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ON THE DIVIDED DIFFERENCE FORM OF FAÀ DI BRUNO'S FORMULA *1)

Xing-hua Wang
(*Department of Mathematics, Zhejiang University, Hangzhou 310028, China*)

He-yu Wang
(*Computer Science School, Zhejiang University, Hangzhou 310027, China; Mathematics Department of Hong Kong Baptist University, Kowloon, Hong Kong*)

Abstract

The n-divided difference of the composite function $h := f \circ g$ of functions f, g at a group of nodes $t_0, t_1, \cdots, t_n$ is shown by the combinations of divided differences of f at the group of nodes $g(t_0), g(t_1), \cdots, g(t_m)$ and divided differences of g at several partial group of nodes $t_0, t_1, \cdots, t_n$, where $m = 1, 2, \cdots, n$. Especially, when the given group of nodes are equal to each other completely, it will lead to Faà di Bruno's formula of higher derivatives of function h.

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1. Introduction

It is studied that a divided difference of a function at a group of given nodes can be shown by combining the divided difference of the same function at another group of nodes[1,6,11]. When the given nodes are closed together, the divided difference is difficult to be computed. If the divided difference can be described by the combination of the divided difference of the same function at another distant node, then can be easily computed [12]. That is to say, it is the generalization of Lagrange numerical derivative method.

It is well-known that the Leibniz formula

$$h^{(n)}(x) = \sum_{\nu=0}^{n} \binom{n}{\nu} f^{(\nu)}(x) g^{(n-\nu)}(x) \tag{1.1}$$

of higher derivative of $h(x) := f(x) \cdot g(x)$, whose divided difference form is as follows:
Steffensen[10] formula. Let $h(x) := f(x) \cdot g(x)$. For any nodes $x_0, x_1, \cdots, x_n$, we have

$$h[x_0, x_1, \cdots, x_n] = \sum_{\nu=0}^{n} f[x_0, x_1, \cdots, x_\nu] \cdot g[x_\nu, x_{\nu+1}, \cdots, x_n]. \tag{1.2}$$

Now, let us begin to study the formulas of divided difference or the form of Faà di Bruno's formula for the composite function. Recently, introductions to the Faà di Bruno's formula of higher derivative of composite function[9, 5] and it's generalizations [3, 7] have been closely noticed by researchers. Johnson [9] stated not only its history, but also its partition description under the view of combination, the Bell polynomial description, determinant description, and various kinds of formulas based on Taylor formulas. One of the aims of this paper is to give

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a supplement to the Johnson's interesting results based on what we obtain in the paper. Of cause, Basic formulas obtained in the paper about the divided difference is undoubtedly useful for numerical analysis and combination analysis.

We will begin from the first order case. It is known that in primary calculus the formulas of the first order derivative for the composite function is as the following:

Chain Rule. Suppose $x = g(t)$ and $y = f(x)$ have derivatives at $t = t_0$ and $x = g(t_0)$, respectively. Then, the composite function $y = h(t) := f(g(t))$ has derivative at $t = t_0$,and

$$h'(t) = f'(x_0)g'(t_0) = f'(g(t_0))g'(t_0). \tag{1.3}$$

The proof of the Chain Rule should be done as followings. Let $t_1 \neq t_0$, and $x_1 = g(t_1)$, we have

$$\frac{h(t_1) - h(t_0)}{t_1 - t_0} = \frac{f(x_1) - f(x_0)}{x_1 - x_0} \cdot \frac{g(t_1) - g(t_0)}{t_1 - t_0}. \tag{1.4}$$

Let $t_1 \to t_0$, (1.3) follows from (1.4) immediately. But the bug of this proof method is that we can not guarantee $x_1 \neq x_0$, even though we can let $t_1 \neq t_0$. So there is something wrong with (1.4). To avoid this tragedy, in Courant & John's well-known calculus course [4], strict condition of $g'(t)$ has no zero point on an interval is used.

