{u}_h)}{a_h(\Pi_h u, \Pi_h u)}. \tag{3.9}$$

For the modified Morley element method in the case of $f = \varepsilon^2\Delta^2 u_1 - \Delta u_1$ and $g = \Delta^2 u_1$, $E_{\varepsilon,h}$ and $\tilde{E}_h$, corresponding to some ε and h, are listed in Table 3.1. In the case that $f = \varepsilon^2\Delta^2 u_2 - \Delta u_2$ and $g = \Delta^2 u_2$, $E_{\varepsilon,h}$ and $\tilde{E}_h$ are listed in Table 3.2.

From Tables 3.1 and 3.2 we see that the modified Morley element method converges for all $\varepsilon \in [0,1]$. More precisely, the result shows that $E_{\varepsilon,h}$ is linear with respect to h as well as $E_{0,h}$ and $\tilde{E}_h$ are.

Table 3.1:

$\epsilon\backslash h$	2	1	2^{-1}	2^{-2}	2^{-3}	2^{-4}
0	0.5800	0.2942	0.1654	0.08072	0.03969	0.01960
2^{-10}	0.5800	0.2942	0.1654	0.08071	0.03966	0.01958
2^{-8}	0.5802	0.2943	0.1654	0.0805	0.03923	0.01874
2^{-6}	0.5844	0.2950	0.1651	0.07802	0.03429	0.01276
2^{-4}	0.6492	0.3082	0.1680	0.06994	0.02814	0.01234
2^{-2}	1.438	0.5122	0.2923	0.1426	0.06951	0.03398
1	3.565	0.8335	0.4097	0.1959	0.09494	0.04634
∞ (Biharmonic)	4.195	0.8872	0.4243	0.2021	0.09781	0.04773

Table 3.2:

$\epsilon\backslash h$	2	1	2^{-1}	2^{-2}	2^{-3}	2^{-4}
0	0.7717	0.3048	0.1778	0.08484	0.04107	0.02009
2^{-10}	0.7717	0.3048	0.1778	0.08483	0.04105	0.02003
2^{-8}	0.7721	0.3049	0.1777	0.08466	0.04063	0.01920
2^{-6}	0.7776	0.3054	0.1777	0.08226	0.03570	0.01316
2^{-4}	0.8643	0.3140	0.1822	0.07345	0.02838	0.01209
2^{-2}	1.919	0.4598	0.2949	0.1401	0.06752	0.03288
1	4.788	0.7376	0.4012	0.1907	0.09203	0.04484
∞ (Biharmonic)	5.646	0.7877	0.4144	0.1966	0.09480	0.04618

4. Convergence Analysis

In this section, we discuss the convergence properties of the modified Morley element methods given in the previous section.

We introduce the following mesh-dependent norm $\|\cdot\|_{m,h}$ and semi-norm $|\cdot|_{m,h}$:

$$\|v\|_{m,h} = \Big(\sum_{T\in\mathcal{T}_h} \|v\|_{m,T}^2\Big)^{1/2}, \quad |v|_{m,h} = \Big(\sum_{T\in\mathcal{T}_h} |v|_{m,T}^2\Big)^{1/2},$$

for $v \in L^2(\Omega)$ that $v|_T \in H^m(T)$, $\forall T \in \mathcal{T}_h$.

Let u and u_h be the solutions of problems (1.4) and (3.4), respectively.

Lemma 4.1. *There exists a constant C independent of h and ε such that for any $v_h \in V_{h0}$, there exists $w_h \in H_0^1(\Omega)$ satisfying*

$$\|v_h - w_h\|_{0,\Omega} + h|v_h - w_h|_{1,h} \le Ch^2|v_h|_{2,h}, \tag{4.1}$$

$$\|\Pi_h^s v_h - w_h\|_{0,\Omega} + h|\Pi_h^s v_h - w_h|_{1,h} \le Ch|\Pi_h^s v_h|_{1,h}. \tag{4.2}$$

