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THE EFFECT OF MEMORY TERMS IN DIFFUSION PHENOMENA*[1)]

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Abstract

In this paper the effect of integral memory terms in the behavior of diffusion phenomena is studied. The energy functional associated with different models is analyzed and stability inequalities are established. Approximation methods for the computation of the solution of the integro-differential equations are constructed. Numerical results are included.

Mathematics subject classifications: 74D99,74S20, 78M20, 80M20.
Key words: Heat propagation, Integro-differential equation, Numerical approximation, Splitting method.

1. Heat Equation and Jeffrey's Kernel

Let us consider the problem of heat conduction in a one dimensional homogeneous and isotropic bar $(0, a)$ in which the heat pulses are transmitted by waves at finite but perhaps high speed. Representing by $q(x, t)$ the heat flux and assuming that holds the Fourier law

$$q = -k_1 \frac{\partial u}{\partial x}\,, \tag{1}$$

where k_1 is the effective thermal conductivity, it can be shown that the temperature u at (x, t) satisfies the classical heat equation

$$\frac{\partial u}{\partial t} = c\frac{\partial^2 u}{\partial x^2}\,, \tag{2}$$

where c represents the thermal diffusivity. It is well known that this equation has the unphysical property that if a sudden change in the temperature is made at some point of the bar, it will be felt instantly everywhere. We say that diffusion gives rise to infinite speeds of propagation.

The problem that unphysical infinite speeds of propagation are generated by diffusion was first treated in [3]. In order to avoid this serious drawback it has been proposed in [3] to define the flux by an integral over the history of the temperature gradient, that is,

$$q(x, t) = -\frac{k}{\tau}\int_{-\infty}^{t} e^{-\frac{t-s}{\tau}} \frac{\partial u}{\partial x}(x, s)\, ds\ , \tag{3}$$

where k represents the thermal conductivity. We note that the Fourier law holds as the limit of Cattaneo's law (3) when $\tau \to 0$. This definition of $q(x, t)$ corresponds to a first order approximation, in τ, of the modified Fourier law

$$q(x, t + \tau) = -k_1 \frac{\partial u}{\partial x}(x, t).$$

In fact, considering the first order approximation

$$q(x, t + \tau) \simeq q(x, t) + \tau \frac{\partial q}{\partial t}(x, t),$$

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and integrating the first order differential equation

$$\frac{1}{\tau} q(x,t) + \frac{\partial q}{\partial t}(x,t) = -\frac{k_1}{\tau}\frac{\partial u}{\partial x}(x,t),$$

we obtain (3).

Considering (3), it can be shown that the temperature u at (x,t) satisfies Cattaneo's equation

$$\frac{\partial u}{\partial t}(x,t) = \frac{k}{\gamma\tau}\int_{-\infty}^{t} e^{-\frac{t-s}{\tau}} \frac{\partial^2 u}{\partial x^2}(x,s)\, ds\,, \tag{4}$$

where γ is the heat capacity. This equation was considered by different authors. For instance, Vernotte, in [15], considered Cattaneo's equation as the simplest that gives rise to finite speed of propagation. In fact, equation (4) is equivalent to the hyperbolic telegraph equation

$$\frac{\partial^2 u}{\partial t^2} + \frac{1}{\tau}\frac{\partial u}{\partial t} = \frac{k}{\gamma\tau}\frac{\partial^2 u}{\partial t^2}\,, \tag{5}$$

which transmits waves with a finite speed $\sqrt{\dfrac{k}{\gamma\tau}}$ and presents a very small attenuation as a consequence of relaxation. The telegraph equation is the simplest mathematical model combining wave propagation and diffusion.

In Figure 1 we show the long time behavior of heat equation and Cattaneo's equation. The plots have been obtained from the discretization with standard numerical methods in a very fine mesh.

However, as pointed out in the engineering literature (see for example [9]), there are no real conductors which exhibit the wave propagation behavior of Cattaneo's model.

In [9] a corrected version of flux (3) is presented. A kernel of Jeffrey's type was then considered by replacing in (3) the exponential kernel by

$$Q(s) = k_1\delta(s) + \frac{k_2}{\tau}e^{-\frac{s}{\tau}}, \tag{6}$$

where $\delta(s)$ is a Dirac delta function, and k_1 and k_2 represent , respectively, the effective thermal conductivity and the elastic conductivity. In this case the Fourier law leads to a flux q defined by

$$q(x,t) = -k_1\frac{\partial u}{\partial x}(x,t) - \frac{k_2}{\tau}\int_{-\infty}^{t} e^{-\frac{t-s}{\tau}}\frac{\partial u}{\partial x}(x,s)\, ds. \tag{7}$$

It can be shown that the temperature, in this case, satisfies Jeffrey's integro-differential equation

$$\frac{\partial u}{\partial t}(x,t) = \frac{k_1}{\gamma}\frac{\partial^2 u}{\partial x^2}(x,t) + \frac{k_2}{\gamma\tau}\int_{-\infty}^{t} e^{-\frac{t-s}{\tau}}\frac{\partial^2 u}{\partial x^2}(x,s)\, ds\,. \tag{8}$$

In recent years several authors gave attention to the introduction of Volterra integrals to model heat propagation (see [2], [5], [13]).