As a matter of fact, the case $x = x_1$ is not a tragedy. it just takes $\dfrac{f(x_1) - f(x_0)}{x_1 - x_0}$ into the existed derivative $f'(x_0)$. If we use the following definition of the first order divided difference of f at nodes x_0, x_1:

$$f[x_0, x_1] := \begin{cases} \dfrac{f(x_1) - f(x_0)}{x_1 - x_0}, & \text{if } x_1 \neq x_0; \\ f'(x_0), & \text{if } x_1 = x_0, \end{cases} \tag{1.5}$$

then (1.4) can be replaced by the following suitable equality

$$\frac{h(t_1) - h(t_0)}{t_1 - t_0} = f[x_0, x_1]\frac{g(t_1) - g(t_0)}{t_1 - t_0}. \tag{1.6}$$

The above equation surely holds for $x_1 \neq x_0$. As long as $y = f(x)$ has derivative at $x = x_0$. Let $t_1 \to t_0$ in (1.6), then the Chain Rule (1.3) follows.

Besides, we can easily find that (1.3) and the key step in its proof can be unified into the following formula:

$$\begin{aligned} h[t_0, t_1] &= f[x_0, x_1]g[t_0, t_1] \\ &= f[g(t_0), g(t_1)]g[t_0, t_1], \end{aligned} \tag{1.7}$$

which are called as *divided difference form of Chain Rule* or *chain rule of the first order divided difference*. When $t_1 \neq t_0$, (1.7) becomes (1.6); while $t_1 = t_0$, (1.7) becomes (1.3).

Studies in the paper shows that similar results hold for higher divided differences. That is, Faà di Bruno's formula for higher derivative of composite function has its relative form of divided difference. It will be given in the following Theorem 2.2 and 2.3, which not only give a suitable generalization to Faà di Bruno's formula, but also give the simplest proof method for the formula. They are basic formulas about divided difference.

2. Main Results

To discuss the n-th order divided difference of f at a group of nodes $x_0, x_1, \cdots, x_n$, we first give its definition of recursion form for nodes which are different from each other. Then, we understand the divided difference at nodes where some nodes are equal as the limit of the divided difference mentioned above. From this point of view, we have (for example, see Isaacson & Keller [8])

Lemma 2.1. *Let $x_0, x_1, \cdots, x_m$ be nodes different from each other, and $x_{m+1}, x_{m+2}, \cdots, x_n$ have $k_i \geq 0$ nodes equal to x_i, respectively, where $i = 0, 1, \cdots, m$, and $k_0+k_1+\cdots+k_m = n-m$. If f has the k_i-th order derivative at x_i, $i = 0, 1, \cdots, m$, then*

$$f[x_0, x_1, \cdots, x_n] = \frac{1}{k_0!k_1!\cdots k_m!}\left(\frac{\partial}{\partial x_0}\right)^{k_0}\left(\frac{\partial}{\partial x_1}\right)^{k_1}\cdots\left(\frac{\partial}{\partial x_m}\right)^{k_m} f[x_0, x_1, \cdots, x_m]. \tag{2.1}$$

Especially, if f has the n-th order derivative at x_0, then

$$f[x_0, x_1, \cdots, x_n] = \frac{1}{n!}f^{(n)}(x_0) \tag{2.1a}$$

holds for the case $x_1 = x_2 = \cdots = x_n = x_0$.

In the following, we briefly call the condition in Lemma 2.1 for the existence of derivative of f as *f has enough higher order derivative at relative nodes.*

The main result is

Theorem 2.2. *Let $h(t) := f(g(t))$. If f has enough higher order derivative at relative nodes, then for a group of different nodes $t_0, t_1, \cdots, t_n$ $(n \geq 1)$, we have*

$$h[t_0, t_1, \cdots, t_n] = \sum_{m=1}^{n} f[g(t_0), g(t_1), \cdots, g(t_m)] \sum_{\nu_0 \leq \nu_1 \leq \nu_2 \leq \cdots \leq \nu_m} \prod_{i=1}^{m} g[t_{i-1}, t_{\nu_{i-1}}, t_{\nu_{i-1}+1}, \cdots, t_{\nu_i}], \tag{2.2}$$

where $\nu_0 := m, \nu_m := n$, and subscript variants in the inner multiple summation of the right hand of (2.3) are $\nu_1, \nu_2 \cdots, \nu_{m-1}$. If function g has enough higher order derivative at relative nodes, then (2.2) holds for any groupe of nodes $t_0, t_1, \cdots, t_n$(it is not necessary to suppose that the nodes are different from each other). Especially, when $t_1 = t_2 = \cdots t_n = t_0$, (2.2) becomes Faà di Brouno's formula defined by

