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SUPERCONVERGENCE AND A POSTERIORI ERROR ESTIMATES FOR BOUNDARY CONTROL GOVERNED BY STOKES EQUATIONS *1)

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Dedicated to the 70th birthday of Professor Lin Qun

Abstract

In this paper, the superconvergence results are derived for a class of boundary control problems governed by Stokes equations. We derive superconvergence results for both the control and the state approximation. Base on superconvergence results, we obtain asymptotically exact a posteriori error estimates.

Mathematics subject classification: 49J20, 65N30.
Key words: Boundary control, Finite element method, Superconvergence, A posteriori error estimates.

1. Introduction

Finite element approximation of optimal control problems plays a very important role in the numerical methods for these problems. The literature in this aspect is huge. A priori error estimates of finite element approximation were established for the distributed optimal control of problems governed by partial differential equations; see, for example, [6] and [10]. A posteriori error estimates of some distributed optimal control problems has been studied, see, [18], [19], [20] and [11]. As one of important kinds of optimal control problems, the boundary control problem is widely used in scientific and engineering computing. A priori error estimates have been provided for linear boundary control problems, see [7] and [9]. A posteriori error estimates have also been obtained for boundary control problems, see [16] and [17]. In recent years, the superconvergence property of some distributed optimal control problems including the distributed optimal control problem governed by Stokes equations have been investigated, see, for example, [5], [12], [15], [24]. Although superconvergence property of finite element approximation is widely used in numerical simulations, it is not yet been utilized in boundary control problems.

In this work, we present the superconvergence analysis and a posteriori error estimates for the finite element approximation of the boundary control problems governed by Stokes equations. The purpose of this work is to derive superconvergence results of the control and the state for the boundary control problems governed by Stokes equations. Based on superconvergence results, we obtain asymptotically exact a posteriori error estimates. The obtained error estimates can then be used as a posteriori error indicators to construct reliable adaptive finite element methods.

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The outline of this paper is as follows: In section 2, we shall give a weak formula for the boundary control problem and then discuss the finite element approximation of the control problem. In section 3, a superconvergence result for the control $\mathbf{u}$ is derived by applying patch recovery operator and the supercovergence analysis technique. In section 4, recovery and superconvergence for state and co-state are derived by using L^2 projection methods. In section 5, recovery type a posteriori error estimates are derived. In the last section, we discuss briefly some possible future work.

Let Ω be a bounded open set in $\mathbb{R}^2$ with Lipschitz boundaries $\partial\Omega$. In this paper we adopt the standard notation $W^{m,q}(\Omega)$ for Sobolev spaces on Ω with the norm $\|.\|_{m,q,\Omega}$ and the seminorm $|.|_{m,q,\Omega}$. We shall extend these (semi)norms to vector functions whose components belong to $W^{m,q}(\Omega)$. We set $W_0^{m,q}(\Omega) \equiv \{w \in W^{m,q}(\Omega) : w|_{\partial\Omega} = 0\}$. We denote $W^{m,2}(\Omega)(W_0^{m,2}(\Omega))$ by $H^m(\Omega)(H_0^m(\Omega))$ equipped with the norm $\|.\|_{m,\Omega}$ and the seminorm $|.|_{m,\Omega}$. In addition, c or C denotes a general positive constant independent of h.

2. Finite Element Approximation of Boundary Control Problems

In this section, we study the finite element approximation of the boundary control problems governed by Stokes equations, where the boundary control is applied on the part of the boundary, and the fixed boundary condition is given on the other part of the boundary. In the rest of the paper, let Ω be a bounded open set in $\mathbb{R}^2$ with boundaries $\partial\Omega$, where $\partial\Omega = \Gamma_a \cup \Gamma_b$, $\Gamma_a \cap \Gamma_b = \emptyset$, $\text{meas}(\Gamma_a) > 0$ and $\text{meas}(\Gamma_b) > 0$. We shall take the state space $\mathbf{V} = \{\mathbf{v} \in (H^1(\Omega))^2 : \mathbf{v}|_{\Gamma_a} = 0\}$, the control space $\mathbf{U} = (L^2(\Gamma_b))^2$, and the observation space $\mathbf{Y} = (L^2(\Gamma_b))^2$. Let B be a linear continuous operator from $\mathbf{U}$ to $\mathbf{U}$. Let $\mathbf{K} = \{\mathbf{u} \in \mathbf{U} : \mathbf{u} \geq 0\}$. We are interested in the following boundary control problem: Give $\mathbf{f}$, $\mathbf{y}_d$, $\mathbf{z}_b$, find $(\mathbf{y},\mathbf{u}) \in \mathbf{V}\times\mathbf{K}$ such that

$$\begin{aligned} \min_{\mathbf{u}\in\mathbf{K}}\{\frac{1}{2}\|\mathbf{y}-\mathbf{y}_d\|^2_{L^2(\Gamma_b)} + \frac{1}{2}\|\mathbf{u}\|^2_{L^2(\Gamma_b)}\}, \\ -\triangle\mathbf{y} + \nabla r = \mathbf{f} \quad \text{in } \Omega, \\ \text{div}\mathbf{y} = 0 \quad \text{in } \Omega, \\ \mathbf{y} = 0 \quad \text{on } \Gamma_a, \\ \frac{\partial\mathbf{y}}{\partial\mathbf{n}} - r\mathbf{n} = \mathbf{z}_b + B\mathbf{u} \quad \text{on } \Gamma_b, \end{aligned} \tag{2.1}$$

where $\mathbf{f} \in (L^2(\Omega))^2$, $\mathbf{y}_d, \mathbf{z}_b \in (L^2(\Gamma_b))^2$. Let $\mathbf{H} = (L^2(\Omega))^2$, $Q = L^2(\Omega)$, and let

$$\begin{aligned} a(\mathbf{y},\mathbf{w}) &= \int_\Omega \nabla\mathbf{y}\cdot\nabla\mathbf{w}, \quad \forall \mathbf{y},\mathbf{w} \in \mathbf{V}, \\ b(\mathbf{v},r) &= \int_\Omega r\text{div}\mathbf{v} \quad \forall(\mathbf{v},r) \in \mathbf{V}\times Q, \\ (\mathbf{f},\mathbf{w}) &= \int_\Omega \mathbf{f}\mathbf{w}, \quad \forall(\mathbf{f},\mathbf{w}) \in \mathbf{H}\times\mathbf{V}, \\ (\mathbf{u},\mathbf{v})_{\mathbf{U}} &= \int_{\Gamma_b} \mathbf{u}\mathbf{v}, \quad \forall \mathbf{u},\mathbf{v} \in \mathbf{U}. \end{aligned}$$

Then the weak formula of the state equation reads: find $(\mathbf{y}(\mathbf{u}), r(\mathbf{u})) \in \mathbf{V}\times Q$ such that

$$\begin{aligned} a(\mathbf{y}(\mathbf{u}),\mathbf{w}) - b(\mathbf{w},r(\mathbf{u})) = (\mathbf{f},\mathbf{w}) + (B\mathbf{u}+\mathbf{z}_b,\mathbf{w})_{\mathbf{U}} \quad \forall\mathbf{w}\in\mathbf{V}. \\ b(\mathbf{y}(\mathbf{u}),\phi) = 0 \quad \forall\phi\in Q. \end{aligned}$$

For the above problem, it is well known that the following Babuška-Brezzi condition holds (see [8]):

$$\sup_{\mathbf{v}\in\mathbf{V}} \frac{b(\mathbf{v},q)}{\|\mathbf{v}\|_{1,\Omega}} \geq C\|q\|_{0,\Omega} \quad \forall q \in Q,$$

where C is a constant independent of $\mathbf{v}$ and q.

Using the weak formula, our boundary control problem can be restated as the following (BCP):

$$\min_{\mathbf{u}\in\mathbf{K}\subset\mathbf{U}}\{\frac{1}{2}\|\mathbf{y}-\mathbf{y}_d\|^2_{L^2(\Gamma_b)}+\frac{1}{2}\|\mathbf{u}\|^2_{L^2(\Gamma_b)}\},$$
$$a(\mathbf{y}(\mathbf{u}),\mathbf{w})-b(\mathbf{w},r(\mathbf{u}))=(\mathbf{f},\mathbf{w})+(\mathbf{z}_b+B\mathbf{u},\mathbf{w})_{\mathbf{U}}\qquad\forall\mathbf{w}\in\mathbf{V},$$
$$b(\mathbf{y}(\mathbf{u}),\phi)=0\qquad\forall\phi\in Q.$$

It is well known (see, e.g., [13]) that the control problem (BCP) has a unique solution $(\mathbf{y},r,\mathbf{u})$ and that $(\mathbf{y},r,\mathbf{u})$ is the solution of (BCP) if and only if there is a co-state $(\mathbf{p},s)\in\mathbf{V}\times Q$ such that $(\mathbf{y},r,\mathbf{p},s,\mathbf{u})$ satisfies the following optimality conditions (BCP-OPT):

$$a(\mathbf{y},\mathbf{w})-b(\mathbf{w},r)=(\mathbf{f}+\mathbf{w})+(\mathbf{z}_b+B\mathbf{u},\mathbf{w})_{\mathbf{U}}\qquad\forall\mathbf{w}\in\mathbf{V},\tag{2.2}$$
$$b(\mathbf{y},\phi)=0\qquad\forall\phi\in Q,\tag{2.3}$$
$$a(\mathbf{q},\mathbf{p})+b(\mathbf{q},s)=(\mathbf{y}-\mathbf{y}_d,\mathbf{q})_{\mathbf{U}}\qquad\forall\mathbf{q}\in\mathbf{V},\tag{2.4}$$
$$b(\mathbf{p},\psi)=0\qquad\forall\psi\in Q,\tag{2.5}$$
$$(\mathbf{u}+B^*\mathbf{p},\mathbf{v}-\mathbf{u})_{\mathbf{U}}\geq 0\qquad\forall\mathbf{v}\in\mathbf{K}\subset\mathbf{U}.\tag{2.6}$$

where B^* is the adjoint operator of B.

Let us consider the finite element approximation of the control problem (BCP).

Let Ω^h be a polygonal approximation to Ω with boundary $\partial\Omega^h$. Let T^h be a partitioning of Ω^h into disjoint regular triangles τ, so that $\bar{\Omega}^h=\cup_{\tau\in T^h}\bar{\tau}$. Let h_τ denote the maximum diameter of the element τ in T^h, Let $h=\max_{\tau\in T^h}h_\tau$. We assume that Ω is a convex polygon so that $\Omega=\Omega^h$.