Proof. Let $v_h \in V_{h0}$. For $T \in \mathcal{T}_h$, denote by Π_T^1 the linear interpolation operator with function values at all vertices of T as degrees of freedom. Define $\Pi_h^1 v$ by

$$\Pi_h^1 v|_T = \Pi_T^1(v|_T), \quad \forall T \in \mathcal{T}_h,$$

for a function $v \in L^2(\Omega)$ and $v|_T \in C^0(T)$, $\forall T \in \mathcal{T}_h$. By the interpolation theory, the following inequality is true:

$$|\Pi_h^s v_h - \Pi_h^1 \Pi_h^s v_h|_{m,h} \le Ch^{2-m}|\Pi_h^s v_h|_{2,h}, \quad 0 \le m \le 1. \tag{4.3}$$

Given a set $B \subset R^n$, let $\mathcal{T}_h(B) = \{\, T \in \mathcal{T}_h \mid B \cap T \neq \emptyset \,\}$ and $N_h(B)$ the number of the elements in $\mathcal{T}_h(B)$. Now we define $w_h \in H_0^1(\Omega)$ as follows: for any $T \in \mathcal{T}_h$,

i) $w_h|_T \in P_1(T)$.

ii) if the vertex a_i of T is in Ω then

$$w_h(a_i) = \frac{1}{N_h(a_i)} \sum_{T' \in \mathcal{T}_h(a_i)} (\Pi_h^s v_h|_{T'})(a_i).$$

Thus, w_h is well defined. We will show that

$$|\Pi_h^s v_h - w_h|_{m,h} \le Ch^{2-m}|\Pi_h^s v_h|_{2,h}, \quad 0 \le m \le 1. \tag{4.4}$$

By the affine technique, we can show that

$$|p|_{m,T}^2 \le Ch^{3-2m} \sum_{i=1}^{4} |p(a_i)|^2, \quad \forall p \in P_1(T), \quad m = 0, 1. \tag{4.5}$$

Set $\varphi = \Pi_h^1 \Pi_h^s v_h - w_h$ and $\psi = \Pi_h^s v_h$. Obviously, $\varphi|_T \in P_1(T)$, $\forall T \in \mathcal{T}_h$. For $T \in \mathcal{T}_h$, let $\varphi_T = \varphi|_T$ and $\psi_T = \psi|_T$.

If the vertex a_i of T is in Ω then by the definition of w_h,

$$\varphi(a_i) = \psi_T(a_i) - \frac{1}{N_h(a_i)} \sum_{T' \in \mathcal{T}_h(a_i)} \psi_{T'}(a_i) = \frac{1}{N_h(a_i)} \sum_{T' \in \mathcal{T}_h(a_i)} \Big(\psi_T(a_i) - \psi_{T'}(a_i)\Big).$$

For $T' \in \mathcal{T}_h(a_i)$ there exist $T_1, \cdots, T_J \in \mathcal{T}_h(a_i)$ such that $T_1 = T$, $T_J = T'$ and $\tilde{F}_j = T_j \cap T_{j+1}$ is a common face of T_j and T_{j+1} and $a_i \in \tilde{F}_j$, $1 \le j < J$. By the inverse inequality, we have

$$\begin{aligned} \big|\psi_T(a_i) - \psi_{T'}(a_i)\big|^2 &= \Big|\sum_{j=1}^{J-1} \big(\psi_{T_j}(a_i) - \psi_{T_{j+1}}(a_i)\big)\Big|^2 \\ \le\ & C\sum_{j=1}^{J-1} \big|\psi_{T_j}(a_i) - \psi_{T_{j+1}}(a_i)\big|^2 \le Ch^{-2} \sum_{j=1}^{J-1} \big|\psi_{T_j} - \psi_{T_{j+1}}\big|_{0,\tilde{F}_j}^2. \end{aligned}$$

On each edge of $\tilde{F}_j$, the integral average of ψ_{T_j} is equal to the one of $\psi_{T_{j+1}}$ by the definition of ψ. Hence

$$\big|\psi_{T_j} - \psi_{T_{j+1}}\big|_{0,\tilde{F}_j}^2 \le Ch^3\big(|\psi|_{2,T_j}^2 + |\psi|_{2,T_{j+1}}^2\big).$$

Then

$$\Big|\psi_T(a_i)-\psi_{T'}(a_i)\Big|^2 \le Ch\sum_{j=1}^{J}|\psi|_{2,T_j}^2.$$

Since $N_h(T)$ is bounded, we get

$$|\varphi(a_i)|^2 \le Ch\sum_{T'\in\mathcal{T}_h(T)}|\psi|_{2,T'}^2. \tag{4.6}$$