For $k_2 = 0$ we have the classical diffusion equation while for $k_1 = 0$ we obtain Cattaneo's equation. In Figure 2 we present the behavior of the three models at different times. We remark that Jeffrey's model allows the selection of parameters k_1 and k_2 such that mathematical models in agreement with experimental behavior of different materials can be obtained.

Cattaneo's equation and Jeffrey's equations predict different quantitative and qualitative behavior for the propagation of heat. This fact can be explained because while Cattaneo's equation is of hyperbolic type, Jeffrey's equation has a parabolic behavior. In the first case, if the initial condition presents discontinuities they will be propagated with constant speed. By the contrary, as Jeffrey's equation is of parabolic type, any discontinuity of the initial condition will be smoothed by diffusion associated with the effective thermal conductivity.

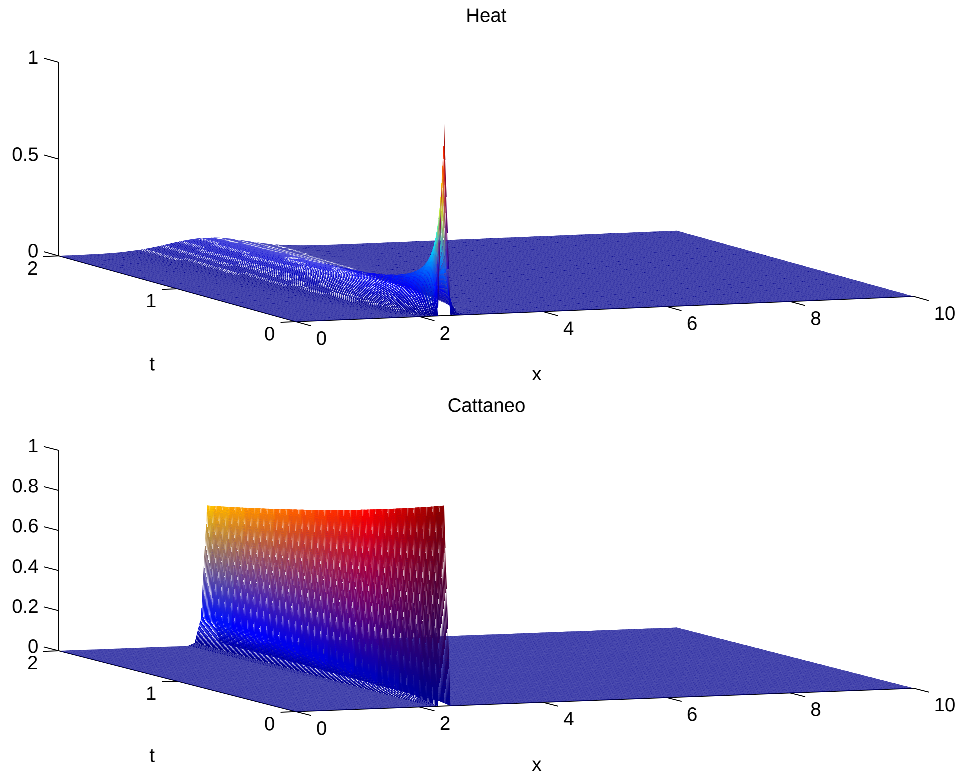

Figure 1: Solutions of the heat equation and Cattaneo's equation with a Dirac delta initial condition.

Finally we remark that for reaction-diffusion equations the approximation of the transport by diffusion gives also rise to infinite speeds of propagation. In [6]-[8], [10]-[12] different models are proposed in order to avoid this drawback induced by the classical Fick's law.

The aim of the present paper is the analytical and numerical study of Jeffrey's equation. From an analytical point of view the novelty of our approach is the establishment of an energy estimate which leads to the stability of the solution. From a numerical viewpoint we propose a new splitting method which simulates the heat transport as the superposition of diffusion and wave propagation.

In Section 2 we study the energy behavior of the solution of models Jeffrey's (8). This result establish its stability. Section 3 is devoted to the study of numerical methods for (8) obtained following two different approaches. The first one is obtained discretizing the partial derivatives and the integral term using a quadrature rule; the second method is established using a splitting technique and Proposition 2. Numerical simulations are included.