$$\frac{1}{n!}h^{(n)}(t_0) = \sum_{m=1}^{n} \frac{1}{m!}f^{(m)}(g(t_0)) \sum_{\nu_0 \leq \nu_1 \leq \nu_2 \leq \cdots \leq \nu_m} \prod_{i=1}^{m} \frac{g^{(\nu_i - \nu_{i-1}+1)}(t_0)}{(\nu_i - \nu_{i-1}+1)!}, \qquad \nu_0 := m,\ \nu_m := n. \tag{2.3}$$

The equation (2.3) is a new version of Faà di Bruno's formula. From (2.1a), it is obvious that (2.2) contains (2.3). Next, we use new subscript variants

$$k_i := \nu_i - \nu_{i-1} + 1,\ i = 1, 2, \cdots, m,$$

in the inner multiple summation of the right hand of (2.2). Then, the former will be replaced by

$$\nu_i = k_1 + k_2 + \cdots + k_i + m - i,\ i = 1, 2, \cdots, m-1.$$

The restricted condition of new subscript variants is

$$k_1 + k_2 + \cdots + k_m = n, \quad k_1 \geq 1, \cdots, k_m \geq 1.$$

Thus, from Theorem 2.2 , we get

Theorem 2.3. *Let $h(t) := f(g(t))$. If functions f and g has enough higher order derivative at relative nodes, respectively, then for any group of nodes $t_0, t_1, \cdots, t_n$ $(n \geq 1)$, it holds that*

$$\begin{aligned} h[t_0, t_1, \cdots, t_n] = &\sum_{m=1}^{n} f[g(t_0), g(t_1), \cdots, g(t_m)] \\ &\times \sum_{\substack{k_1+k_2+\cdots+k_m=n \\ k_1,\cdots,k_m \geq 1}} \prod_{i=1}^{m} g[t_{i-1}, t_{\nu_{i-1}}, t_{\nu_{i-1}+1}, \cdots, t_{\nu_i}], \end{aligned} \tag{2.4}$$

where

$$\nu_0 := m,\ \nu_i := k_1 + k_2 + \cdots + k_i + m - i \quad (i = 1, 2, \cdots, m-1), \quad \nu_m := n.$$

Especially, when $t_1 = \cdots = t_n = t_0$, (2.3) becomes Faà di Bruno's formula defined by

$$\frac{1}{n!} h^{(n)}(t_0) = \sum_{m=1}^{n} \frac{1}{m!} f^{(m)}(g(t_0)) \sum_{\substack{k_1+k_2+\cdots+k_m=n \\ k_1,\cdots,k_m \geq 1}} \prod_{i=1}^{m} \frac{g^{(k_i)}(t_0)}{k_i!}. \tag{2.5}$$

The inner multiple summation of the right hand of (2.5) is just the value of the general partial Bell polynomial $\hat{B}_{n,m}(x_1, x_2, \cdots, x_n)$ at

$$x_k = \frac{g^{(k)}(t_0)}{k!}, \quad k = 1, 2 \cdots, n.$$

In fact, the (general) productive function of $\hat{B}_{n,m}$ is

$$(\sum_{k=1}^{\infty} x_k t^k)^m = \sum_{n=m}^{\infty} \hat{B}_{n,m} t^n, \quad m = 1, 2, \cdots, n. \tag{2.6}$$

See, for example, Comter [2]. If we expand the left hand of (2.6) according to the multiplication method of m power series, and compare coefficients of t^n, we get

$$\hat{B}_{n,m}(x_1, x_2, \cdots, x_n) = \sum_{\substack{k_1+k_2+\cdots+k_m=n \\ k_1,\cdots,k_m \geq 1}} x_{k_1} x_{k_2} \cdots, x_{k_m}. \tag{2.7}$$

So, (2.5) can be rewritten as

$$\frac{1}{n!} h^{(n)}(t_0) = \sum_{m=1}^{n} \frac{1}{m!} f^{(m)}(g(t_0)) \hat{B}_{n,m}(\frac{g'(t_0)}{1!}, \frac{g''(t_0)}{2!}, \cdots, \frac{g^{(n)}(t_0)}{n!}). \tag{2.8}$$