If the vertex a_i of T is on $\partial\Omega$ then there exists $T'\in\mathcal{T}_h(a_i)$ with a face F of T' belonging to $\partial\Omega$ and $a_i\in F$. By the definition of w_h,

$$\begin{aligned}|\varphi(a_i)| &= |\psi_T(a_i)-\psi_{T'}(a_i)+\psi_{T'}(a_i)|\\ &\le |\psi_T(a_i)-\psi_{T'}(a_i)|+|\psi_{T'}(a_i)|.\end{aligned}$$

Since the integral average of $\psi_{T'}$ on each edge of F vanishes,

$$|\psi_{T'}(a_i)|^2 \le Ch^{-2}|\psi_{T'}|_{0,F}^2 \le Ch|\psi|_{2,T'}^2$$

by the inverse inequality. By similar analysis for $|\psi_T(a_i)-\psi_{T'}(a_i)|$, we conclude that (4.6) is also true in this case.

Combining (4.5) and (4.6), we obtain

$$h^{2m}|\varphi|_{m,T}^2 \le Ch^4\sum_{T'\in\mathcal{T}_h(T)}|\psi|_{2,T'}^2.$$

Summing the above inequality over all $T\in\mathcal{T}_h$ gives

$$h^{2m}|\varphi|_{m,h}^2 \le Ch^4\sum_{T\in\mathcal{T}_h}\sum_{T'\in\mathcal{T}_h(T)}|\psi|_{2,T'}^2.$$

Consequently,

$$h^{2m}|\varphi|_{m,h}^2 \le Ch^4|\psi|_{2,h}^2. \tag{4.7}$$

Inequality (4.4) follows from (4.7) and (4.3).

We obtain (4.2) by (4.4) and the inverse inequality, and (4.1) by (4.4) and Lemma 2.1. This completes the proof of Lemma 4.1. □

Lemma 4.2. *There exists a constant C independent of h and ε such that for any $v_h\in V_{h0}$,*

$$|\,b_h(\Pi_h^s u,\Pi_h^s v_h)+(\Delta u,\Pi_h^s v_h)| \le Ch|u|_{2,\Omega}|\Pi_h^s v_h|_{1,h}, \tag{4.8}$$

$$|\,a_h(u,v_h)-(\Delta^2 u,\Pi_h^s v_h)| \le C(h|u|_{3,\Omega}+h^2\|\Delta^2 u\|_{0,\Omega})|v_h|_{2,h}, \tag{4.9}$$

when $u\in H^3(\Omega)$.

Proof. Let $v_h\in V_{h0}$. By Green's formula,

$$\begin{aligned}&b_h(\Pi_h^s u,\Pi_h^s v_h)+(\Delta u,\Pi_h^s v_h)\\ &= b_h(\Pi_h^s u-u,\Pi_h^s v_h)+\sum_{T\in\mathcal{T}_h}\int_{\partial T}\frac{\partial u}{\partial\nu}\Pi_h^s v_h.\end{aligned}$$

Given $T \in \mathcal{T}_h$ and a face F of T, and let P_F^0 be the orthogonal projection operator from $L^2(F)$ to $P_0(F)$. By Lemma 2.2, we have

$$\sum_{T\in\mathcal{T}_h}\int_{\partial T}\frac{\partial u}{\partial\nu}\Pi_h^s v_h = \sum_{T\in\mathcal{T}_h}\sum_{F\subset\partial T}\int_F\Big(\frac{\partial u}{\partial\nu} - P_F^0\frac{\partial u}{\partial\nu}\Big)(\Pi_h^s v_h - P_F^0\Pi_h^s v_h).$$

By the interpolation theory and the Schwarz inequality we obtain

$$\Big|\sum_{T\in\mathcal{T}_h}\int_{\partial T}\frac{\partial u}{\partial\nu}\Pi_h^s v_h\Big| \le Ch|u|_{2,\Omega}|\Pi_h^s v_h|_{1,h}. \tag{4.10}$$

On the other hand,

$$|b_h(\Pi_h^s u - u, \Pi_h^s v_h)| \le Ch|u|_{2,\Omega}|\Pi_h^s v_h|_{1,h}.$$

Hence (4.8) follows.