2. An Energy Estimate for the Jeffrey's Equation

We consider in what follows the initial boundary value problem (IBVP) associated with (8) but where the integral is computed in $(0,t)$, that is

$$\frac{\partial u}{\partial t}(x,t) = \frac{k_1}{\gamma}\frac{\partial^2 u}{\partial x^2}(x,t) + \frac{k_2}{\gamma\tau}\int_0^t e^{-\frac{t-s}{\tau}}\frac{\partial^2 u}{\partial x^2}(x,s)\,ds,\ \ x\in(0,a), t>0, \tag{9}$$

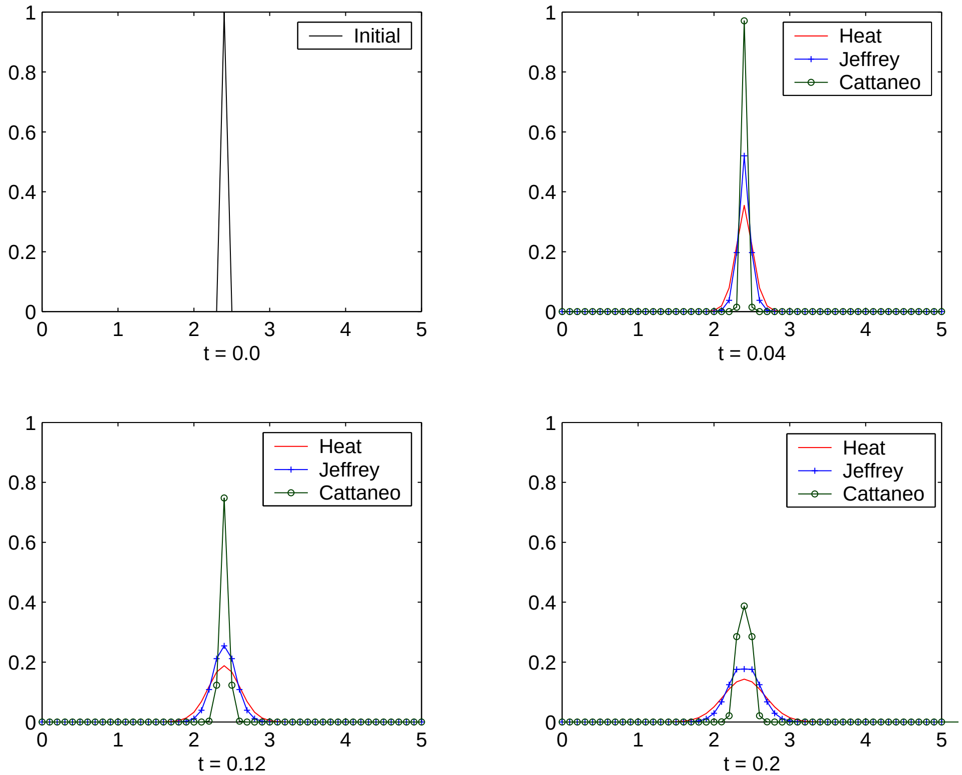

Figure 2: Behavior of the solutions of heat equation, Cattaneo's equation and Jeffreys equation.

associated with initial and boundary conditions

$$\begin{aligned} &u(x,0) = u_0(x),\ x \in (0,a), \\ &u(0,t) = u(a,t) = 0,\ t > 0. \end{aligned} \tag{10}$$

We establish, in the following result, an estimate for the energy of the functional

$$\|u\|_{L^2} + \frac{k_2}{\gamma\tau}\|\int_0^t e^{-\frac{t-s}{\tau}}\frac{\partial u}{\partial x}(.,s)\,ds\|^2_{L^2},$$

for $t > 0$, where $\|.\|_{L^2}$ represents the usual L^2 norm.

Proposition 1. *Let u be a solution of (9). Then*

$$\|u\|^2_{L^2} + \frac{k_2}{\gamma\tau}\|\int_0^t e^{-\frac{t-s}{\tau}}\frac{\partial u}{\partial x}(.,s)\,ds\|^2_{L^2} \le e^{-2\min\{\frac{k_1}{\gamma a^2},\frac{1}{\tau}\}t}\|u_0\|^2_{L^2}. \tag{11}$$

Proof. Multiplying Jeffrey's equation (9) by u, with respect to the L^2 inner product $(.,.)$, we easily get

$$\frac{1}{2}\frac{d}{dt}\|u\|^2_{L^2} = -\frac{k_1}{\gamma}\|\frac{\partial u}{\partial x}\|^2_{L^2} - \frac{k_2}{\gamma\tau}(\int_0^t e^{-\frac{t-s}{\tau}}\frac{\partial u}{\partial x}(.,s)\,ds, \frac{\partial u}{\partial x}). \tag{12}$$