Let $\mathbf{V}^h\subset\mathbf{V}$ and $Q^h\subset L^2(\Omega)$ be two finite element spaces for velocity and pressure, respectively, associated with the partition T^h. Let P_r be the set of polynomial of degree no more than r with $r\geq 0$. Assume that the polynomial space in the construction of $\mathbf{V}^h$ contains P_1, and that of Q^h contains P_0. The two finite element spaces $\mathbf{V}^h$ and Q^h are assumed to satisfy the following properties:

Property P1. (Approximation property of $\mathbf{V}^h$): For all $\mathbf{y}\in(H^{m+1}(\Omega))^2$,

$$\inf_{\mathbf{v}\in\mathbf{V}^h}(\|\mathbf{y}-\mathbf{v}\|_{0,\Omega}+h\|\mathbf{y}-\mathbf{v}\|_{1,\Omega})\leq Ch^{m+1}\|\mathbf{y}\|_{m+1,\Omega},\quad m=0,1.$$

Property P2. (Approximation property of Q^h): For all $r\in H^m(\Omega)$,

$$\inf_{q\in Q^h}\|r-q\|_{0,\Omega}\leq Ch^m\|r\|_{m,\Omega},\quad m=0,1.$$

Property P3. (Uniform Babuška-Brezzi condition):

$$\sup_{\mathbf{v}\in\mathbf{V}^h}\frac{b(\mathbf{v},q)}{\|\mathbf{v}\|_{1,\Omega}}\geq C\|q\|_{0,\Omega},\quad\forall q\in Q^h.$$

These above assumptions are satisfied by some finite elements, e.g., the mini element and the Bernardi-Raugel element of order one.

Let $T^h_{\mathbf{U}}$ be a partitioning of Γ_b, so that $\Gamma_b=\cup_{s\in T^h_U}\bar{s}$. $\bar{s}$ and $\bar{s}'$ have at most one common vertex if $\bar{s}$ and $\bar{s}'\in T^h_{\mathbf{U}}$. Associated with $T^h_{\mathbf{U}}$ is another finite dimensional subspace $W^h_{\mathbf{U}}$ of $L^2(\Gamma_b)$, such that $\chi|_s$ are polynomial of order zero for all $\chi|_s\in W^h_{\mathbf{U}}$ and $s\in T^h_{\mathbf{U}}$. Here there is no requirement for the continuity. Let $\mathbf{U}^h=(W^h_{\mathbf{U}})^2$. Let h_s denote the length of the element s in $T^h_{\mathbf{U}}$, Let $h_{\mathbf{U}}=\max_{s\in T^h_{\mathbf{U}}}\{h_s\}$.

Let

$$\mathbf{K}^h=\{\mathbf{v}_h\in\mathbf{U}^h:\mathbf{v}_h\geq 0\}.$$

Then a possible finite element approximation of (BCP), which we shall label $(\mathrm{BCP})^h$, is as follows:

$$\min_{\mathbf{u}_h\in\mathbf{K}^h\subset\mathbf{U}^h}\{\frac{1}{2}\|\mathbf{y}_h-\mathbf{y}_d\|^2_{L^2(\Gamma_b)}+\frac{1}{2}\|\mathbf{u}_h\|^2_{L^2(\Gamma_b)}\},$$

$$a(\mathbf{y}_h,\mathbf{w}_h)-b(\mathbf{w}_h,r_h)=(\mathbf{f},\mathbf{w}_h)+(\mathbf{z}_b+B\mathbf{u}_h,\mathbf{w}_h)_{\mathbf{U}}\quad\forall\mathbf{w}_h\in\mathbf{V}^h,$$

$$b(\mathbf{y}_h,\phi_h)=0\quad\forall\phi_h\in Q^h,$$

where $\mathbf{K}^h$ is a closed convex set in $\mathbf{U}^h$.

It follows that the control problem (BCP)^h has a solution $(\mathbf{y}_h,r_h,u_h)$ and that if a pair $(\mathbf{y}_h,r_h,\mathbf{u}_h)\in\mathbf{V}^h\times Q^h\times\mathbf{U}^h$ is the solution of (BCP)^h, then there is a co-state $(\mathbf{p}_h,s_h)\in\mathbf{V}^h\times Q^h$ such that $(\mathbf{y}_h,r_h,\mathbf{p}_h,s_h,\mathbf{u}_h)$ satisfies the following optimality conditions, which we shall label (BCP-OPT)^h:

$$a(\mathbf{y}_h,\mathbf{w}_h)-b(\mathbf{w}_h,r_h)=(\mathbf{f}+\mathbf{w}_h)+(\mathbf{z}_b+B\mathbf{u}_h,\mathbf{w}_h)_{\mathbf{U}}\qquad\forall\mathbf{w}_h\in\mathbf{V}^h,\tag{2.7}$$

$$b(\mathbf{y}_h,\phi_h)=0\qquad\forall\phi_h\in Q^h,\tag{2.8}$$

$$a(\mathbf{q}_h,\mathbf{p}_h)+b(\mathbf{q}_h,s_h)=(\mathbf{y}_h-\mathbf{y}_d,\mathbf{q}_h)_{\mathbf{U}}\qquad\forall\mathbf{q}_h\in\mathbf{V}^h,\tag{2.9}$$

$$b(\mathbf{p}_h,\psi_h)=0\qquad\forall\psi_h\in Q^h,\tag{2.10}$$

$$(\mathbf{u}_h+B^*\mathbf{p}_h,\mathbf{v}_h-\mathbf{u}_h)_{\mathbf{U}}\geq 0\qquad\forall\mathbf{v}_h\in\mathbf{K}^h\subset\mathbf{U}^h.\tag{2.11}$$

It is well known that for the problem (2.2)-(2.6) and its finite element approximation (2.7)-(2.11), the following error estimate holds:

$$e\equiv\|\mathbf{u}-\mathbf{u}_h\|_{0,\Gamma_b}+\|\mathbf{y}-\mathbf{y}_h\|_{1,\Omega}+\|r-r_h\|_{0,\Omega}+\|\mathbf{p}-\mathbf{p}_h\|_{1,\Omega}+\|s-s_h\|_{0,\Omega}\leq C(h+h_{\mathbf{U}}),\tag{2.12}$$

if $\mathbf{y},\mathbf{p}\in(H^2(\Omega))^2$ and $r,s\in H^1(\Omega)$, $\mathbf{u}\in(H^1(\Gamma_b))^2$.

Assume that Stokes equations have H^k-regularity $(k=1,2)$ in the sense that for any give $\mathbf{f}\in(H^{k-2}(\Omega))^2$, $g\in H^{k-1}(\Omega)$ and $\mathbf{z}_b\in(H^{k-\frac{3}{2}}(\Gamma_b))^2$, the following problem

$$a(\mathbf{y},\mathbf{w})-b(\mathbf{w},r)=(\mathbf{f},\mathbf{w})+(\mathbf{z}_b,\mathbf{w})_{\mathbf{U}}\quad\forall\mathbf{w}\in\mathbf{V}.\tag{2.13}$$

$$b(\mathbf{y},\phi)=(g,\phi)\quad\forall\phi\in Q.\tag{2.14}$$

has a unique solution $\mathbf{y}\in\mathbf{V}\cap(H^k(\Omega))^2$ and $r\in H^{k-1}(\Omega)$ such that

$$\|\mathbf{y}\|_{k,\Omega}+\|r\|_{k-1,\Omega}\leq C(\|\mathbf{f}\|_{k-2,\Omega}+\|g\|_{k-1,\Omega}+\|\mathbf{z}_b\|_{k-\frac{3}{2},\Gamma_b}).\tag{2.15}$$

The following lemmas are important in deriving superconvergence results.

Lemma 2.1. *(see [1]) Assume that Ω is a convex polygon. Then,*

$$\|\gamma_0 v\|_{0,\partial\Omega}\leq C(\Omega)\|v\|^{\frac{1}{2}}_{0,\Omega}\cdot\|v\|^{\frac{1}{2}}_{1,\Omega}\quad\forall v\in H^1(\Omega),$$

where γ_0 is the trace operator, $C(\Omega)$ is the constant dependent on Ω.

Lemma 2.2. *(see [8]) Let Ω be a convex polygon and let $p\geq 1$ and $s\geq 0$ be two real numbers, $s-\frac{1}{p}=l+\sigma$ where $l\geq 0$ is an integer and $0<\sigma<1$. Then*

$$\|\gamma_0 v\|_{s-1,p,\partial\Omega}\leq\|\gamma_0 v\|_{s-\frac{1}{p},p,\partial\Omega}\leq C\|v\|_{s,p,\Omega}.$$

Lemma 2.3. *(see [8]) Under hypotheses* **P1**, **P2** *and* **P3**, *assume that Stokes equations (2.13)-(2.14) have H^2-regularity . Let $(\mathbf{\Psi},\rho)$ be the solution of Stokes equations (2.13)-(2.14), and let $\mathbf{\Psi}\in\mathbf{V}\cap(H^2(\Omega))^2$, $\rho\in H^1(\Omega)$. Then*

$$\|\mathbf{\Psi}-\mathbf{\Psi}_h\|_{0,\Omega}\leq Ch^2(\|\mathbf{\Psi}\|_{2,\Omega}+\|\rho\|_{1,\Omega}),\tag{2.16}$$

$$\|\mathbf{\Psi}-\mathbf{\Psi}_h\|_{1,\Omega}+\|\rho-\rho_h\|_{0,\Omega}\leq Ch(\|\mathbf{\Psi}\|_{2,\Omega}+\|\rho\|_{1,\Omega}),\tag{2.17}$$

where $(\mathbf{\Psi}_h,\rho_h)\in\mathbf{V}^h\times Q^h$ is the finite element approximation of $(\mathbf{\Psi},\rho)$.

3. Superconvergence Analysis and Recovery for the Control u

In this section, we will provide the superconvergence results. Firstly, let us consider the superconvergence analysis for the control $\mathbf{u}$. Let

$$\Gamma_b^+=\{\cup s:\ s\subset\Gamma_b,\ \mathbf{u}|_s>0\},$$

$$\Gamma_b^0=\{\cup s:\ s\subset\Gamma_b,\ \mathbf{u}|_s=0\},$$
$$\Gamma_b^b=\Gamma_b\setminus(\Gamma_b^+\cup\Gamma_b^0).$$

In this paper, we assume that $\mathbf{u}$ and $T_{\mathbf{U}}^h$ are regular such that $meas(\Gamma_b^b)\le Ch_{\mathbf{U}}$.