Now let $\phi \in H^1(\Omega)$. Let $i, j \in \{1,2,3\}$. It is known that the integral average of $\frac{\partial}{\partial x_j}v_h$ on F is continuous through F and vanishes when $F \subset \partial\Omega$. Then Green's formula gives

$$\begin{aligned}
&\sum_{T\in\mathcal{T}_h}\int_T\Big(\phi\frac{\partial^2 v_h}{\partial x_i\partial x_j} + \frac{\partial\phi}{\partial x_i}\frac{\partial v_h}{\partial x_j}\Big)\\
&= \sum_{T\in\mathcal{T}_h}\int_{\partial T}\phi\frac{\partial v_h}{\partial x_j}\nu_i = \sum_{T\in\mathcal{T}_h}\sum_{F\subset\partial T}\int_F\phi\frac{\partial v_h}{\partial x_j}\nu_i\\
&= \sum_{T\in\mathcal{T}_h}\sum_{F\subset\partial T}\int_F\phi\Big(\frac{\partial v_h}{\partial x_j} - P_F^0\frac{\partial v_h}{\partial x_j}\Big)\nu_i\\
&= \sum_{T\in\mathcal{T}_h}\sum_{F\subset\partial T}\int_F(\phi - P_F^0\phi)\Big(\frac{\partial v_h}{\partial x_j} - P_F^0\frac{\partial v_h}{\partial x_j}\Big)\nu_i.
\end{aligned}$$

From the Schwarz inequality and the interpolation theory we obtain

$$\begin{aligned}
&\Big|\sum_{T\in\mathcal{T}_h}\sum_{F\subset\partial T}\int_F(\phi - P_F^0\phi)\Big(\frac{\partial v_h}{\partial x_j} - P_F^0\frac{\partial v_h}{\partial x_j}\Big)\nu_i\Big|\\
&\le \sum_{T\in\mathcal{T}_h}\sum_{F\subset\partial T}\|\phi - P_F^0\phi\|_{0,F}\Big\|\frac{\partial v_h}{\partial x_j} - P_F^0\frac{\partial v_h}{\partial x_j}\Big\|_{0,F}\\
&\le C\sum_{T\in\mathcal{T}_h}h|\phi|_{1,T}|v_h|_{2,T} \le Ch|\phi|_{1,\Omega}|v_h|_{2,h}.
\end{aligned}$$

Consequently, we obtain that for any $\phi \in H^1(\Omega),\ v_h \in V_{h0},\ i,j \in \{1,2,3\}$,

$$\Big|\sum_{T\in\mathcal{T}_h}\int_T\Big(\phi\frac{\partial^2 v_h}{\partial x_i\partial x_j} + \frac{\partial\phi}{\partial x_i}\frac{\partial v_h}{\partial x_j}\Big)\Big| \le Ch|\phi|_{1,\Omega}|v_h|_{2,h}. \tag{4.11}$$

Let $w_h \in H_0^1(\Omega)$ be as in (4.1) and (4.2). Then

$$\begin{aligned}
&a_h(u, v_h) - (\Delta^2 u, \Pi_h^s v_h)\\
&= (\Delta^2 u, w_h - \Pi_h^s v_h) + \sum_{i=1}^{3} \sum_{T\in\mathcal{T}_h} \int_T \frac{\partial \Delta u}{\partial x_i} \frac{\partial (w_h - v_h)}{\partial x_i}\\
&\quad + \sum_{i=1}^{3} \sum_{T\in\mathcal{T}_h} \int_T \Big(\Delta u \frac{\partial^2 v_h}{\partial x_i^2} + \frac{\partial \Delta u}{\partial x_i} \frac{\partial v_h}{\partial x_i} \Big)\\
&\quad + \sum_{1\le i\ne j\le 3} \sum_{T\in\mathcal{T}_h} \int_T \Big(\frac{\partial^2 u}{\partial x_i \partial x_j} \frac{\partial^2 v_h}{\partial x_i \partial x_j} + \frac{\partial^3 u}{\partial x_i^2 \partial x_j} \frac{\partial v_h}{\partial x_j} \Big)\\
&\quad - \sum_{1\le i\ne j\le 3} \sum_{T\in\mathcal{T}_h} \int_T \Big(\frac{\partial^2 u}{\partial x_i^2} \frac{\partial^2 v_h}{\partial x_j^2} + \frac{\partial^3 u}{\partial x_i^2 \partial x_j} \frac{\partial v_h}{\partial x_j} \Big). \qquad (4.12)
\end{aligned}$$