As

$$(\int_0^t e^{-\frac{t-s}{\tau}}\frac{\partial u}{\partial x}(.,s)\,ds, \frac{\partial u}{\partial x}) = \frac{1}{2}\frac{d}{dt}\|\int_0^t e^{-\frac{t-s}{\tau}}\frac{\partial u}{\partial x}(.,s)\,ds\|^2_{L^2} + \frac{1}{\tau}\|\int_0^t e^{-\frac{t-s}{\tau}}\frac{\partial u}{\partial x}(.,s)\,ds\|^2_{L^2},$$

we deduce from (12) the differential inequality

$$\frac{d}{dt}\left(\|u\|^2_{L^2}+\frac{k_2}{\gamma\tau}\|\int_0^t e^{-\frac{t-s}{\tau}}\frac{\partial u}{\partial x}(.,s)\,ds\|^2_{L^2}\right)$$
$$\leq 2\max\{-\frac{k_1}{\gamma a^2},-\frac{1}{\tau}\}\left(\|u\|^2_{L^2}+\frac{k_2}{\gamma\tau}\|\int_0^t e^{-\frac{t-s}{\tau}}\frac{\partial u}{\partial x}(.,s)\,ds\|^2_{L^2}\right) \tag{13}$$

which allow us to obtain (11).

Remark 1. Let us compare estimate (11) with estimates that can be established for heat equation (2) or Cattaneo's equation (4). From Proposition 1 for Cattaneo's model (4) holds the estimate

$$\|u\|^2_{L^2}+\frac{k}{\gamma\tau}\|\int_0^t e^{-\frac{t-s}{\tau}}\frac{\partial u}{\partial x}(.,s)\,ds\|^2_{L^2}\leq\|u_0\|^2_{L^2}.$$

Comparing Jeffrey's model with Cattaneo's model we conclude that for the first model

$$\|u\|^2_{L^2}+\frac{k_2}{\gamma\tau}\|\int_0^t e^{-\frac{t-s}{\tau}}\frac{\partial u}{\partial x}(,.s)\,ds\|^2_{L^2}\to 0$$

as $t\to\infty$, while for the second model the energy is bounded by $\|u_0\|_{L^2}$.

As for the classical heat model holds the estimate

$$\|u\|_{L^2}\leq e^{-\frac{k}{\gamma a^2}t}\|u_0\|_{L^2}$$

we conclude that $\|u\|_{L^2}\to 0$ as $t\to\infty$. No further information on the behavior of the "average" in time of the gradient of u can be obtained for this equation.

Remark 2. Let us now consider the stability of the previous models. Let $\tilde{u}$ be the solution corresponding to the initial condition $\tilde{u}_0$. For Jeffrey's equation (8) and Cattaneo's equation (4) hold respectively the following estimates

$$\|u-\tilde{u}\|^2_{L^2}+\frac{k_2}{\gamma\tau}\|\int_0^t e^{-\frac{t-s}{\tau}}\frac{\partial}{\partial x}(u-\tilde{u})\,ds\|^2_{L^2}\leq e^{-2\min\{\frac{k_1}{\gamma a^2},\frac{1}{\tau}\}t}\|u_0-\tilde{u}_0\|^2_{L^2},$$

and

$$\|u-\tilde{u}\|^2_{L^2}+\frac{k}{\gamma\tau}\|\int_0^t e^{-\frac{t-s}{\tau}}\frac{\partial}{\partial x}(u-\tilde{u})\,ds\|^2_{L^2}\leq\|u_0-\tilde{u}_0\|^2_{L^2}. \tag{14}$$

Then for the first model the L_2 norm of the perturbation of u and of an "average" in time of its gradient decreases as t increases going to zero as $t\to+\infty$; for the second model the quantity defined in the first member of (14) is bounded by the perturbation in the initial state. As for the classical heat model (2) holds

$$\|u-\tilde{u}\|_{L^2}\leq e^{-\frac{k}{\gamma a^2}t}\|u_0-\tilde{u}_0\|_{L^2}$$

we conclude that $\|u-\tilde{u}\|_{L^2}\to 0$, as $t\to+\infty$.

3. Numerical Discretizations of Jeffreys's Equation

3.1. Non-splitting Methods

In this subsection we present a simple first order numerical method for solving the integro-differential equation (9).

We consider a spatial uniform grid x_i such that $x_{i+1}-x_i=h$ and a uniform temporal grid t_n such that $t_{n+1}-t_n=\Delta t$. By u_j^n we denote a numerical approximation of $u(x_j,t_n)$.

Let us consider equation (9) at (x_j, t_n). Using the trapezoidal rule in the discretization of the integral term and discretizing the partial derivative with respect to t with backward differences and the partial derivative with respect to the space variable with second order centered differences we obtain

$$\frac{u_j^n - u_j^{n-1}}{\Delta t} = \frac{k_1}{\gamma} D_{2,x} u_j^{n-1} + \frac{k_2 \Delta t}{2\gamma\tau}\left(e^{-\frac{t_{n-1}}{\tau}} D_{2,x} u_j^0 + 2\sum_{\ell=1}^{n-2} e^{-\frac{t_{n-1}-t_\ell}{\tau}} D_{2,x} u_j^\ell + D_{2,x} u_j^{n-1}\right), \tag{15}$$