Lemma 3.1. *Let $\mathbf{u}$ and $\mathbf{u}_h$ be the solutions of (2.2)-(2.6) and (2.7)-(2.11), respectively. Assume that $\mathbf{p}\in(H^2(\Omega))^2$, $\gamma_0\mathbf{p}\in(W^{1,\infty}(\partial\Omega))^2$, and $\mathbf{u}\in(W^{1,\infty}(\Gamma_b))^2$. Let $\mathbf{u}_I\in\mathbf{K}^h$ be the L^2-project of $\mathbf{u}$, such that*

$$\mathbf{u}_I|_s=\frac{\int_s\mathbf{u}}{\int_s 1},\qquad\forall s\in T_{\mathbf{U}}^h.$$

Assume that Ω is convex. Then,

$$\|\mathbf{u}_h-\mathbf{u}_I\|_{0,\Gamma_b}\le C(h^{\frac32}+h_{\mathbf{U}}^{\frac32}).\tag{3.1}$$

Proof. Note that $\mathbf{u}_h,\mathbf{u}_I\in\mathbf{K}^h\subset\mathbf{K}$. It follows from (2.6) and (2.11) that

$$(\mathbf{u}+B^*\mathbf{p},\mathbf{u}-\mathbf{u}_h)_{\mathbf{U}}\le 0,$$

and

$$(\mathbf{u}_h+B^*\mathbf{p}_h,\mathbf{u}_h-\mathbf{u}_I)_{\mathbf{U}}\le 0.$$

It follows that

$$\begin{aligned}
&\|\mathbf{u}_h-\mathbf{u}_I\|_{0,\Gamma_b}^2=(\mathbf{u}_h-\mathbf{u}_I,\mathbf{u}_h-\mathbf{u}_I)_{\mathbf{U}}\le-(B^*\mathbf{p}_h,\mathbf{u}_h-\mathbf{u}_I)_{\mathbf{U}}-(\mathbf{u}_I,\mathbf{u}_h-\mathbf{u}_I)_{\mathbf{U}}\\
=\;&(B^*\mathbf{p},\mathbf{u}-\mathbf{u}_h)_{\mathbf{U}}+(B^*\mathbf{p},\mathbf{u}_I-\mathbf{u})_{\mathbf{U}}+(\mathbf{u}_I,\mathbf{u}_I-\mathbf{u}_h)_{\mathbf{U}}+(B^*(\mathbf{p}-\mathbf{p}_h),\mathbf{u}_h-\mathbf{u}_I)_{\mathbf{U}}\\
\le\;&-(\mathbf{u},\mathbf{u}-\mathbf{u}_h)_{\mathbf{U}}+(B^*\mathbf{p},\mathbf{u}_I-\mathbf{u})_{\mathbf{U}}+(\mathbf{u}_I,\mathbf{u}_I-\mathbf{u}_h)_{\mathbf{U}}+(B^*(\mathbf{p}-\mathbf{p}_h),\mathbf{u}_h-\mathbf{u}_I)_{\mathbf{U}}\\
=\;&(\mathbf{u}_I-\mathbf{u},\mathbf{u}_I-\mathbf{u}_h)_{\mathbf{U}}+(\mathbf{u}+B^*\mathbf{p},\mathbf{u}_I-\mathbf{u})_{\mathbf{U}}+(B^*(\mathbf{p}-\mathbf{p}(\mathbf{u}_h)),\mathbf{u}-\mathbf{u}_I)_U\\
&+(B^*(\mathbf{p}-\mathbf{p}(\mathbf{u}_h)),\mathbf{u}_h-\mathbf{u})_{\mathbf{U}}+(B^*(\mathbf{p}(\mathbf{u}_h)-\mathbf{p}_h),\mathbf{u}_h-\mathbf{u}_I)_{\mathbf{U}},
\end{aligned}\tag{3.2}$$

where $\mathbf{p}(\mathbf{u}_h)$ is the solution of the auxiliary equation:

$$a(\mathbf{y}(\mathbf{u}_h),\mathbf{w})-b(\mathbf{w},r(\mathbf{u}_h))=(\mathbf{f},\mathbf{w})+(\mathbf{z}_b+B\mathbf{u}_h,\mathbf{w})_{\mathbf{U}}\qquad\forall\mathbf{w}\in\mathbf{V},\tag{3.3}$$
$$b(\mathbf{y}(\mathbf{u}_h),\phi)=0\qquad\forall\phi\in Q,\tag{3.4}$$
$$a(\mathbf{q},\mathbf{p}(\mathbf{u}_h))+b(\mathbf{q},s(\mathbf{u}_h))=(\mathbf{y}(\mathbf{u}_h)-\mathbf{y}_d,\mathbf{q})_{\mathbf{U}}\qquad\forall\mathbf{q}\in\mathbf{V},\tag{3.5}$$
$$b(\mathbf{p}(\mathbf{u}_h),\psi)=0\qquad\forall\psi\in Q.\tag{3.6}$$

It is clear that for any given $\mathbf{u}_h\in\mathbf{K}^h$, this system has an unique solution.

Let π^c be the integral average operator such that $\pi^c\mathbf{u}=\mathbf{u}_I$. It follows from the definition of $\mathbf{u}_I$ that

$$(\mathbf{u}_I-\mathbf{u},\mathbf{u}_I-\mathbf{u}_h)_{\mathbf{U}}=\sum_s(\mathbf{u}_I-\mathbf{u}_h)\int_s(\pi^c\mathbf{u}-\mathbf{u})=0,\tag{3.7}$$

and

$$\begin{aligned}
(B^*(\mathbf{p}-\mathbf{p}(\mathbf{u}_h)),\mathbf{u}-\mathbf{u}_I)_{\mathbf{U}}&=\sum_s\int_s\Big(B^*(\mathbf{p}-\mathbf{p}(\mathbf{u}_h))-\pi^c(B^*(\mathbf{p}-\mathbf{p}(\mathbf{u}_h)))\Big)(\mathbf{u}-\pi^c\mathbf{u})\\
&\le C\sum_s h_s^2|B^*(\mathbf{p}-\mathbf{p}(\mathbf{u}_h))|_{1,s}|\mathbf{u}|_{1,s}\le Ch_{\mathbf{U}}^2\|\mathbf{p}-\mathbf{p}(\mathbf{u}_h)\|_{1,\partial\Omega}\|\mathbf{u}\|_{1,\Gamma_b}.
\end{aligned}\tag{3.8}$$

Using Lemma 2.2, (2.2)-(2.5) and (3.3)-(3.6), we have that

$$\begin{aligned}
\|\mathbf{p}-\mathbf{p}(\mathbf{u}_h)\|_{1,\partial\Omega}&\le C\|\mathbf{p}-\mathbf{p}(\mathbf{u}_h)\|_{2,\Omega}\le C\|\mathbf{y}-\mathbf{y}(\mathbf{u}_h)\|_{\frac12,\partial\Omega}\le C\|\mathbf{y}-\mathbf{y}(\mathbf{u}_h)\|_{1,\Omega}\\
&\le C\|B(\mathbf{u}-\mathbf{u}_h)\|_{0,\Gamma_b}\le C\|\mathbf{u}-\mathbf{u}_h\|_{0,\Gamma_b}\\
&\le C\|\mathbf{u}-\mathbf{u}_I\|_{0,\Gamma_b}+C\|\mathbf{u}_I-\mathbf{u}_h\|_{0,\Gamma_b}\\
&\le Ch_{\mathbf{U}}|\mathbf{u}|_{1,\Gamma_b}+C\|\mathbf{u}_I-\mathbf{u}_h\|_{0,\Gamma_b}
\end{aligned}\tag{3.9}$$

and

$$(B^*(\mathbf{p}-\mathbf{p}(\mathbf{u}_h)),\mathbf{u}_h-\mathbf{u})_{\mathbf{U}} = (\mathbf{p}-\mathbf{p}(\mathbf{u}_h),B(\mathbf{u}_h-\mathbf{u}))_{\mathbf{U}} = a(\mathbf{y}(\mathbf{u}_h)-\mathbf{y},\mathbf{p}-\mathbf{p}(\mathbf{u}_h)) = (\mathbf{y}-\mathbf{y}(\mathbf{u}_h),\mathbf{y}(\mathbf{u}_h)-\mathbf{y})_{\mathbf{U}} \le 0. \quad (3.10)$$

It follows from Schwarz inequality that

$$(B^*(\mathbf{p}(\mathbf{u}_h)-\mathbf{p}_h),\mathbf{u}_h-\mathbf{u}_I)_{\mathbf{U}} \le C\|B^*(\mathbf{p}(\mathbf{u}_h)-\mathbf{p}_h)\|_{0,\partial\Omega}\|\mathbf{u}_h-\mathbf{u}_I\|_{0,\Gamma_b} \le C\|\mathbf{p}(\mathbf{u}_h)-\mathbf{p}_h\|^2_{0,\partial\Omega}+C\delta\|\mathbf{u}_h-\mathbf{u}_I\|^2_{0,\Gamma_b}, \quad (3.11)$$

where δ is an arbitrary small positive constant. Then, it follows from (3.7)-(3.11) that

$$\begin{aligned}\|\mathbf{u}_h-\mathbf{u}_I\|^2_{0,\Gamma_b} &\le (\mathbf{u}+B^*\mathbf{p},\mathbf{u}_I-\mathbf{u})_{\mathbf{U}}+Ch^2_{\mathbf{U}}\|\mathbf{u}\|_{1,\Gamma_b}(h_{\mathbf{U}}\|\mathbf{u}\|_{1,\Gamma_b}+\|\mathbf{u}_I-\mathbf{u}_h\|_{0,\Gamma_b})\\ &\quad +C\|\mathbf{p}(\mathbf{u}_h)-\mathbf{p}_h\|^2_{0,\partial\Omega}+C\delta\|\mathbf{u}_I-\mathbf{u}_h\|^2_{0,\Gamma_b}\\ &\le (\mathbf{u}+B^*\mathbf{p},\mathbf{u}_I-\mathbf{u})_{\mathbf{U}}+Ch^3_{\mathbf{U}}\|\mathbf{u}\|^2_{1,\Gamma_b}+C\|\mathbf{p}(\mathbf{u}_h)-\mathbf{p}_h\|^2_{0,\partial\Omega}\\ &\quad +C\delta\|\mathbf{u}_I-\mathbf{u}_h\|^2_{0,\Gamma_b}.\end{aligned}$$

Then we have,

$$\|\mathbf{u}_h-\mathbf{u}_I\|^2_{0,\Gamma_b} \le C(\mathbf{u}+B^*\mathbf{p},\mathbf{u}_I-\mathbf{u})_{\mathbf{U}}+Ch^3_{\mathbf{U}}\|\mathbf{u}\|^2_{1,\Gamma_b}+C\|\mathbf{p}(\mathbf{u}_h)-\mathbf{p}_h\|^2_{0,\partial\Omega}. \quad (3.12)$$