If we expand the m power of the left hand of (2.6) directly according to the formula of multinomial terms, then

$$\begin{aligned} \hat{B}_{n,m}(x_1, x_2, \cdots, x_n) &= \sum_{\substack{a_1+2a_2+\cdots+na_n=n \\ a_1+a_2+\cdots+a_n=m}} \frac{m!}{a_1! a_2! \cdots a_n!} x_1^{a_1} x_2^{a_2} \cdots, x_n^{a_n} \\ &= \frac{m!}{n!} B_{n,m}(1!x_1, 2!x_2, \cdots, n!x_n) \end{aligned} \tag{2.9}$$

follows, where

$$B_{n,m}(x_1, x_2, \cdots, x_n) := \sum_{\substack{a_1+2a_2+\cdots+na_n=n \\ a_1+a_2+\cdots+a_n=m}} \frac{n!}{a_1!(1!)^{a_1} a_2!(2!)^{a_2} \cdots a_n!(n!)^{a_n}} x_1^{a_1} x_2^{a_2} \cdots, x_n^{a_n} \tag{2.10}$$

is an exponent form of partial Bell polynomial. Based on (2.10), (2.5) can be rewritten as

$$h^{(n)}(t_0) = \sum_{m=1}^{n} f^{(m)}(g(t_0)) B_{n,m}(g'(t_0), g''(t_0), \cdots, g^{(n)}(t_0)). \tag{2.11}$$

Example 2.4. *When $n = 4$, it follows from (2.2) that*

$$\begin{aligned}
&h[t_0, t_1, t_2, t_3, t_4] \\
&= f[g(t_0), g(t_1)]g[t_0, t_1, t_2, t_3, t_4] \\
&+ f[g(t_0), g(t_1), g(t_2)](g[t_0, t_1]g[t_1, t_2, t_3, t_4] \\
&\qquad + g[t_0, t_2, t_3]g[t_1, t_3, t_4] \\
&\qquad + g[t_0, t_2, t_3, t_4]g[t_1, t_4]) \\
&+ f[g(t_0), g(t_1), g(t_2), g(t_3)](g[t_0, t_3]g[t_1, t_3]g[t_2, t_3, t_4] \\
&\qquad + g[t_0, t_3]g[t_1, t_3, t_4]g[t_2, t_4] \\
&\qquad + g[t_0, t_3, t_4]g[t_1, t_4]g[t_2, t_4]) \\
&+ f[g(t_0), g(t_1), g(t_2), g(t_3), g(t_4)]g[t_0, t_4]g[t_1, t_4]g[t_2, t_4]g[t_3, t_4].
\end{aligned} \tag{2.2a}$$

When $t_1 = t_2 = t_3 = t_4 = t_0$, it becomes

$$\begin{aligned}
\frac{h^{(4)}(t_0)}{4!} &= \frac{f'(g(t_0))}{1!}\frac{g^{(4)}(t_0)}{4!} \\
&+ \frac{f''(g(t_0))}{2!}\Big(2(\frac{g'(t_0)}{1!})(\frac{g'''(t_0)}{3!}) + (\frac{g''(t_0)}{2!})^2\Big) \\
&+ \frac{f'''(g(t_0))}{3!}\Big(3(\frac{g'(t_0)}{1!})^2(\frac{g''(t_0)}{2!})\Big) \\
&+ \frac{f^{(4)}(g(t_0))}{4!}\Big(\frac{g'(t_0)}{1!}\Big)^4.
\end{aligned} \tag{2.3a}$$

Now we see that

$$\hat{B}_{4,1} = x_4,\ \hat{B}_{4,2} = 2x_1x_3 + x_2^2,\ \hat{B}_{4,3} = 3x_1^2x_2,\ \hat{B}_{4,4} = x_1^4.$$

And (2.3a) can be rewritten as

$$\begin{aligned}
h^{(4)}(t_0) &= f'(g(t_0))g^{(4)}(t_0) + f''(g(t_0))(4g'(t_0)g'''(t_0) + 3g''(t_0)^2) \\
&+ 6g'''(g(t_0))g'(t_0)^2g''(t_0) + f^{(4)}(g(t_0))g'(t_0)^4.
\end{aligned}$$

Again, we see that

$$B_{4,1} = x_4,\ B_{4,2} = 4x_1x_3 + 3x_2^2,\ B_{4,3} = 6x_1^2x_2,\ B_{4,4} = x_1^4.$$

Example 2.4 shows how the complexity relation in Faà di Bruno's formula evolves from the simple relation of divided difference of composite function in Theorem 2.2. Indeed, the divided difference of composite function given by Theorem 2.2 is not only much more general than the Faà di Bruno's formula, but also much clearer.