We obtain (4.9) from (4.12), (4.11), (4.1) and Lemma 2.1. □

Theorem 4.1. *There exists a constant C independent of h and ε such that*

$$\varepsilon \|u - u_h\|_{2,h} + \|u - \Pi_h^s u_h\|_{1,h} \le Ch(|u|_{2,\Omega} + \varepsilon |u|_{3,\Omega} + \varepsilon h \|\Delta^2 u\|_{0,\Omega}) \qquad (4.13)$$

when $u \in H^3(\Omega)$.

Proof. Let $\varphi_h = \Pi_h u$. Then

$$\begin{aligned}
&\varepsilon \|u - u_h\|_{2,h} + \|u - \Pi_h^s u_h\|_{1,h}\\
&\le \varepsilon \|u - \varphi_h\|_{2,h} + \|u - \Pi_h^s \varphi_h\|_{1,h} + \varepsilon \|u_h - \varphi_h\|_{2,h} + \|\Pi_h^s (u_h - \varphi_h)\|_{1,h}. \qquad (4.14)
\end{aligned}$$

Set $v_h = u_h - \varphi_h$. From (3.4) and (1.1), we derive that

$$\begin{aligned}
&\varepsilon^2 a_h(v_h, v_h) + b_h(\Pi_h^s v_h, \Pi_h^s v_h)\\
&= \varepsilon^2 a_h(u - \varphi_h, v_h) + b_h(\Pi_h^s (u - \varphi_h), \Pi_h^s v_h)\\
&\quad + \varepsilon^2 \Big((\Delta^2 u, \Pi_h^s v_h) - a_h(u, v_h) \Big) - \Big((\Delta u, \Pi_h^s v_h) + b_h(\Pi_h^s u, \Pi_h^s v_h) \Big).
\end{aligned}$$

By the interpolation theory, Lemma 2.1, (4.8) and (4.9), we have

$$\begin{aligned}
&\varepsilon^2 a_h(v_h, v_h) + b_h(\Pi_h^s v_h, \Pi_h^s v_h)\\
&\le Ch\big(|u|_{2,\Omega} + \varepsilon |u|_{3,\Omega} + \varepsilon h \|\Delta^2 u\|_{0,\Omega} \big) \big(\varepsilon |v_h|_{2,h} + |\Pi_h^s v_h|_{1,h} \big).
\end{aligned}$$

Since

$$\varepsilon^2 \|v_h\|_{2,h}^2 + \|\Pi_h^s v_h\|_{1,h}^2 \le C \Big(\varepsilon^2 a_h(v_h, v_h) + b_h(\Pi_h^s v_h, \Pi_h^s v_h) \Big)$$

we obtain that

$$\varepsilon \|u_h - \varphi_h\|_{2,h} + \|\Pi_h^s (u_h - \varphi_h)\|_{1,h} \le Ch\big(|u|_{2,\Omega} + \varepsilon |u|_{3,\Omega} + \varepsilon h \|\Delta^2 u\|_{0,\Omega} \big). \qquad (4.15)$$

The theorem follows from the interpolation theory, (4.14) and (4.15). □

Similar to Lemma 5.1 in [10], we can prove the following lemma.

Lemma 4.3. *If Ω is convex, then there exists a constant C independent of ε such that*

$$\varepsilon^{-1/2} |u - u^0|_{1,\Omega} + \varepsilon^{1/2} |u|_{2,\Omega} + \varepsilon^{3/2} |u|_{3,\Omega} \le C \|f\|_{0,\Omega} \qquad (4.16)$$

for all $f \in L^2(\Omega)$.

Lemma 4.4. *There exists a constant C independent of ε and h such that*

$$\|v\|_{0,\partial T} \le C\Big(h^{-1/2}\|v\|_{0,T} + \|v\|_{0,T}^{1/2}\|v\|_{1,T}^{1/2}\Big), \tag{4.17}$$

$$\sum_{F\subset\partial T} \|v - P_F^0 v\|_{0,F} \le C\|v\|_{0,T}^{1/2}|v|_{1,T}^{1/2}, \tag{4.18}$$

for all $v \in H^1(T)$ and $T \in \mathcal{T}_h$.