Let $\mathbf{p}(\mathbf{y}_h)\in\mathbf{V}\cap(H^2(\Omega))^2$ be the solution of the equation:

$$a(\mathbf{q},\mathbf{p}(\mathbf{y}_h))+b(\mathbf{q},s(\mathbf{y}_h)) = (\mathbf{y}_h-\mathbf{y}_d,\mathbf{q})_{\mathbf{U}} \quad \forall\mathbf{q}\in\mathbf{V}, \quad (3.13)$$

$$b(\mathbf{p}(\mathbf{y}_h),\psi)=0 \quad \forall\psi\in Q. \quad (3.14)$$

Then it follows from (3.5), (3.13) and (2.15) that

$$\|\mathbf{p}(\mathbf{y}_h)-\mathbf{p}(\mathbf{u}_h)\|_{0,\partial\Omega} \le C\|\mathbf{p}(\mathbf{y}_h)-\mathbf{p}(\mathbf{u}_h)\|_{1,\Omega} \le C\|\mathbf{y}_h-\mathbf{y}(\mathbf{u}_h)\|_{-\frac{1}{2},\Gamma_b} \le C\|\mathbf{y}_h-\mathbf{y}(\mathbf{u}_h)\|_{0,\partial\Omega}. \quad (3.15)$$

Note that $\mathbf{y}_h$ and $\mathbf{p}_h$ are the standard finite element approximations of $\mathbf{y}(\mathbf{u}_h)$ and $\mathbf{p}(\mathbf{y}_h)$, respectively. We have that (see, e.g., [3])

$$\|\mathbf{y}_h-\mathbf{y}(\mathbf{u}_h)\|_{0,\Omega}+h\|\mathbf{y}_h-\mathbf{y}(\mathbf{u}_h)\|_{1,\Omega} \le Ch^2(|\mathbf{y}(\mathbf{u}_h)|_{2,\Omega}+|r(\mathbf{u}_h)|_{1,\Omega}) \le Ch^2, \quad (3.16)$$

and

$$\|\mathbf{p}_h-\mathbf{p}(\mathbf{y}_h)\|_{0,\Omega}+h\|\mathbf{p}_h-\mathbf{p}(\mathbf{y}_h)\|_{1,\Omega} \le Ch^2(|\mathbf{p}(\mathbf{y}_h)|_{2,\Omega}+|s(\mathbf{y}_h)|_{1,\Omega}) \le Ch^2. \quad (3.17)$$

Therefore, it follows from (3.15)-(3.17) and Lemma 2.1 that

$$\begin{aligned}&\|\mathbf{p}(\mathbf{u}_h)-\mathbf{p}_h\|_{0,\partial\Omega} \le \|\mathbf{p}(\mathbf{u}_h)-\mathbf{p}(\mathbf{y}_h)\|_{0,\partial\Omega}+\|\mathbf{p}(\mathbf{y}_h)-\mathbf{p}_h\|_{0,\partial\Omega}\\ \le\ & C\|\mathbf{y}_h-\mathbf{y}(\mathbf{u}_h)\|_{0,\partial\Omega}+C\|\mathbf{p}_h-\mathbf{p}(\mathbf{y}_h)\|^{\frac{1}{2}}_{0,\Omega}\|\mathbf{p}_h-\mathbf{p}(\mathbf{y}_h)\|^{\frac{1}{2}}_{1,\Omega} \quad (3.18)\\ \le\ & C\|\mathbf{y}_h-\mathbf{y}(\mathbf{u}_h)\|^{\frac{1}{2}}_{0,\Omega}\|\mathbf{y}_h-\mathbf{y}(\mathbf{u}_h)\|^{\frac{1}{2}}_{1,\Omega}+Ch^{\frac{3}{2}}\\ \le\ & Ch^{\frac{3}{2}}.\end{aligned}$$

Note that

$$(\mathbf{u}+B^*\mathbf{p},\mathbf{u}_I-\mathbf{u})_{\mathbf{U}} = \int_{\Gamma_b^+}(\mathbf{u}+B^*\mathbf{p})(\mathbf{u}_I-\mathbf{u})+\int_{\Gamma_b^0}(\mathbf{u}+B^*\mathbf{p})(\mathbf{u}_I-\mathbf{u})+\int_{\Gamma_b^b}(\mathbf{u}+B^*\mathbf{p})(\mathbf{u}_I-\mathbf{u}),$$

and

$$(\mathbf{u}+B^*\mathbf{p})|_{\Gamma_b^+}=0, \quad (\mathbf{u}_I-\mathbf{u})|_{\Gamma_b^0}=0.$$

Then,

$$\begin{aligned}&(\mathbf{u}+B^*\mathbf{p},\mathbf{u}_I-\mathbf{u})_{\mathbf{U}} = \int_{\Gamma_b^b}(\mathbf{u}+B^*\mathbf{p})(\mathbf{u}_I-\mathbf{u})\\ =\ & \sum_{s\subset\Gamma_b^b}\int_s\Big(\mathbf{u}+B^*\mathbf{p}-\pi^c(\mathbf{u}+B^*\mathbf{p})\Big)(\pi^c\mathbf{u}-\mathbf{u}) \le C\sum_{s\subset\Gamma_b^b}h_s^2|\mathbf{u}+B^*\mathbf{p}|_{1,s}|\mathbf{u}|_{1,s}\\ \le\ & Ch^2_{\mathbf{U}}(\|\mathbf{u}\|^2_{1,\infty,\Gamma_b}+\|\mathbf{p}\|^2_{1,\infty,\partial\Omega})meas(\Gamma_b^b) \le Ch^3_{\mathbf{U}}. \quad (3.19)\end{aligned}$$

Therefore, (3.1) follows from (3.12), (3.18) and (3.19).

In order to provide the global superconvergence for the control $\mathbf{u}$, we construct the recovery operator $\mathcal{R}_h$ on the mesh $T^h_{\mathbf{U}}$. Define the set of the piecewise linear functions for control as follows:

$$U^h_{lin} = \{v \in C(\Gamma_b) : \ v \in P_1(s) \ \ \forall s \in T^h_{\mathbf{U}}\},$$

where $P_1(s)$ stands for the set of the linear functions on the element $s \in T^h_{\mathbf{U}}$. Set $\mathcal{R}_h v \in U^h_{lin}$. The values of $\mathcal{R}_h v$ on the nodes are defined by least-squares argument on an element patches surrounding the nodes as follows. Let z_i be a node, $\omega_{z_i} = \{\cup s : \ s \in T^h_{\mathbf{U}}, \ z_i \in \bar{s}\}$, V_{z_i} be the linear function space on ω_{z_i}. Set $\mathcal{R}_h v(z_i) = \sigma_{z_i}(z_i)$, where σ_{z_i} satisfies that

$$E(\sigma_{z_i}) = \min_{w \in V_{z_i}} E(w),$$

where

$$E(w) = \sum_{s \in \omega_{z_i}} \left(\int_s (w - v) \right)^2.$$

Lemma 3.2. *Assume that* $\mathbf{u} \in (W^{1,\infty}(\Gamma_b))^2$ *and* $\mathbf{u}|_{\Gamma_b^+} \in (H^2(\Gamma_b^+))^2$. *Then,*

$$\|\mathcal{R}_h \mathbf{u} - \mathbf{u}\|_{0,\Gamma_b} \le C h^{\frac{3}{2}}_{\mathbf{U}}, \tag{3.20}$$

where $\mathcal{R}_h$ *is the recovery operator defined above.*

Proof. Note that $\mathbf{u} \in (W^{1,\infty}(\Gamma_b))^2$, and $\mathbf{u} \in (H^2(\Gamma_b^+ \cup \Gamma_b^0))^2$. Let

$$\Gamma_b^{++} = \{\cup s : \ \omega_z \subset \Gamma_b^+, \forall z \in \bar{s}\}, \ \ \Gamma_b^{00} = \{\cup s : \ \omega_z \subset \Gamma_b^0, \forall z \in \bar{s}\}, \ \ \Gamma_b^{bb} = \Gamma_b \setminus (\Gamma_b^{++} \cup \Gamma_b^{00}).$$

Then,

$$\mathcal{R}_h \mathbf{u}(x) = \mathbf{u}(x) = 0 \quad \forall x \in \Gamma_b^{00}. \tag{3.21}$$

It can be proved by the standard technique (see, e.g., [3]) that

$$\|\mathcal{R}_h \mathbf{u} - \mathbf{u}\|_{0,\Gamma_b^{++}} \le C h^2_{\mathbf{U}} \|\mathbf{u}\|_{2,\Gamma_b^+}, \tag{3.22}$$

and

$$\|\mathcal{R}_h \mathbf{u} - \mathbf{u}\|^2_{0,\Gamma_b^{bb}} \le C h^2_{\mathbf{U}} \|\mathbf{u}\|^2_{1,\Gamma_b^{bb}} \le C h^2_{\mathbf{U}} \|\mathbf{u}\|^2_{1,\infty,\Gamma_b} meas(\Gamma_b^{bb}).$$

Note that $meas(\Gamma_b^b) = O(h_{\mathbf{U}})$ and hence $meas(\Gamma_b^{bb}) = O(h_{\mathbf{U}})$. We have that

$$\|\mathcal{R}_h \mathbf{u} - \mathbf{u}\|_{0,\Gamma_b^{bb}} \le C h^{\frac{3}{2}}_{\mathbf{U}}. \tag{3.23}$$

Therefore, It follows from (3.21)-(3.23) that

$$\begin{aligned} \|\mathcal{R}_h \mathbf{u} - \mathbf{u}\|^2_{0,\Gamma_b} &= \|\mathcal{R}_h \mathbf{u} - \mathbf{u}\|^2_{0,\Gamma_b^{++}} + \|\mathcal{R}_h \mathbf{u} - \mathbf{u}\|^2_{0,\Gamma_b^{00}} + \|\mathcal{R}_h \mathbf{u} - \mathbf{u}\|^2_{0,\Gamma_b^{bb}} \\ &\le C h^4_{\mathbf{U}} + 0 + C h^3_{\mathbf{U}} \le C h^3_{\mathbf{U}}. \end{aligned}$$

This proves (3.20).