3. Proof of the Main Results

First, we need the following transformation form of Steffensen formula

Lemma 3.1. *Let $t_0, t_1, \cdots, t_n$ be a group of nodes different from each other, and $\Phi(t) := \phi(t) \cdot \psi(t)$, where ϕ and ψ satisfy*

$$\phi(t_0) = 0 \quad \text{and} \quad \psi(t_n) = 0,$$

respectively. Then it holds that

$$\Phi[t_0, t_1, \cdots, t_n] = \sum_{\nu=1}^{n-1} \phi[t_0, t_1, \cdots, t_\nu]\psi[t_\nu, t_{\nu+1}, \cdots, t_n]. \tag{3.1}$$

Proof. The equation (3.1) follows from Steffenson formula

$$\Phi[t_0,t_1,\cdots,t_n]=\phi[t_0]\psi[t_0,t_1,\cdots,t_n]$$
$$+\sum_{\nu=1}^{n-1}\phi[t_0,t_1,\cdots,t_\nu]\psi[t_\nu,t_{\nu+1},\cdots,t_n]+\phi[t_0,t_1,\cdots,t_n]\psi[t_n],$$

and

$$\phi[t_0]=\phi(t_0)=0,\ \psi[t_n]=\psi(t_n)=0$$

by the assumption.

Lemma 3.1 can be generalized to the following more general conclusion.

Lemma 3.2. *Let $t_0,t_1,\cdots,t_n$ be a group of nodes different from each other, and function ϕ_i satisfies*

$$\phi_i(t_i)=0,\qquad i=0,1,\cdots,m-1.$$

Suppose $\Phi(t):=\phi_0(t)\phi_1(t)\cdots\phi_{m-1}(t)$. Then when $n\geq m$, we have

$$\Phi[t_0,t_1,\cdots,t_n]=\sum_{\nu_0\leq\nu_1\leq\nu_2\leq\cdots\leq\nu_m}\prod_{i=1}^{m}\phi_{i-1}[t_{i-1},t_{\nu_{i-1}},t_{\nu_{i-1}+1},\cdots,t_{\nu_i}],\quad \nu_0:=m,\ \nu_m:=n. \tag{3.2}$$

Proof. We use induction for m. It is true that (3.2) holds for $m=1$.

For $m=2$, it follows

$$\Phi[t_0,t_1,\cdots,t_n]=\Phi[t_0,t_2,t_3,\cdots,t_n,t_{n+1}]$$
$$=\sum_{\nu=2}^{n}\phi_0[t_0,t_2,t_3,\cdots,t_\nu]\phi_1[t_\nu,t_{\nu+1},\cdots,t_{n+1}]$$

from Lemma 3.1(here $t_{n+1}:=t_1$) and the symmetry of the divided differences. If we replace t_{n+1} with t_1 and change the sequence order of the nodes, then we have

$$\Phi[t_0,t_1,\cdots,t_n]=\sum_{\nu=2}^{n}\phi_0[t_0,t_2,t_3,\cdots,t_\nu]\phi_1[t_1,t_\nu,t_{\nu+1},\cdots,t_n].$$

The lemma holds for $m=2$.