Proof. Let $\hat{T}$ be the reference tetrahedron. From [7] we know that

$$\|\hat{v}\|_{0,\partial\hat{T}} \le C\|\hat{v}\|_{0,\hat{T}}^{1/2}\|\hat{v}\|_{1,\hat{T}}^{1/2}, \quad \forall \hat{v} \in H^1(\hat{T}). \tag{4.19}$$

Then we obtain (4.17) by the affine technique.

Now let $T \in \mathcal{T}_h$ and let P_T^0 be the orthogonal projection operator from $L^2(T)$ to $P_0(T)$. For each $\hat{F} \subset \partial\hat{T}$ and $\hat{v} \in H^1(\hat{T})$, we have by (4.19) and the interpolation theory,

$$\begin{aligned}\|\hat{v} - P_{\hat{F}}^0\hat{v}\|_{0,\hat{F}} &\le \|\hat{v} - P_{\hat{T}}^0\hat{v} - P_{\hat{F}}^0(\hat{v} - P_{\hat{T}}^0\hat{v})\|_{0,\hat{F}}\\ &\le C\|\hat{v} - P_{\hat{T}}^0\hat{v}\|_{0,\hat{T}}^{1/2}\|\hat{v} - P_{\hat{T}}^0\hat{v}\|_{1,\hat{T}}^{1/2} \le \|\hat{v}\|_{0,\hat{T}}^{1/2}|\hat{v}|_{1,\hat{T}}^{1/2}.\end{aligned}$$

Consequently, we obtain (4.18) by the affine technique. □

Theorem 4.2. *If Ω is convex, then there exists a constant C independent of h and ε such that*

$$\varepsilon\|u - u_h\|_{2,h} + \|u - \Pi_h^s u_h\|_{1,h} \le Ch^{1/2}\|f\|_{0,\Omega}. \tag{4.20}$$

Proof. From the interpolation theory, it is true that

$$\|u - \Pi_h u\|_{2,h}^2 \le C|u|_{2,\Omega}\|u - \Pi_h u\|_{2,h} \le Ch|u|_{2,\Omega}|u|_{3,\Omega}.$$

By Lemma 4.3, we have

$$\varepsilon\|u - \Pi_h u\|_{2,h} \le Ch^{1/2}\|f\|_{0,\Omega}. \tag{4.21}$$

Similar to (4.4) in [10], we can show that

$$\|v - \Pi_h^s v\|_{1,h}^2 \le Ch|v|_{1,\Omega}|v|_{2,\Omega}, \quad \forall v \in H_0^2(\Omega). \tag{4.22}$$

Using (4.22), we obtain

$$\|u - u^0 - \Pi_h^s(u - u^0)\|_{1,h}^2 \le Ch|u - u^0|_{1,\Omega}|u - u^0|_{2,\Omega},$$

and we have, by the interpolation theory,

$$\|u^0 - \Pi_h^s u^0\|_{1,h} \le Ch|u^0|_{2,\Omega}.$$

By Lemma 4.3 and the following inequalities,

$$\begin{aligned}&\|u^0\|_{2,\Omega} \le C\|f\|_{0,\Omega},\\ &\|u - \Pi_h^s u\|_{1,h} \le \|u - u^0 - \Pi_h^s(u - u^0)\|_{1,h} + \|u^0 - \Pi_h^s u^0\|_{1,h},\end{aligned} \tag{4.23}$$

we have

$$\|u - \Pi_h^s u\|_{1,h} \le Ch^{1/2}\|f\|_{0,\Omega}. \tag{4.24}$$

Set $v_h = u_h - \Pi_h u$. Lemma 2.2 and Green's formula give

$$b_h(\Pi_h^s u, \Pi_h^s v_h) + (\Delta u, \Pi_h^s v_h) = b_h(\Pi_h^s u - u, \Pi_h^s v_h)$$
$$+ \sum_{T\in\mathcal{T}_h}\sum_{F\subset\partial T}\int_F \Big(\frac{\partial(u-u^0)}{\partial\nu} - P_F^0\frac{\partial(u-u^0)}{\partial\nu}\Big)(\Pi_h^s v_h - P_F^0\Pi_h^s v_h)$$
$$+ \sum_{T\in\mathcal{T}_h}\sum_{F\subset\partial T}\int_F \Big(\frac{\partial u^0}{\partial\nu} - P_F^0\frac{\partial u^0}{\partial\nu}\Big)(\Pi_h^s v_h - P_F^0\Pi_h^s v_h).$$