Theorem 3.1. *Suppose all conditions of Lemma 3.1 and Lemma 3.2 are valid. Then,*

$$\|\mathcal{R}_h \mathbf{u}_h - \mathbf{u}\|_{0,\Gamma_b} \le C(h^{\frac{3}{2}} + h^{\frac{3}{2}}_{\mathbf{U}}). \tag{3.24}$$

Proof. Let $\mathbf{u}_I$ be defined in Lemma 3.1. Then,

$$\|\mathcal{R}_h \mathbf{u}_h - \mathbf{u}\|_{0,\Gamma_b} \le \|\mathbf{u} - \mathcal{R}_h \mathbf{u}\|_{0,\Gamma_b} + \|\mathcal{R}_h \mathbf{u} - \mathcal{R}_h \mathbf{u}_I\|_{0,\Gamma_b} + \|\mathcal{R}_h \mathbf{u}_I - \mathcal{R}_h \mathbf{u}_h\|_{0,\Gamma_b}. \tag{3.25}$$

It follows from Lemma 3.2 that

$$\|\mathbf{u} - \mathcal{R}_h \mathbf{u}\|_{0,\Gamma_b} \le C h^{\frac{3}{2}}_{\mathbf{U}}. \tag{3.26}$$

Noting the definition of $\mathcal{R}_h$, we have that

$$\mathcal{R}_h \mathbf{u} = \mathcal{R}_h \mathbf{u}_I, \tag{3.27}$$

and

$$\|\mathcal{R}_h \mathbf{u}_I - \mathcal{R}_h \mathbf{u}_h\|_{0,\Gamma_b} \le C \|\mathbf{u}_I - \mathbf{u}_h\|_{0,\Gamma_b}. \tag{3.28}$$

It has been proved in Lemma 3.1 that

$$\|\mathbf{u}_I - \mathbf{u}_h\|_{0,\Gamma_b} \le C(h^{\frac{3}{2}} + h^{\frac{3}{2}}_{\mathbf{U}}). \tag{3.29}$$

Therefore, (3.24) follows from (3.25)-(3.29).

Corollary 3.1. *Let* $\mathbf{u}$ *and* $\mathbf{u_h}$ *be the solutions of (2.2)-(2.6) and (2.7)-(2.11), respectively. Assume that* Ω *is convex. Then,*

$$\|\mathbf{u}-\mathbf{u}_h\|_{-\frac{1}{2},\Gamma_b} \le C(h^{\frac{3}{2}} + h_{\mathbf{U}}^{\frac{3}{2}}). \tag{3.30}$$

Proof. For any function $\mathbf{\Phi} \in (H^{\frac{1}{2}}(\Gamma_b))^2$, let $\mathbf{\Phi}_I \in \mathbf{U}^h$ be the L^2-project of $\mathbf{\Phi}$, such that

$$\mathbf{\Phi}_I|_s = \frac{\int_s \mathbf{\Phi}}{\int_s 1}.$$

we have

$$(\mathbf{u}-\mathbf{u}_h, \mathbf{\Phi})_{\mathbf{U}} = (\mathbf{u}-\mathbf{u}_h, \mathbf{\Phi}-\mathbf{\Phi}_I)_{\mathbf{U}} + (\mathbf{u}-\mathbf{u}_h, \mathbf{\Phi}_I)_{\mathbf{U}}. \tag{3.31}$$

Note that

$$\begin{aligned}
&(\mathbf{u}-\mathbf{u}_h, \mathbf{\Phi}-\mathbf{\Phi}_I)_{\mathbf{U}} \le \|\mathbf{u}-\mathbf{u}_h\|_{0,\Gamma_b}\|\mathbf{\Phi}-\mathbf{\Phi}_I\|_{0,\Gamma_b} \\
\le\ & C(h_{\mathbf{U}}+h)h_{\mathbf{U}}^{\frac{1}{2}}\|\mathbf{\Phi}\|_{\frac{1}{2},\Gamma_b} \le C(h^{\frac{3}{2}} + h_{\mathbf{U}}^{\frac{3}{2}})\|\mathbf{\Phi}\|_{\frac{1}{2},\Gamma_b}.
\end{aligned} \tag{3.32}$$

It follows from Lemma 3.1 that

$$(\mathbf{u}-\mathbf{u}_h, \mathbf{\Phi}_I)_{\mathbf{U}} = (\mathbf{u}_I-\mathbf{u}_h, \mathbf{\Phi})_{\mathbf{U}} \le \|\mathbf{u}_I-\mathbf{u}_h\|_{0,\Gamma_b}\|\mathbf{\Phi}\|_{0,\Gamma_b} \le C(h^{\frac{3}{2}} + h_{\mathbf{U}}^{\frac{3}{2}})\|\mathbf{\Phi}\|_{\frac{1}{2},\Gamma_b}. \tag{3.33}$$

Therefore, it follows from (3.31)-(3.33) that

$$\|\mathbf{u}-\mathbf{u}_h\|_{-\frac{1}{2},\Gamma_b} = \sup_{\mathbf{\Phi}\in H^{\frac{1}{2}}(\Gamma_b)} \frac{(\mathbf{u}-\mathbf{u}_h, \mathbf{\Phi})_{\mathbf{U}}}{\|\mathbf{\Phi}\|_{\frac{1}{2},\Gamma_b}} \le C(h^{\frac{3}{2}} + h_{\mathbf{U}}^{\frac{3}{2}}).$$

This proves the Corollary.

4. Superconvergence Analysis for State and Co-state

In this section, we will discuss the superconvergence for the state and the co-state. The technique used in this section can be find in, e.g., [23]. Firstly, we will construct the recovery on coarse meshes for the state $\mathbf{y}$, r and the co-state $\mathbf{p}$, s. Let $\mathcal{T}^{H_i}$, $i=1,2$, be another two finite element partitions with mesh sizes H_i, where $h < H_i$ $(i=1,2)$. It will be essential to our argument to allow H_i to be sufficiently large compared to h. In this paper, we construct the partition $\mathcal{T}^{H_i}$, $i=1,2$, such that they are quasi-uniform and regular (see, [3]). Let $\mathbf{V}^{H_1} \subset (L^2(\Omega))^2$ and $Q^{H_2} \subset H^1(\Omega)$ be finite element spaces consisting of piecewise polynomials of degree two and order one associated with the meshes $\mathcal{T}^{H_1}$ and $\mathcal{T}^{H_2}$, respectively. Define $\mathcal{P}_{H_1}$ and $\mathcal{P}_{H_2}$ to be the L^2 projectors from $L^2(\Omega)$ onto the finite element spaces $\mathbf{V}^{H_1}$ and Q^{H_2} respectively. Then, we can find that $\mathcal{P}_{H_1}\mathbf{y}_h$, $\mathcal{P}_{H_1}\mathbf{p}_h$, $\mathcal{P}_{H_2}r_h$, $\mathcal{P}_{H_2}s_h$ are good recovery of $\mathbf{y}_h$, $\mathbf{p}_h$, r_h and s_h.

In the following, we will provide some important lemmas, which will be useful for establishing superconvergence results.

Lemma 4.1. *Suppose the mesh* $\mathcal{T}^{H_1}$ *is quasi-uniform (see, [3]), and* $\mathbf{V}^{H_1} \subset (L^2(\Omega))^2$. *Suppose that all conditions of Lemmas 3.1 and 3.2 are valid. Let* $(\mathbf{y}, r, \mathbf{p}, s)$ *and* $(\mathbf{y}(\mathbf{u}_h), r(\mathbf{u}_h), \mathbf{p}(\mathbf{u}_h), s(\mathbf{u}_h))$ *be the solutions of (2.2)-(2.6) and (3.3)-(3.6), respectively. Assume* $(\mathbf{y}, r)$ *and* $(\mathbf{p}, s)$ *belong to* $(\mathbf{V}\cap(H^3(\Omega))^2)\times H^1(\Omega)$, *and* $(\mathbf{y}(\mathbf{u}_h),\ r(\mathbf{u}_h))$ *and* $(\mathbf{p}(\mathbf{u}_h), s(\mathbf{u}_h))$ *belong to* $(\mathbf{V}\cap(H^2(\Omega))^2)\times H^1(\Omega)$. *Let* $(\mathbf{y}_h, r_h,\ \mathbf{p}_h, s_h)$ *be the solutions of (2.7)-(2.11). Then*

$$\|\nabla(\mathbf{y}-\mathcal{P}_{H_1}\mathbf{y}_h)\|_{0,\Omega} + \|\nabla(\mathbf{p}-\mathcal{P}_{H_1}\mathbf{p}_h)\|_{0,\Omega} \le C(H_1^2 + h^2H_1^{-1} + h^{\frac{3}{2}} + h_{\mathbf{U}}^{\frac{3}{2}}). \tag{4.1}$$

Proof. We first estimate $\|\nabla(\mathbf{y}-\mathcal{P}_{H_1}\mathbf{y}_h)\|_{0,\Omega}$. It is easy to see that

$$\begin{aligned}
\|\nabla(\mathbf{y}-\mathcal{P}_{H_1}\mathbf{y}_h)\|_{0,\Omega} \le\ & \|\nabla\mathbf{y}-\nabla\mathcal{P}_{H_1}\mathbf{y}\|_{0,\Omega} + \|\nabla\mathcal{P}_{H_1}\mathbf{y}-\nabla\mathcal{P}_{H_1}\mathbf{y}(\mathbf{u}_h)\|_{0,\Omega} \\
& + \|\nabla\mathcal{P}_{H_1}\mathbf{y}(\mathbf{u}_h)-\nabla\mathcal{P}_{H_1}\mathbf{y}_h\|_{0,\Omega}.
\end{aligned} \tag{4.2}$$

It is well known (see, e.g., [2] and [21]) that

$$\|\nabla\mathbf{y}-\nabla\mathcal{P}_{H_1}\mathbf{y}\|_{0,\Omega} \le \|\mathbf{y}-\mathcal{P}_{H_1}\mathbf{y}\|_{1,\Omega} \le CH_1^2|\mathbf{y}|_{3,\Omega}. \tag{4.3}$$

Using (2.2)-(2.3) and (3.3)-(3.4), we know $\mathbf{y}-\mathbf{y}(\mathbf{u}_h)$ is the solution of the following equation:

$$a(\mathbf{y}-\mathbf{y}(\mathbf{u}_h),\mathbf{w})-b(\mathbf{w},r-r(\mathbf{u}_h))=(B(\mathbf{u}-\mathbf{u}_h),\mathbf{w})_{\mathbf{U}} \qquad \forall \mathbf{w}\in\mathbf{V}, \tag{4.4}$$
$$b(\mathbf{y}-\mathbf{y}(\mathbf{u}_h),\phi)=0 \qquad \forall\phi\in Q. \tag{4.5}$$

Hence, it follows from (2.15) with $k=1$ and Corollary 3.1 that

$$\|\mathbf{y}-\mathbf{y}(\mathbf{u}_h)\|_{1,\Omega}\le C\|B(\mathbf{u}-\mathbf{u}_h)\|_{-\frac{1}{2},\Gamma_b}\le C\|\mathbf{u}-\mathbf{u}_h\|_{-\frac{1}{2},\Gamma_b}\le C(h^{\frac{3}{2}}+h_{\mathbf{U}}^{\frac{3}{2}}). \tag{4.6}$$