Suppose that divided difference of multiplication of $m-1$ functions, whose values at $t_0,t_1,\cdots,t_{m-2}$ are zero, can be computed by (3.2) for $m\geq 3$. Write

$$\Phi(t):=\phi_0(t)\phi_1(t)\cdots\phi_{m-3}(t)\overline{\phi}_{m-2}(t),$$
$$\overline{\phi}_{m-2}(t):=\phi_{m-2}(t)\phi_{m-1}(t),\qquad t_{n+1}:=t_{m-1}.$$

Then we get

$$\Phi[t_0,t_1,\cdots,t_n]=\Phi[t_0,t_1,\cdots t_{m-2},t_m,t_{m+1}\cdots,t_n,t_{n+1}]$$
$$=\sum_{\nu_0\leq\nu_1\leq\cdots\leq\nu_{m-2}\leq n+1}\prod_{i=1}^{m-2}\phi_{i-1}[t_{i-1},t_{\nu_{i-1}},t_{\nu_{i-1}+1},\cdots,t_{\nu_i}]\overline{\phi}_{m-2}[t_{m-2},t_{\nu_{m-2}},t_{\nu_{m-2}+1},\cdots,t_{n+1}]. \tag{3.3}$$

Note that the terms with $\nu_{m-2}=n+1$ in the multiple summation of the right hand of (3.3) is not necessary to be computed, because they all have factors

$$\overline{\phi}_{m-2}[t_{m-2},t_{n+1}]=\overline{\phi}_{m-2}[t_{m-2},t_{m-1}]=0.$$

So, the restriction on the index of multiple summation becomes

$$m=\nu_0\leq\nu_1\leq\nu_2\leq\cdots\leq\nu_{m-2}\leq n. \tag{3.4}$$

By Lemma 3.1 and $\phi_{m-1}(t_{n+1}) = \phi_{m-1}(t_{m-1}) = 0$, we get

$$\begin{aligned}
&\overline{\phi}_{m-2}[t_{m-2}, t_{\nu_{m-2}}, t_{\nu_{m-2}+1}, \cdots, t_{n+1}] \\
&= \sum_{\nu=\nu_{m-2}}^{n} \phi_{m-2}[t_{m-2}, t_{\nu_{m-2}}, t_{\nu_{m-2}+1}, \cdots, t_\nu]\phi_{m-1}[t_\nu, t_{\nu+1}, \cdots, t_{n+1}] \\
&= \sum_{\nu_{m-2} \le \nu \le n} \phi_{m-2}[t_{m-2}, t_{\nu_{m-2}}, t_{\nu_{m-2}+1}, \cdots, t_\nu]\phi_{m-1}[t_{n+1}, t_\nu, t_{\nu+1}, \cdots, t_n].
\end{aligned} \tag{3.5}$$

Put it into (3.3), and combine the multiple summation at the end of (3.3) with that of (3.5) into a new multiple summation with the index restriction (3.4) and $\nu_{m-2} \le \nu \le n$. It is obvious that the index restriction on multiple summation is

$$\nu_0 \le \nu_1 \le \nu_2 \le \cdots \le \nu_{m-2} \le \nu_{m-1} \le \nu_m, \nu_{m-1} := \nu, \nu_0 := m, \nu_m := n.$$

So, (3.2) holds for all positive integer m.

The Proof of Theorem 2.2. Let $t_0, t_1, \cdots, t_n$ be nodes different from each other, Newton interpolation polynomial of function h on it is

$$N(t; h, \{t_i\}_{i=0}^n) = h(t_0) + \sum_{\nu=1}^{n} h[t_0, t_1, \cdots, t_\nu]\omega_\nu(t), \tag{3.6}$$

where

$$\omega_\nu(t) := \omega_\nu(t; \{t_i\}_{i=0}^n) := \prod_{i=0}^{\nu-1}(t - t_i).$$

On the other hand, Newton interpolation polynomial of function f at nodes $x_0, x_1, \cdots, x_n$ is

$$N(x; f, \{x_i\}_{i=0}^n) = f(x_0) + \sum_{m=1}^{n} f[x_0, x_1, \cdots, x_m]\omega_m(x; \{x_i\}_{i=0}^n). \tag{3.7}$$

Let $x = g(t),\ x_i = g(t_i), i = 0, 1, \cdots, n$ in (3.7) and replace

$$\omega_m(g(t); \{g(t_i)\}_{i=0}^n) = \prod_{i=0}^{m-1}(g(t) - g(t_i)) =: \Omega_m(t) \tag{3.8}$$

with its Newton interpolation polynomial

$$N(t; \Omega_m, \{t_i\}_{i=0}^n) = \sum_{\nu=m}^{n} \Omega_m[t_0, t_1, \cdots, t_\nu]\omega_\nu(t)$$