By the Schwarz inequality and the interpolation theory, we have

$$|\, b_h(\Pi_h^s u, \Pi_h^s v_h) + (\Delta u, \Pi_h^s v_h)| \le C\sum_{T\in\mathcal{T}_h}\Big(\ |u-\Pi_h^s u|_{1,T} + h|u^0|_{2,T}$$
$$+h^{1/2}\sum_{F\subset\partial T}\Big|\frac{\partial(u-u^0)}{\partial\nu} - P_F^0\frac{\partial(u-u^0)}{\partial\nu}\Big|_{0,F}\Big)|\Pi_h^s v_h|_{1,T}.$$

It follows from (4.24), (4.18), (4.23) and Lemma 4.3 that

$$|\, b_h(\Pi_h^s u, \Pi_h^s v_h) + (\Delta u, \Pi_h^s v_h)| \le Ch^{1/2}\|f\|_{0,\Omega}\|\Pi_h^s v_h\|_{1,h}. \tag{4.25}$$

Now let $\phi\in H^1(\Omega)$ and $i,j\in\{1,2\}$. From the proof of Lemma 4.2, we have

$$\Big|\sum_{T\in\mathcal{T}_h}\int_T\Big(\phi\frac{\partial^2 v_h}{\partial x_i\partial x_j} + \frac{\partial\phi}{\partial x_i}\frac{\partial v_h}{\partial x_j}\Big)\Big|$$
$$\le \sum_{T\in\mathcal{T}_h}\sum_{F\subset\partial T}\|\phi - P_F^0\phi\|_{0,F}\Big\|\frac{\partial v_h}{\partial x_j} - P_F^0\frac{\partial v_h}{\partial x_j}\Big\|_{0,F}.$$

By the interpolation theory and (4.17), we have

$$\Big|\sum_{T\in\mathcal{T}_h}\int_T\Big(\phi\frac{\partial^2 v_h}{\partial x_i\partial x_j} + \frac{\partial\phi}{\partial x_i}\frac{\partial v_h}{\partial x_j}\Big)\Big| \le Ch^{1/2}\|\phi\|_{0,\Omega}^{1/2}\|\phi\|_{1,\Omega}^{1/2}|v_h|_{2,h}. \tag{4.26}$$

Let $w_h\in H_0^1(\Omega)$ such that (4.1) and (4.2) are true. If $\varepsilon\le h$, then by Green's formula we get

$$\sum_{T\in\mathcal{T}_h}\int_T\frac{\partial\phi}{\partial x_i}\frac{\partial(w_h-v_h)}{\partial x_i} = \sum_{T\in\mathcal{T}_h}\int_{\partial T}\phi\frac{\partial(w_h-v_h)}{\partial x_i}\nu_i - \sum_{T\in\mathcal{T}_h}\int_T\phi\frac{\partial^2(w_h-v_h)}{\partial x_i^2}.$$

By the Schwarz inequality, (4.1) and (4.17), we obtain

$$\Big|\sum_{T\in\mathcal{T}_h}\int_T\frac{\partial\phi}{\partial x_i}\frac{\partial(w_h-v_h)}{\partial x_i}\Big|$$
$$\le \sum_{T\in\mathcal{T}_h}\|\phi\|_{0,\partial T}\Big\|\frac{\partial(w_h-v_h)}{\partial x_i}\Big\|_{0,\partial T} + \sum_{T\in\mathcal{T}_h}\|\phi\|_{0,T}|w_h-v_h|_{2,T}$$
$$\le C\big(h^{1/2}\|\phi\|_{0,\Omega}^{1/2}\|\phi\|_{1,\Omega}^{1/2} + \|\phi\|_{0,\Omega}\big)|v_h|_{2,h}.$$

Hence when $\varepsilon\le h$,

$$\varepsilon^2\Big|\sum_{T\in\mathcal{T}_h}\int_T\frac{\partial\phi}{\partial x_i}\frac{\partial(w_h-v_h)}{\partial x_i}\Big| \le Ch^{1/2}\Big(\varepsilon^2\|\phi\|_{0,\Omega}^{1/2}\|\phi\|_{1,\Omega}^{1/2} + \varepsilon^{3/2}\|\phi\|_{0,\Omega}\Big)|v_h|_{2,h}. \tag{4.27}$$