Therefore, using the H^1-stability of the L^2-projection into finite element space (see, e.g., [2] and [4]), we have

$$\|\nabla\mathcal{P}_{H_1}\mathbf{y}-\nabla\mathcal{P}_{H_1}\mathbf{y}(\mathbf{u}_h)\|_{0,\Omega}\le\|\mathcal{P}_{H_1}(\mathbf{y}-\mathbf{y}(\mathbf{u}_h))\|_{1,\Omega}\le C\|\mathbf{y}-\mathbf{y}(\mathbf{u}_h)\|_{1,\Omega}\le C(h^{\frac{3}{2}}+h_{\mathbf{U}}^{\frac{3}{2}}). \tag{4.7}$$

Moreover, by the inverse inequality and (3.16), we have that

$$\|\nabla\mathcal{P}_{H_1}\mathbf{y}(\mathbf{u}_h)-\nabla\mathcal{P}_{H_1}\mathbf{y}_h\|_{0,\Omega}\le CH_1^{-1}\|\mathcal{P}_{H_1}\mathbf{y}(\mathbf{u}_h)-\mathcal{P}_{H_1}\mathbf{y}_h)\|_{0,\Omega}\le Ch^2H_1^{-1}. \tag{4.8}$$

Therefore, it follows from (4.2), (4.3), (4.7) and (4.8) that

$$\|\nabla(\mathbf{y}-\mathcal{P}_{H_1}\mathbf{y}_h)\|_{0,\Omega}\le C(H_1^2+h^2H_1^{-1}+h^{\frac{3}{2}}+h_{\mathbf{U}}^{\frac{3}{2}}). \tag{4.9}$$

Similarly, we can estimate $\|\nabla(\mathbf{p}-\mathcal{P}_{H_1}\mathbf{p}_h)\|_{0,\Omega}$. Again, we have

$$\begin{aligned}\|\nabla(\mathbf{p}-\mathcal{P}_{H_1}\mathbf{p}_h)\|_{0,\Omega} &\le \|\nabla\mathbf{p}-\nabla\mathcal{P}_{H_1}\mathbf{p}\|_{0,\Omega}+\|\nabla\mathcal{P}_{H_1}\mathbf{p}-\nabla\mathcal{P}_{H_1}\mathbf{p}(\mathbf{u}_h)\|_{0,\Omega}\\ &\quad+\|\nabla\mathcal{P}_{H_1}\mathbf{p}(\mathbf{u}_h)-\nabla\mathcal{P}_{H_1}\mathbf{p}_h\|_{0,\Omega},\end{aligned} \tag{4.10}$$

and

$$\|\nabla\mathbf{p}-\nabla\mathcal{P}_{H_1}\mathbf{p}\|_{0,\Omega}\le\|\mathbf{p}-\mathcal{P}_{H_1}\mathbf{p}\|_{1,\Omega}\le CH_1^2|\mathbf{p}|_{3,\Omega}. \tag{4.11}$$

It follows from (2.4)-(2.5) and (3.5)-(3.6) that $\mathbf{p}-\mathbf{p}(\mathbf{u}_h)$ satisfies the following equation:

$$a(\mathbf{p}-\mathbf{p}(\mathbf{u}_h),\mathbf{q})+b(\mathbf{q},s-s(\mathbf{u}_h))=(\mathbf{y}-\mathbf{y}(\mathbf{u}_h),\mathbf{q})_{\mathbf{U}} \qquad \forall\mathbf{q}\in\mathbf{V}, \tag{4.12}$$
$$b(\mathbf{p}-\mathbf{p}(\mathbf{u}_h),\psi)=0 \qquad \forall\psi\in Q. \tag{4.13}$$

Using (2.15), Lemma 2.2 and Corollary 3.1 , and noting that $\mathbf{p}-\mathbf{p}(\mathbf{u}_h)$ is the solution of (4.12)-(4.13), it can be deduced that

$$\begin{aligned}\|\mathbf{p}-\mathbf{p}(\mathbf{u}_h)\|_{1,\Omega} &\le C\|\mathbf{y}-\mathbf{y}(\mathbf{u}_h)\|_{-\frac{1}{2},\Gamma_b}\le C\|\mathbf{y}-\mathbf{y}(\mathbf{u}_h)\|_{0,\partial\Omega}\le C\|\mathbf{y}-\mathbf{y}(\mathbf{u}_h)\|_{1,\Omega}\\ &\le C\|B(\mathbf{u}-\mathbf{u}_h)\|_{-\frac{1}{2},\Gamma_b}\le C\|\mathbf{u}-\mathbf{u}_h\|_{-\frac{1}{2},\Gamma_b}\le C(h^{\frac{3}{2}}+h_{\mathbf{U}}^{\frac{3}{2}}).\end{aligned} \tag{4.14}$$

Thus,

$$\|\nabla\mathcal{P}_{H_1}\mathbf{p}-\nabla\mathcal{P}_{H_1}\mathbf{p}(\mathbf{u}_h)\|_{0,\Omega}\le\|\mathcal{P}_{H_1}(\mathbf{p}-\mathbf{p}(\mathbf{u}_h))\|_{1,\Omega}\le C\|\mathbf{p}-\mathbf{p}(\mathbf{u}_h)\|_{1,\Omega}\le C(h^{\frac{3}{2}}+h_{\mathbf{U}}^{\frac{3}{2}}). \tag{4.15}$$

By the inverse inequality and (3.18), we have that

$$\|\nabla\mathcal{P}_{H_1}\mathbf{p}(\mathbf{u}_h)-\nabla\mathcal{P}_{H_1}\mathbf{p}_h\|_{0,\Omega}\le CH_1^{-1}\|\mathcal{P}_{H_1}\mathbf{p}(\mathbf{u}_h)-\mathcal{P}_{H_1}\mathbf{p}_h\|_{0,\Omega}\le Ch^2H_1^{-1}. \tag{4.16}$$

Therefore, it follows from (4.10), (4.11), (4.15) and (4.16) that

$$\|\nabla(\mathbf{p}-\mathcal{P}_{H_1}\mathbf{p}_h)\|_{0,\Omega}\le C(H_1^2+h^2H_1^{-1}+h^{\frac{3}{2}}+h_{\mathbf{U}}^{\frac{3}{2}}). \tag{4.17}$$

Then (4.1) follows from (4.9) and (4.17), and the proof of the theorem has been completed.

Similarly, we have the following result for r and s.

Lemma 4.2. *Assume the mesh $\mathcal{T}^{H_2}$ is quasi-uniform, and $Q^{H_2}\in H^1(\Omega)$. Suppose that all conditions of Lemmas 3.1 and 3.2 are valid. Let $(\mathbf{y}_h,r_h,\mathbf{p}_h,s_h)$ and $(\mathbf{y}(\mathbf{u}_h),r(\mathbf{u}_h),\mathbf{p}(\mathbf{u}_h),s(\mathbf{u}_h))$ be the solutions of (2.7)-(2.11) and (3.3)-(3.6), respectively. Assume $(\mathbf{y}(\mathbf{u}_h),r(\mathbf{u}_h))$ and $(\mathbf{p}(\mathbf{u}_h),$ $s(\mathbf{u}_h))$ belong to $(\mathbf{V}\cap H^2(\Omega))^2\times(H^1(\Omega)$, Then*

$$\|\mathcal{P}_{H_2}r(\mathbf{u}_h)-\mathcal{P}_{H_2}r_h\|_{0,\Omega}\le Ch^2H_2^{-1}, \tag{4.18}$$
$$\|\mathcal{P}_{H_2}s(\mathbf{u}_h)-\mathcal{P}_{H_2}s_h\|_{0,\Omega}\le C(h^2H_2^{-1}+h^{\frac{3}{2}}). \tag{4.19}$$

Proof. Note that r_h is the standard finite element approximation of $r(\mathbf{u}_h)$. The definitions of $\|\cdot\|_{0,\Omega}$ and $\mathcal{P}_{H_2}$ give

$$\|\mathcal{P}_{H_2}r(\mathbf{u}_h)-\mathcal{P}_{H_2}r_h\|_{0,\Omega}=\sup_{\phi\in L^2(\Omega),\|\phi\|_{0,\Omega}=1}|(\mathcal{P}_{H_2}r(\mathbf{u}_h)-\mathcal{P}_{H_2}r_h,\phi)|,$$

and

$$(\mathcal{P}_{H_2}r(\mathbf{u}_h)-\mathcal{P}_{H_2}r_h,\phi)=(r(\mathbf{u}_h)-r_h,\mathcal{P}_{H_2}\phi).$$

Then

$$\|\mathcal{P}_{H_2}r(\mathbf{u}_h)-\mathcal{P}_{H_2}r_h\|_{0,\Omega}=\sup_{\phi\in L^2(\Omega),\|\phi\|_{0,\Omega}=1}|(r(\mathbf{u}_h)-r_h,\mathcal{P}_{H_2}\phi)|.$$

Consider the following problem that seeks $(\mathbf{\Psi},\rho)\in\mathbf{V}\times Q$ such that

$$a(\mathbf{\Psi},\mathbf{w})-b(\mathbf{w},\rho)=0 \qquad \forall\mathbf{w}\in\mathbf{V}, \tag{4.20}$$

$$b(\mathbf{\Psi},q)=(\mathcal{P}_{H_2}\phi,q) \qquad \forall q\in Q. \tag{4.21}$$