(Obviously, $\Omega_m[t_0, t_1, \cdots, t_\nu] = 0$ for $\nu < m$), it follows the Newton interpolation polynomial for composite function $f \circ g$ defined by

$$N(t; f \circ g, \{t_i\}_{i=0}^n) = f(g(t_0)) + \sum_{m=1}^{n} f[g(t_0), g(t_1), \cdots, g(t_m)] \sum_{\nu=m}^{n} \Omega_m[t_0, t_1, \cdots, t_\nu]\omega_\nu(t).$$

Change the order of the summation, we have

$$N(t; f \circ g, \{t_i\}_{i=0}^n) = f(g(t_0)) + \sum_{\nu=1}^{n}\sum_{m=1}^{\nu} f[g(t_0), g(t_1), \cdots, g(t_m)]\Omega_m[t_0, t_1, \cdots, t_\nu]\omega_\nu(t). \tag{3.9}$$

Since (3.6) and (3.9) are the same according to the uniqueness of interpolation polynomial of $h = f \circ g$, the coefficients of the term including $\omega_n(t)$ are equal. So,

$$h[t_0, t_1, \cdots, t_n] = \sum_{m=1}^{n} f[g(t_0), g(t_1), \cdots, g(t_m)]\Omega_m[t_0, t_1, \cdots, t_n] \tag{3.10}$$

follows. If we use Lemma 3.2 to compute the divided difference of function Ω_m defined by (3.8), we get

$$\phi_i(t) = g(t) - g(t_i), \quad i = 0, 1, \cdots, m-1.$$

Since the order of divided difference of ϕ_i included in the right hand of (3.2) is at least one, these divided differences are ones of g, which induces

$$\begin{aligned} \Omega_m[t_0, t_1, \cdots, t_n] = &\sum_{\nu_0 \le \nu_1 \le \nu_2 \le \cdots \le \nu_m} \prod_{i=1}^{m} g[t_{i-1}, t_{\nu_{i-1}}, t_{\nu_{i-1}+1}, \cdots, t_{\nu_i}], \\ &\nu_0 := m, \ \nu_m := n. \end{aligned} \tag{3.11}$$

Hence (2.2) follows form (3.11)and (3.10).

If function g has enough higher order derivative at relative nodes, (2.2) holds for general nodes automatically.

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References

[1] C. de Boor, A divided difference expansion of a divided difference, *J. Approx. Theory*, **122** (2003), 10–12.

[2] L. Comtet, Analyse Combinatoire, Tome Premir, Presses Universitaires de France, Paris, 1970.

[3] G. M. Constantine & T. H. Savits, A multivariate Faà di Bruno formula with applicaiotns, *Trans. Amer. Math. Soc.*, **348** (1996), 503–520.

[4] R. Courant & F. John, Introduction to Calculus and Analysis, Vol. I, Springer-Verleg, New York Inc., 1989.

[5] A. D. D. Craik, Prehistory of Faà di Bruno's formula, *Amer. Math. Monthly*, **112** (2005), 119–130.

[6] M. Floater, Error formulas for divided difference expansions and numerical differentiation, *J. Approx. Theory*, **122** (2003), 1–9.

[7] L. H. Encinas & J. M. Masque, A short proof of the generalized Faà di Bruno's formula, *Appl. Math. Lett.*, **16** (2003), 975–979.

[8] E. Isaacson & H. B. Keller, Analysis of Numerical Methods, Wiley, New York, 1966.

[9] W. P. Johnson, The curious history of Faà di Bruno's formula, *Amer. Math. Monthly*, **109** (2002), 217–231.

[10] J. F. Steffensen, Note on divided differences, *Danske Vid. Selsk. Math.-Fys. Medd.*, **17** (1939) 3, 1–12.

[11] X. Wang, M. J. Lai & S. Yang., On the divided differences of the remainder in polynomial interpolation, *J. Approx. Theory*, **127** (2004), 193–197.

[12] X. Wang, H. Wang & M. J. Lai, Some results on numerical divided difference formulas, *Sci. China Ser. A*, **28**:11 (2005), 1441–1450.