where $j = 1, \ldots, N-1$, and

$$D_{2,x} v_j = \frac{1}{h^2}(v_{j+1} - 2v_j + v_{j-1}).$$

In order to reduce the computational effort due to the use of the trapezoidal rule in the discretization of the memory term we rewrite method (15) as a three stage method

$$\begin{aligned} u_j^{n+1} &= \left(e^{-\frac{\Delta t}{\tau}} + 1\right) u_j^n + \Delta t \left(\frac{k_1}{\gamma} + \frac{k_2 \Delta t}{2\gamma\tau}\right) D_{2,x} u_j^n \\ &+ \Delta t e^{-\frac{\Delta t}{\tau}} \left(-\frac{k_1}{\gamma} + \frac{k_2 \Delta t}{2\gamma\tau}\right) D_{2,x} u_j^{n-1} - e^{-\frac{\Delta t}{\tau}} u_j^{n-1}. \end{aligned} \tag{16}$$

The numerical solution obtained with method (16) is plotted in Figure 3 and experimentally we observed an instable behavior of this method for reasonable values of stepsizes h and Δt. This behavior is illustrated in Figure 4. To overcome this drawback we study in Section 3.2 a splitting method.

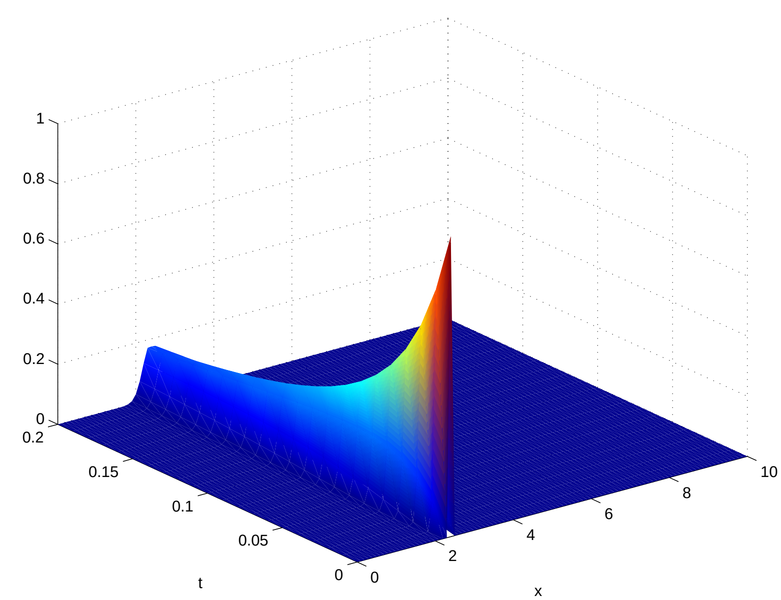

Figure 3: Numerical solution of Jeffrey's equation obtained using method (15) with $\frac{k_i}{\gamma} = 0.1, \tau = 1, h = 0.1, \Delta t = 0.03$.

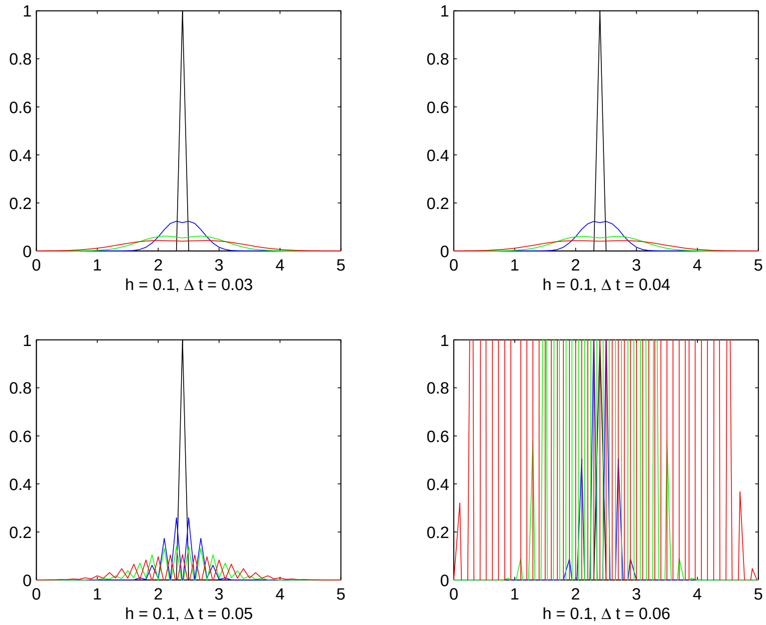

Figure 4: Stability behavior of method (15) with $\dfrac{k_i}{\gamma} = 0.1, i = 1, 2, \tau = 1, h = 0.1$ at $T = 2$.