Replacing q in (4.21) by $r(\mathbf{u}_h)-r_h$ and using (2.7)-(2.11), (3.3)-(3.6) and (4.20), we have

$$\begin{aligned}
&(r(\mathbf{u}_h)-r_h,\mathcal{P}_{H_2}\phi)\\
=\;& b(\mathbf{\Psi},r(\mathbf{u}_h)-r_h)=b(\mathbf{\Psi}-\mathbf{\Psi}_I,r(\mathbf{u}_h)-r_h)+b(\mathbf{\Psi}_I,r(\mathbf{u}_h)-r_h)\\
=\;& b(\mathbf{\Psi}-\mathbf{\Psi}_I,r(\mathbf{u}_h)-r_h)+a(\mathbf{y}(\mathbf{u}_h)-\mathbf{y}_h,\mathbf{\Psi}_I)\\
=\;& b(\mathbf{\Psi}-\mathbf{\Psi}_I,r(\mathbf{u}_h)-r_h)+a(\mathbf{y}(\mathbf{u}_h)-\mathbf{y}_h,\mathbf{\Psi}_I-\mathbf{\Psi})+a(\mathbf{y}(\mathbf{u}_h)-\mathbf{y}_h,\mathbf{\Psi})\\
=\;& b(\mathbf{\Psi}-\mathbf{\Psi}_I,r(\mathbf{u}_h)-r_h)+a(\mathbf{y}(\mathbf{u}_h)-\mathbf{y}_h,\mathbf{\Psi}_I-\mathbf{\Psi})+b(\mathbf{y}(\mathbf{u}_h)-\mathbf{y}_h,\rho-\rho_I),
\end{aligned}$$

where $\Psi_I\in\mathbf{V}^h$ and $\rho_I\in Q^h$ are two arbitrary functions in finite element spaces. Using Schwarz inequality and the approximation properties P1 and P2, (2.15) and (2.17), we obtain

$$\begin{aligned}
(r(\mathbf{u}_h)-r_h,\mathcal{P}_{H_2}\phi) \;\le\;& \|\mathbf{\Psi}-\mathbf{\Psi}_I\|_{1,\Omega}\|r(\mathbf{u}_h)-r_h\|_{0,\Omega}+\|\mathbf{y}(\mathbf{u}_h)-\mathbf{y}_h\|_{1,\Omega}\|\mathbf{\Psi}_I-\mathbf{\Psi}\|_{1,\Omega}\\
&+\|\mathbf{y}(\mathbf{u}_h)-\mathbf{y}_h\|_{1,\Omega}\|\rho-\rho_I\|_{0,\Omega}\\
\le\;& Ch^2(\|\mathbf{\Psi}\|_{2,\Omega}+\|\rho\|_{1,\Omega})(\|\mathbf{y}(\mathbf{u}_h)\|_{2,\Omega}+\|r(\mathbf{u}_h)\|_{1,\Omega})\\
\le\;& Ch^2\|\mathcal{P}_{H_2}\phi\|_{1,\Omega}(\|\mathbf{y}(\mathbf{u}_h)\|_{2,\Omega}+\|r(\mathbf{u}_h)\|_{1,\Omega})\\
\le\;& Ch^2H_2^{-1}\|\phi\|_{0,\Omega}.
\end{aligned}$$

Therefore

$$\|\mathcal{P}_{H_2}r(\mathbf{u}_h)-\mathcal{P}_{H_2}r_h\|_{0,\Omega}\le Ch^2H_2^{-1}.$$

This is (4.18).

Consider the auxiliary equation (3.13)-(3.14), it is clear that s_h is the standard finite element approximation of $s(\mathbf{y}_h)$, where $s(\mathbf{y}_h)$ is the solution of (3.13)-(3.14). Then, using the similar way, we can prove that

$$\|\mathcal{P}_{H_2}s(\mathbf{y}_h)-\mathcal{P}_{H_2}s_h\|_{0,\Omega}\le Ch^2H_2^{-1}. \tag{4.22}$$

It can been shown from (3.16), (4.22) and the property of L^2 -projectors that

$$\begin{aligned}
\|\mathcal{P}_{H_2}s(\mathbf{u}_h)-\mathcal{P}_{H_2}s_h\|_{0,\Omega} \;=\;& \|\mathcal{P}_{H_2}s(\mathbf{u}_h)-\mathcal{P}_{H_2}s(\mathbf{y}_h)+\mathcal{P}_{H_2}s(\mathbf{y}_h)-\mathcal{P}_{H_2}s_h\|_{0,\Omega}\\
\le\;& \|s(\mathbf{u}_h)-s(\mathbf{y}_h)\|_{0,\Omega}+\|\mathcal{P}_{H_2}s(\mathbf{y}_h)-\mathcal{P}_{H_2}s_h\|_{0,\Omega}\\
\le\;& C\|\mathbf{y}_h-\mathbf{y}(\mathbf{u}_h)\|_{-\frac{1}{2},\Gamma_b}+Ch^2H_2^{-1}\\
\le\;& C\|\mathbf{y}_h-\mathbf{y}(\mathbf{u}_h)\|_{0,\Gamma_b}+Ch^2H_2^{-1}\\
\le\;& C\|\mathbf{y}_h-\mathbf{y}(\mathbf{u}_h)\|_{0,\Omega}^{\frac{1}{2}}\|\mathbf{y}_h-\mathbf{y}(\mathbf{u}_h)\|_{1,\Omega}^{\frac{1}{2}}+Ch^2H_2^{-1}\\
\le\;& C(h^{\frac{3}{2}}+h^2H_2^{-1}).
\end{aligned}$$

This proves (4.19).

Lemma 4.3. *Suppose that all conditions of Lemma 4.2 are valid. Let* $(\mathbf{y},r,\mathbf{p},s)$ *and* $(\mathbf{y}_h,r_h,\mathbf{p}_h,s_h)$ *be the solutions of (2.2)-(2.6) and (2.7)-(2.11), respectively. Assume* $(\mathbf{y},r)$ *and* $(\mathbf{p},s)$ *belongs to* $(\mathbf{V}\cap(H^2(\Omega))^2)\times H^2(\Omega)$*, then*

$$\|r-\mathcal{P}_{H_2}r_h\|_{0,\Omega}+\|s-\mathcal{P}_{H_2}s_h\|_{0,\Omega}\le C(H_2^2+h^2H_2^{-1}+h^{\frac{3}{2}}+h_{\mathbf{U}}^{\frac{3}{2}}). \tag{4.23}$$

Proof. First, we estimate $\|r-\mathcal{P}_{H_2}r_h\|_{0,\Omega}$. It can be show that

$$\|r-\mathcal{P}_{H_2}r_h\|_{0,\Omega}\leq\|r-\mathcal{P}_{H_2}r\|_{0,\Omega}+\|\mathcal{P}_{H_2}r-\mathcal{P}_{H_2}r(\mathbf{u}_h)\|_{0,\Omega}+\|\mathcal{P}_{H_2}r(\mathbf{u}_h)-\mathcal{P}_{H_2}r_h\|_{0,\Omega}, \quad (4.24)$$

where $r(\mathbf{u}_h)$ is defined by (3.3)-(3.6). It is well known that

$$\|r-\mathcal{P}_{H_2}r\|_{0,\Omega}\leq CH_2^2|r|_{2,\Omega}. \quad (4.25)$$

Using Corollary 3.1 and (2.15) with $k=1$, and noting that $r-r(\mathbf{u}_h)$ is an analytic solution of (4.4)-(4.5), it can be shown that

$$\begin{aligned}\|\mathcal{P}_{H_2}r-\mathcal{P}_{H_2}r(\mathbf{u}_h)\|_{0,\Omega} &\leq \|r-r(\mathbf{u}_h)\|_{0,\Omega}\leq C\|B(\mathbf{u}-\mathbf{u}_h)\|_{-\frac12,\Gamma_b}\\ &\leq C\|\mathbf{u}-\mathbf{u}_h\|_{-\frac12,\Gamma_b}\leq C(h^{\frac32}+h_{\mathbf{U}}^{\frac32}). \end{aligned} \quad (4.26)$$

Combing (4.24)-(4.26) and (4.18) gives

$$\|r-\mathcal{P}_{H_2}r_h\|_{0,\Omega}\leq C(H_2^2+h^2H_2^{-1}+h^{\frac32}+h_{\mathbf{U}}^{\frac32}). \quad (4.27)$$

Similarly, we can estimate $\|s-\mathcal{P}_{H_2}s_h\|_{0,\Omega}$. Again, it is easy to show that

$$\|s-\mathcal{P}_{H_2}s_h\|_{0,\Omega}\leq\|s-\mathcal{P}_{H_2}s\|_{0,\Omega}+\|\mathcal{P}_{H_2}s-\mathcal{P}_{H_2}s(\mathbf{u}_h)\|_{0,\Omega}+\|\mathcal{P}_{H_2}s(\mathbf{u}_h)-\mathcal{P}_{H_2}s_h\|_{0,\Omega}, \quad (4.28)$$

and

$$\|s-\mathcal{P}_{H_2}s\|_{0,\Omega}\leq CH_2^2|s|_{2,\Omega}. \quad (4.29)$$

Moreover,

$$\begin{aligned}\|\mathcal{P}_{H_2}s-\mathcal{P}_{H_2}s(\mathbf{u}_h)\|_{0,\Omega} &\leq \|s-s(\mathbf{u}_h)\|_{0,\Omega}\leq C\|\mathbf{y}-\mathbf{y}(\mathbf{u}_h)\|_{-\frac12,\Gamma_b}\\ &\leq C\|\mathbf{y}-\mathbf{y}(\mathbf{u}_h)\|_{1,\Omega}\leq C\|B(\mathbf{u}-\mathbf{u}_h)\|_{-\frac12,\Gamma_b}\\ &\leq C\|\mathbf{u}-\mathbf{u}_h\|_{-\frac12,\Gamma_b}\leq C(h^{\frac32}+h_{\mathbf{U}}^{\frac32}). \end{aligned} \quad (4.30)$$

Combing (4.28)-(4.30) and (4.19) gives

$$\|s-\mathcal{P}_{H_2}s_h\|_{0,\Omega}\leq C(H_2^2+h^2H_2^{-1}+h^{\frac32}+h_{\mathbf{U}}^{\frac32}). \quad (4.31)$$

Then (4.23) follows from (4.27) and (4.31). The theorem has been proved.

In this section, the recovery operator $\mathcal{P}_{H_1}$ (for $\mathbf{y}_h$ and $\mathbf{p}_h$) and $\mathcal{P}_{H_2}$ (for r_h and s_h) are provided. Our recovery technique (the postprocessing technique of least-squares surface fitting discussed in this paper) is to project the finite element solution to another finite element space of higher order with a different mesh. It is proved in Lemma 4.1 and 4.3 that after recovery, the error is improved to be $O(H_1^2+H_2^2+h^2H_1^{-1}+h^2H_2^{-1}+h^{\frac32}+h_{\mathbf{U}}^{\frac32})$, which is better than the original error $O(h+h_{\mathbf{U}})$, if H_1 and H_2 are chosen suitably. In the following, let us consider how to choose H_1 and H_2 to get global superconvergence. It also will be discussed in the next section for a posteriori error estimate. Let

$$H_i=h^{\alpha_i},\quad i=1,2,$$

with $\alpha_1,\alpha_2\in(0,1)$. The parameter α_i will play an important role later in achieving a superconvergence for the finite element approximation $(\mathbf{y}_h,r_h,\mathbf{p}_h,s_h)$.