When $\varepsilon > h$, by the Schwarz inequality and (4.1) we have,

$$\varepsilon^2\Big|\sum_{T\in\mathcal{T}_h}\int_T \frac{\partial\phi}{\partial x_i}\frac{\partial(w_h-v_h)}{\partial x_i}\Big| \le Ch\varepsilon^2|\phi|_{1,\Omega}|v_h|_{2,h} \le Ch^{1/2}\varepsilon^{5/2}|\phi|_{1,\Omega}|v_h|_{2,h}. \tag{4.28}$$

It follows from (1.1) and (1.5) that

$$\varepsilon^2(\Delta^2 u, w_h-\Pi_h^s v_h) = (\Delta(u-u^0), w_h-\Pi_h^s v_h). \tag{4.29}$$

When $\varepsilon > h$, we have by (4.2) and Lemma 2.2,

$$\begin{aligned}&|(\Delta(u-u^0), w_h-\Pi_h^s v_h)| \le Ch|u-u^0|_{2,\Omega}|\Pi_h^s v_h|_{1,h}\\ \le\, &Ch^{1/2}\varepsilon^{1/2}|u-u^0|_{2,\Omega}|\Pi_h^s v_h|_{1,h}.\end{aligned}$$

By Lemma 4.3 and (4.23) we get that

$$|\varepsilon^2(\Delta^2 u, w_h-\Pi_h^s v_h)| \le Ch^{1/2}\|f\|_{0,\Omega}|\Pi_h^s v_h|_{1,h}, \quad \text{for } \varepsilon > h. \tag{4.30}$$

On the other hand, we have

$$\begin{aligned}&(\Delta(u-u^0), w_h-\Pi_h^s v_h)\\ =&\sum_{j=1}^3\sum_{T\in\mathcal{T}_h}\Big(\int_{\partial T}\frac{\partial(u-u^0)}{\partial x_j}(w_h-\Pi_h^s v_h)\nu_j - \int_T \frac{\partial(u-u^0)}{\partial x_j}\frac{\partial(w_h-\Pi_h^s v_h)}{\partial x_j}\Big).\end{aligned}$$

Then

$$\begin{aligned}&|(\Delta(u-u^0), w_h-\Pi_h^s v_h)|\\ \le&\sum_{j=1}^3\sum_{T\in\mathcal{T}_h}\Big(\Big\|\frac{\partial(u-u^0)}{\partial x_j}\Big\|_{0,\partial T}\|w_h-\Pi_h^s v_h\|_{0,\partial T} + \|u-u^0\|_{1,T}\|w_h-\Pi_h^s v_h\|_{1,T}\Big).\end{aligned}$$

By (4.17), (4.2) and the Schwarz inequality, we obtain

$$\begin{aligned}&|(\Delta(u-u^0), w_h-\Pi_h^s v_h)|\\ \le\, &C\Big(h^{1/2}\|u-u^0\|_{1,\Omega}^{1/2}\|u-u^0\|_{2,\Omega}^{1/2} + \|u-u^0\|_{1,\Omega}\Big)|\Pi_h^s v_h|_{1,h}.\end{aligned}$$

From Lemma 4.3 and (4.29) we get

$$|\varepsilon^2(\Delta^2 u, w_h-\Pi_h^s v_h)| \le C\big(h^{1/2}+\varepsilon^{1/2}\big)\|f\|_{0,\Omega}|\Pi_h^s v_h|_{1,h}.$$

That is, (4.30) is also true when $\varepsilon \le h$.

From Lemma 4.3, (4.12), (4.26)-(4.28) and (4.30) we obtain

$$\varepsilon^2|\,a_h(u,v_h) - (\Delta^2 u, \Pi_h^s v_h)| \le Ch^{1/2}\|f\|_{0,\Omega}\big(\varepsilon|v_h|_{2,h} + |\Pi_h^s v_h|_{1,h}\big). \tag{4.31}$$

Combining (4.21), (4.24), (4.25), (4.31) and the proof of Theorem 4.1, we complete the proof of the theorem. □

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