3.2. Splitting Methods

3.2.1. The Functional Splitting

In this section we present a discretized procedure based on a functional splitting suggested by the decomposition of Jeffrey's heat flux (7) into two parts: Fourier's heat flux (1) and modified heat flux (3) where the first one updated by the second one. This assumption is equivalent to consider the IBVP (9) in the interval $[t, t + \Delta t]$ splitted into two following subproblems:

1.
$$\begin{cases} \dfrac{\partial v_1}{\partial t}(x,t) = \dfrac{k_1}{\gamma}(x,t)\dfrac{\partial^2 v_1}{\partial x^2}(x,t)\,, x \in (0,a)\, t \in (t, t+\Delta t]\,, \\ \\ v_1(x,t) = u(x,t)\,, x \in (0,a), \end{cases} \tag{17}$$

2.
$$\begin{cases} \dfrac{\partial v_2}{\partial t}(x,t) = \dfrac{k_2}{\gamma\tau}\displaystyle\int_0^t e^{-\frac{t-s}{\tau}}\dfrac{\partial^2 u}{\partial x^2}(x,s)\,ds\,, x \in (0,a),\ t \in (t, t+\Delta t], \\ \\ v_2(x,t) = v_1(x, t+\Delta t)\,, x \in (0,a). \end{cases} \tag{18}$$

Then the temperature $u(x, t + \Delta t)$ is approximated by $v_2(x, t + \Delta t)$.

In order to replace the integro-differential equation in (18) by an equivalent partial differential equation, we remark that it is equivalent to the telegraph equation

$$\frac{\partial^2 v_2}{\partial t^2} + \frac{1}{\tau}\frac{\partial v_2}{\partial t} = \frac{k_2}{\gamma\tau}\frac{\partial^2 v_2}{\partial x^2}.$$

This last assertion follows immediately for Proposition 2.

Proposition 2. *Let u be the solution of (9) with initial condition $u(x,0) = u_0(x), x \in (0,a)$, and v be the solution of*

$$\frac{\partial^2 u}{\partial t^2} + \frac{1}{\tau}\frac{\partial u}{\partial t} = \frac{k_1}{\gamma}\frac{\partial^3 u}{\partial t \partial x^2} + \frac{k}{\gamma\tau}\frac{\partial^2 u}{\partial x^2}\,. \tag{19}$$

with initial conditions

$$\begin{cases} \dfrac{\partial v}{\partial t}(x,0) = f(x),\ x \in (0,a), \\ \\ v(x,0) = v_0(x),\ x \in (0,a)\,. \end{cases}$$

Then $u = v$ if and only if $u_0 = v_0$ and $f = \frac{k_1}{\gamma}u_0^{''}$.

Proof. Let u be a solution of (9). Differentiating (9) with respect to time we obtain

$$\frac{\partial^2 u}{\partial t^2}(x,t) = \frac{k_1}{\gamma}\frac{\partial^3 u}{\partial t \partial x^2}(x,t) - \frac{k_2}{\gamma\tau^2}\int_0^t e^{-\frac{t-s}{\tau}}\frac{\partial^2 u}{\partial x^2}\,ds + \frac{k_2}{\gamma\tau}\frac{\partial^2 u}{\partial x^2}(x,t)\,, \tag{20}$$

which allow us to conclude that u is solution of (19). As u satisfies (10) we have

$$\begin{cases} \dfrac{\partial u}{\partial t}(x,0) = \dfrac{k_1}{\gamma}u_0^{''}(x)\,, \\ \\ u(x,0) = u_0(x)\,. \end{cases} \tag{21}$$

Otherwise, if u a solution of (19), (21) then u also satisfies (9), (10).

Let $v_2(x,t) = v_1(x,t+\Delta t)$. We have

$$\frac{\partial v_2}{\partial t}(x,t) = \frac{k_1}{\gamma}\frac{\partial^2 v_1}{\partial x^2}(x,t+\Delta t).$$

Consequently (18) can be replaced by

$$\begin{cases} \dfrac{\partial^2 v_2}{\partial t^2}(x,t) + \dfrac{1}{\tau}\dfrac{\partial v_2}{\partial t}(x,t) = \dfrac{k_2}{\gamma\tau}\dfrac{\partial^2 v_2}{\partial x^2}(x,t)\,,\ x \in (0,a),\ t \in (t, t+\Delta t]\,, \\ \dfrac{\partial v_2}{\partial t}(x,t) = \dfrac{k_1}{\gamma}\dfrac{\partial^2 v_1}{\partial x^2}(x,t+\Delta t)\, x \in (0,a), \\ \\ v_2(x,t) = v_1(x,t+\Delta t). \end{cases} \tag{22}$$

The functional splitting (17), (22) corresponds to considering the solution of the parabolic part of Jeffrey's equation followed by the solution of its hyperbolic part. From physical point of view this splitting states that the heat transmission is viewed as the superposition of diffusion and wave propagation. In (22) the solution obtained only by diffusion is corrected and updated.