Theorem 4.1. *Suppose that all conditions of Lemmas 4.1 and 4.3 are valid. Let $\alpha_1=\frac23$ and $\alpha_2=\frac23$, i.e., $H_1=h^{2/3}$, $H_2=h^{2/3}$. Then*

$$\|\nabla\mathbf{y}-\mathcal{G}_{H_1}\mathbf{y}_h\|_{0,\Omega}+\|\nabla\mathbf{p}-\mathcal{G}_{H_1}\mathbf{p}_h\|_{0,\Omega}\leq C(h^{\frac43}+h_{\mathbf{U}}^{\frac32}), \quad (4.32)$$

$$\|r-\mathcal{P}_{H_2}r_h\|_{0,\Omega}+\|s-\mathcal{P}_{H_2}s_h\|_{0,\Omega}\leq C(h^{\frac43}+h_{\mathbf{U}}^{\frac32}). \quad (4.33)$$

where $\mathcal{G}_{H_1}=\nabla\mathcal{P}_{H_1}$, $\mathcal{P}_{H_1}$ and $\mathcal{P}_{H_2}$ are recovery operators defined in this section.

Proof. Let $H_1=h^{\alpha_1}$, it follows from Lemma 4.1 that

$$\begin{aligned}\|\nabla(\mathbf{y}-\mathcal{P}_{H_1}\mathbf{y}_h)\|_{0,\Omega}+\|\nabla(\mathbf{p}-\mathcal{P}_{H_1}\mathbf{p}_h)\|_{0,\Omega} &\leq C(H_1^2+h^2H_1^{-1}+h^{\frac32}+h_{\mathbf{U}}^{\frac32})\\ &= C(h^{2\alpha_1}+h^{2-\alpha_1}+h^{\frac32}+h_{\mathbf{U}}^{\frac32}). \end{aligned} \quad (4.34)$$

We optimize the above estimate by choosing α_1 such that

$$2\alpha_1 = (2 - \alpha_1).$$

Solving the above equation gives $\alpha_1 = \frac{2}{3}$. Substituting it to (4.34) implies (4.32). Similarly we can prove (4.33).

Remark 4.1. It is proved by Theorem 3.1 and Theorem 4.1 that choosing the suitable H_i and using the recovery process defined in Section 3 and 4, we have following global superconvergence result:

$$\begin{aligned}&\|\mathcal{R}_h\mathbf{u}_h - \mathbf{u}\|_{0,\Gamma_b} + \|\nabla\mathbf{y} - \mathcal{G}_{H_1}\mathbf{y}_h\|_{0,\Omega} + \|\nabla\mathbf{p} - \mathcal{G}_{H_1}\mathbf{p}_h\|_{0,\Omega}\\ &+\|r - \mathcal{P}_{H_2}r_h\|_{0,\Omega} + \|s - \mathcal{P}_{H_2}s_h\|_{0,\Omega} \le C(h^{\frac{4}{3}} + h_{\mathbf{U}}^{\frac{3}{2}}),\end{aligned}$$

which is better than the standard error estimate:

$$\|\mathbf{u}_h - \mathbf{u}\|_{0,\Gamma_b} + \|\nabla(\mathbf{y} - \mathbf{y}_h)\|_{0,\Omega} + \|\nabla(\mathbf{p} - \mathbf{p}_h)\|_{0,\Omega} + \|r - r_h\|_{0,\Omega} + \|s - s_h\|_{0,\Omega} \le C(h + h_{\mathbf{U}}).$$

Remark 4.2. Recalling the fitting finite element space $\mathbf{V}^{H_1} \subset (L^2(\Omega))^2$ in Lemma 4.1, we see that $\mathbf{V}^{H_1}$ could be chosen as the finite element space consisting of discontinuous piecewise quadratic polynomial. From Lemma 4.2, we know that the fitting finite element space $Q^{H_2} \subset H^1(\Omega)$. Then Q^{H_2} is a surface fitting space consisting of continuous piecewise linear polynomial. Note that $\mathcal{P}_{H_1}$ and $\mathcal{P}_{H_2}$ are the L^2 projectors from $L^2(\Omega)$ onto the finite element spaces $\mathbf{V}^{H_1}$ and Q^{H_2}, respectively, and $\mathbf{V}^{H_1} \subset (L^2(\Omega))^2$, $Q^{H_2} \subset H^1(\Omega)$. It is easy to see that the cost of computation for $\mathcal{P}_{H_1}$ is much smaller than the one for $\mathcal{P}_{H_2}$, because that we can calculate $\mathcal{P}_{H_1}\mathbf{y}_h$ and $\mathcal{P}_{H_1}\mathbf{p}_h$ piecewisely. Considering our optimal control problem, it is less important to get the better approximation for r and s. We can only use $\mathcal{P}_{H_1}$ to improve the accuracy of $\mathbf{y}_h$ and $\mathbf{p}_h$, and ignore the error of r_h and s_h. Then the cost of computation for the recovery of r_h and s_h can be removed, which is much more than the one for $\mathbf{y}_h$ and $\mathbf{p}_h$.

5. Recovery Type a Posteriori Error Estimates

Based on the recovery operator $\mathcal{G}_{H_1}$, $\mathcal{P}_{H_2}$ and $\mathcal{R}_h$, where $\mathcal{R}_h$ is defined in Section 3, $\mathcal{G}_{H_1}$ and $\mathcal{P}_{H_2}$ are defined in Section 4. we can defined the recovery type a posteriori error estimator:

$$\begin{aligned}\eta_g \quad = \quad & \|\nabla\mathbf{y}_h - \mathcal{G}_{H_1}\mathbf{y}_h\|_{0,\Omega} + \|\nabla\mathbf{p}_h - \mathcal{G}_{H_1}\mathbf{p}_h\|_{0,\Omega} + \|r_h - \mathcal{P}_{H_2}r_h\|_{0,\Omega}\\ &+\|s_h - \mathcal{P}_{H_2}s_h\|_{0,\Omega} + \|\mathbf{u}_h - \mathcal{R}_h\mathbf{u}_h\|_{0,\Gamma_b}.\end{aligned}$$

Then we have following a posteriori error estimate.

Theorem 5.1. *Let $H_i(i = 1, 2) > h$. Suppose that the error enjoys the "non-degeneracy" condition:*

$$c(h + h_{\mathbf{U}}) \le e, \tag{5.1}$$

where e is defined in (2.12), and all conditions of Lemma 4.1-Lemma 4.3 are valid. Then, we have the following results,

$$\begin{aligned}\|\nabla\mathbf{y} - \mathcal{G}_{H_1}\mathbf{y}_h\|_{0,\Omega} \quad &+ \quad \|\nabla\mathbf{p} - \mathcal{G}_{H_1}\mathbf{p}_h\|_{0,\Omega} + \|r - \mathcal{P}_{H_2}r_h\|_{0,\Omega}\\ &+ \quad \|s - \mathcal{P}_{H_2}s_h\|_{0,\Omega} + \|\mathbf{u} - \mathcal{R}_h\mathbf{u}_h\|_{0,\Gamma_b} \le Me,\end{aligned} \tag{5.2}$$

where e is the exact error defined by (2.12) and

$$M = C(H_1^2/h + H_2^2/h + h/H_1 + h/H_2 + h^{\frac{1}{2}} + h_{\mathbf{U}}^{\frac{1}{2}}).$$

Then we have that

$$\frac{\eta_g}{1 + M} \le e, \tag{5.3}$$

and if $M < 1$,

$$\frac{\eta_g}{1 + M} \le e \le \frac{\eta_g}{1 - M}. \tag{5.4}$$

If $H_i = H_i(h)$ is chosen so that $M \to 0$ as $h \to 0$ and $h_{\mathbf{U}} \to 0$, the estimator is asymptotically exact, i.e.,

$$\lim_{h\to 0,\, h_{\mathbf{U}}\to 0} \frac{\eta_g}{e} = 1. \tag{5.5}$$

Proof. Using Lemma 4.1-Lemma 4.3 and the "non-degeneracy" condition (5.1), we have

$$\begin{aligned}
&\|\nabla \mathbf{y} - \mathcal{G}_{H_1}\mathbf{y}_h\|_{0,\Omega} + \|\nabla \mathbf{p} - \mathcal{G}_{H_1}\mathbf{p}_h\|_{0,\Omega} + \|r - \mathcal{P}_{H_2} r_h\|_{0,\Omega} + \|s - \mathcal{P}_{H_2} s_h\|_{0,\Omega} \\
&\quad + \|\mathbf{u} - \mathcal{R}_h \mathbf{u}_h\|_{0,\Gamma_b} \\
&\quad \le C(H_1^2 + H_2^2 + h^2 H_1^{-1} + h^2 H_2^{-1} + h^{\frac{3}{2}} + h_{\mathbf{U}}^{\frac{3}{2}}) \\
&\quad \le \frac{C}{c}\Big(\frac{H_1^2}{h} + \frac{H_2^2}{h} + \frac{h}{H_1} + \frac{h}{H_2} + h^{\frac{1}{2}} + h_{\mathbf{U}}^{\frac{1}{2}}\Big) c(h + h_{\mathbf{U}}) \le Me.
\end{aligned}$$

This completes the proof of (5.2). Using triangle inequality, it is easy to prove (5.3) and (5.4) from (5.2). The asymptotically exactness estimate (5.5) is the direct result of (5.4).

Remark 5.1. It is clear that M plays an important role in Theorem 5.1. If M is bounded, we thus see that $\frac{\eta_g}{1+M}$ furnishes a lower bound for the real error. If $M \le \bar{M} < 1$, as $h, h_{\mathbf{U}}$ and $H_i (i = 1, 2)$ vary, η_g is the lower and upper bound of the error e. Then we call η_g an equivalent estimator. This would be the case, e.g., $H_i = K_i h$ with K_i fixed and sufficiently large (and $h, h_{\mathbf{U}}$ sufficiently small). We call η_g an asymptotically exact estimator if $M \to 0$ as $h \to 0$ and $h_{\mathbf{U}} \to 0$ so that η_g is the approximation of e. This would be the case, e.g., $H_i = H_i(h)$ were chosen so that $h/H_i \to 0$, and $(H_i/h)h^{1/2} \to 0$ as $h \to 0$. In other word, η_g is an asymptotically exact estimator if $H_i/h \to \infty$ but not faster comparing $h^{\frac{1}{2}} \to 0$.

6. Discussions

In this paper, we discussed the recovery type superconvergence and a posteriori error estimates for the boundary control problems governed by the Stokes equations. It is shown that if the solution is smooth enough and the Stoke equations satisfy some regularity assumption, recovery type superconvergence and a posteriori error estimates for the control, the state and the co-state can be proved. Our results are applicable to many conforming finite element on the regular meshes. It should be pointed that there is no any requirement for the uniform partitions, which is usually required in the most superconvergence analysis.

There are many important issues still to be addressed in this area, for example, deriving the global superconvergence analysis and a recovery type a posteriori error estimate for more complicated control problems and finite element schemes. It is also interesting and very important to investigate the more complicated constrained control problems, i.e., the closed convex set K is more complicated. Finally, many computational issues have to be studied.

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