It is easy to show that, for the functional splitting (17), (22), holds the following proposition.

Proposition 3. *Let u be the solution of the integro-differential (9)and $\bar{u}$ its approximation computed using the functional splitting (17), (22). Then*

$$\|u(t+\Delta t) - \bar{u}(t+\Delta t)\|_\infty = O(\Delta t)\,. \tag{23}$$

If in (22) we assume $\dfrac{\partial v_2}{\partial t}(t) = 0$ then

$$\|u(t+\Delta t) - v_2(t+\Delta t)\|_\infty = O(\Delta t^2)\,. \tag{24}$$

3.2.2. The Splitting Method

Discretizing (17) and (22) we obtain

1.
$$\left\{\begin{array}{l} D_t v^n_{1,j} = \dfrac{k_1}{\gamma} D_{2,x} v^{n+1}_{1,j}\,, \;, \quad j = 1, \ldots, N-1, \\ \\ v^n_{1,j} = u^n_j\,, \quad j = 1, \ldots, N-1, \end{array}\right. \tag{25}$$

2.
$$\begin{array}{l} \left\{\begin{array}{l} D_t w^n_j = \dfrac{k_2}{\gamma\tau} D_{2,x} v^n_{2,j} - \dfrac{1}{\tau} w^n_j\,, \quad j = 1, \ldots, N-1, \\ \\ D_t v^n_{2,j} = w^{n+1}_j\,, \quad j = 1, \ldots, N-1, \end{array}\right. \\ \left\{\begin{array}{l} v^n_{2,j} = v^{n+1}_{1,j}, \quad j = 1, \ldots, N-1, \\ w^n_j = \dfrac{k_1}{\gamma} D_{2,x} v^{n+1}_{1,j}, \quad j = 1, \ldots, N-1. \end{array}\right. \end{array} \tag{26}$$

where $u(x_j, t^{n+1}) \simeq v^{n+1}_{2,j}$, $j = 1, \ldots, N-1$. We remark that we are considering homogeneous Dirichlet boundary conditions.

Using matrix notation, the splitting method (25), (26) has the form

1. $(I - \dfrac{k_1}{\gamma} \Delta t A_2) V^{n+1}_{1,h} = U^n_h$,
2. $W^{n+1}_h = \left(\dfrac{k_1}{\gamma} + \dfrac{\Delta t}{\tau} \dfrac{k_2 - k_1}{\gamma}\right) A_2 V^{n+1}_{1,h}$
3. $U^{n+1}_h = V^{n+1}_{1,h} + \Delta t W^{n+1}_h$,

which lead to

$$U^{n+1}_h = \left(I + \Delta t \left(\frac{k_1}{\gamma} + \frac{\Delta t}{\tau} \frac{k_2 - k_1}{\gamma}\right) A_2\right) \left(I - \frac{k_1}{\gamma} \Delta t A_2\right)^{-1} U^n_h, \tag{27}$$

where A_2 is the matrix associated with $D_{2,x}$.

Let us now proceed to a stability analysis using the L^2- discrete norm.

Proposition 4. *Let U^n_h and U^{n+1}_h be the numerical approximations at time levels n and $n+1$ defined by the splitting method (25), (26). If $\epsilon \in (0, 1)$ and Δt and h are such such that*

$$4 \frac{\Delta t}{h^2} \left(\frac{k_1}{\gamma} + \frac{\Delta t}{\tau} \frac{|k_2 - k_1|}{\gamma}\right) \leq \epsilon, \tag{28}$$

then

$$\|U^{n+1}_h\|_{L^2} \leq \frac{1 - \epsilon}{1 + \frac{k_1}{\gamma} \frac{\Delta t}{a^2}} \|U^n_h\|_{L^2}. \tag{29}$$

Proof. By v_h we denote a grid function defined on the spatial grid such that $v_h(x_0) = v_h(x_N) = 0$. As

$$\begin{array}{rcl} \|I - \dfrac{k_1}{\gamma} \dfrac{\Delta t}{\tau} A_2\|_{L^2} & = & \sup\limits_{0 \neq v_h} \dfrac{\|(I - \frac{k_1}{\gamma} \frac{\Delta t}{\tau} A_2) v_h\|_{L^2}}{\|v_h\|_{L^2}} \\ & \geq & \dfrac{|((I - \frac{k_1 \Delta t}{\gamma} D_{2,x}) v_h, v_h)_{L^2}|}{\|v_h\|^2_{L^2}} \\ & \geq & \dfrac{\|v_h\|^2_{L^2} + \frac{k_1 \Delta t}{\gamma} \sum_{i=1}^N h (D_{-x} v_h(x_i))^2}{\|v_h\|^2_{L^2}}\,, \end{array}$$

and considering the discrete Poincaré-Friedrichs inequality

$$\|v_h\|_{L^2}^2 \leq a^2 \sum_{i=1}^{N} h(D_{-x} v_h(x_i))^2,$$

we conclude

$$\|I - \frac{k_1 \Delta t}{\gamma} A_2\|_2 \geq 1 + \frac{k_1}{\gamma} \frac{\Delta t}{a^2}. \tag{30}$$

Inequality (30) enable us to conclude that

$$\| \left(I - \frac{k_1 \Delta t}{\gamma} A_2 \right)^{-1} \|_{L^2} \leq \frac{1}{1 + \frac{k_1}{\gamma} \frac{\Delta t}{a^2}}. \tag{31}$$

For symmetric matrices an upper bound to the L^2- norm can be obtained using the maximum of the absolute value of the eigenvalues. Using (28) we easily deduce

$$\| \left(I + \Delta t \left(\frac{k_1}{\gamma} + \frac{\Delta t}{\tau} \frac{k_2 - k_1}{\gamma} \right) A_2 \right) \|_{L^2} \leq 1 - \epsilon \tag{32}$$

Combining (27) with the upper bounds (31), (32) we conclude the proof.

A numerical simulation obtained using the splitting method is plotted in Figure 5. We remark that for the same values of h and Δt method (15) is unstable (see Figure 3).

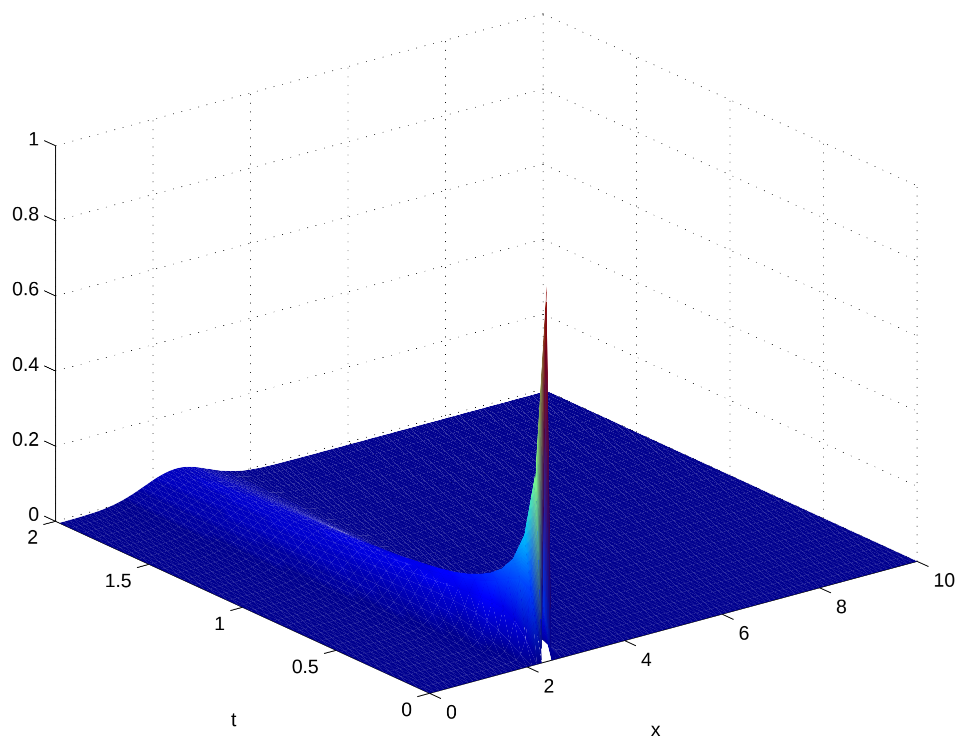

Figure 5: Numerical solutions for the solution of Jeffrey's equation obtained using the splitting method (25), (26) ($\frac{k_i}{\gamma} = 0.1, i = 1, 2, \tau = 1,\ h = 0.1, \Delta t = 0.06$).

The numerical solution obtained by the last method illustrates the theoretical behavior of the solution of Jeffrey's equation: the solution is "between" the solutions of the classical heat equation and of Cattaneo's equation.

4. Conclusion

In this paper we studied the solution of the integro-differential equation (9) that describes the heat propagation of a large class of materials. This equation is based on a generalization of Fourier law for heat flux. We analyse the qualitative properties of the theoretical solution of this problem estimating, in Proposition 1, the energy of the solution. This result enable us to conclude the stability of the IBVP (9), which has the serious drawback of imposing severe stability restrictions.

As the exact solution u of (9) can't be computed analytically, the study of two numerical methods has been carried on. We start by considering the direct discretization of the integro-differential equation - method (16).

The second method - (25), (26) - is obtained using a splitting approach. In this approach problem (9) was splitted into two subproblems taking into account the different behavior of the components of the integro-differential equation (9).

In order to avoid the discretization of the integral term presented in (18) we established the equivalence between the IBVP and an telegraph equation. The stability of the method was established with respect to the discrete L^2 norm.

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