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ON QUADRATURE OF HIGHLY OSCILLATORY FUNCTIONS *1)

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Abstract

Some quadrature methods for integration of $\int_a^b f(x)e^{i\omega g(x)}dx$ for rapidly oscillatory functions are presented. These methods, based on the lower order remainders of Taylor expansion and followed the thoughts of Stetter [9], Iserles and Nørsett [5], are suitable for all ω and the decay of the error can be increased arbitrarily in case that $g'(x) \neq 0$ for $x \in [a,b]$, and easy to be implemented and extended to the improper integration and the general case $I[f] = \int_a^b f(x)e^{ig(\omega,x)}dx$.

Mathematics subject classification: 65D32, 65D30.
Key words: Oscillatory integral, quadrature, Filon-type method, Taylor expansion.

1. Introduction

The quadrature of highly oscillating integrals is important in many areas of applied mathematics. The standard integration formulas such as the trapezoid rule, Simpson's rule or Gaussian integration may suffer from difficulty. Many methods have been developed since Filon [2], such as Price [8], Stetter [9], Longman [6], Levin [7], Iserles [3,4] and Iserles and Nørsett [5], etc.

For the Filon-type quadrature of the form $\int_0^h f(x)e^{i\omega x}w(x)dx$, Iserles [3] analyzed the convergent behavior in a range of frequency regimes and showed that the accuracy increases when oscillation becomes faster. Recently Iserles and Nørsett [5] extended the approach of Iserles [3,4] and defined the generalized Filon-type method for integral $\int_0^1 f(x)e^{i\omega g(x)}dx$ and showed that the rate of decay of the error, once frequency grows, can be increased arbitrarily by the inclusion of higher derivatives.

Both the Filon-type and the generalized Filon-type, an approach f(x) by splines, are efficient for suitably smooth functions under the condition that the moments $\int_0^1 x^k e^{i\omega g(x)}dx$ can be accurately calculated.

Price's numerical approximation of Fourier transforms [8] is considered the integration between the zeros, for example,

$$\int_0^{2\pi} f(x)\sin nx dx.$$

Each

$$\int_{k\pi/n}^{(k+1)\pi/n} f(x)\sin nx dx$$

may be expeditiously computed by use of a Labatto rule. The Price method completely fails when ω is significantly larger than the number of quadrature points.

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Here we present some methods, basing on the lower order remainders of Taylor expansion, which transfer highly oscillatory functions into 'nice' functions—non-highly oscillating functions. And following the thoughts of Setter [9], Iserles and Nørsett [5] we show that these methods are suitable for all ω and the decay of the error can be increased arbitrarily the same as the generalized Filon-type method for large ω in case that $g'(x) \neq 0$ for $x \in [a, b]$. These methods are easy to be implemented and extended to the improper integration and the integral

$$I[f] = \int_a^b f(x)e^{ig(\omega,x)}dx,$$

where $e^{ig(\omega,x)}$ with highly oscillation and $\lim_{\omega\to\infty} g'_x(\omega, x) = \infty$ for all x in $[a, b]$.

2. Quadrature of Integral $\int_a^b f(x)e^{i\omega g(x)}dx$

Let $I[f]$ denote the following integral

$$I[f] = \int_a^b f(x)e^{i\omega g(x)}dx, \tag{2.1}$$

where f and g are suitably smooth functions. Suppose that the function g has at most finite stationary points in $[a, b]$. Without loss of generality, assume g has only one stationary point x_0 in $[a, b]$. Otherwise, we will partition the interval into finite subintervals such that each subinterval only contains one stationary point. The nth order Taylor polynomial of $e^{i\omega g(x)}$ is

$$F_0 = 1, \quad F_n(i\omega g(x)) = 1 + i\omega g(x) + \frac{(i\omega g(x))^2}{2!} + \frac{(i\omega g(x))^3}{3!} + \cdots + \frac{(i\omega g(x))^n}{n!}$$

and the nth order remainder of Taylor expansion is

$$T_n(x) = e^{i\omega g(x)} - F_n(i\omega g(x)). \tag{2.2}$$

$T_n(x)$ can be written as

$$T_n(x) = \begin{cases} \left(\cos(\omega g(x)) - 1 + \frac{(\omega g(x))^2}{2!} + \cdots + \frac{(-1)^{k+1}(\omega g(x))^{2k}}{(2k)!}\right) & + \\ i\left(\sin(\omega g(x)) - \omega g(x) + \cdots + \frac{(-1)^k(\omega g(x))^{2k-1}}{(2k-1)!}\right), & \text{n=2k}, \\ \left(\cos(\omega g(x)) - 1 + \frac{(\omega g(x))^2}{2!} + \cdots + \frac{(-1)^{k+1}(\omega g(x))^{2k}}{(2k)!}\right) & + \\ i\left(\sin(\omega g(x)) - \omega g(x) + \cdots + \frac{(-1)^k(\omega g(x))^{2k+1}}{(2k+1)!}\right), & \text{n=2k+1}. \end{cases}$$

Note that $U_n := \cos(x) - 1 + \frac{x^2}{2!} + \cdots + \frac{(-1)^{k+1}x^{2k}}{(2k)!}$, $V_n := \sin(x) - x + \cdots + \frac{(-1)^k x^{2k-1}}{(2k-1)!}$ are monotonic and smooth in $[0, +\infty)$ or $(-\infty, 0]$ for all $n = 1, 2, \ldots$. Hence, for monotonic and smooth function $g(x)$, $U_n(\omega g(x))$ and $V_n(\omega g(x))$ are smooth and monotonic in $[a, b] \cap [0, \infty)$ and $[a, b] \cap (-\infty, 0]$. Therefore $T_n(x)$ are not oscillatory even if $e^{i\omega g(x)}$ is highly oscillatory for large ω. For example, $\cos(1000x^{\frac{1}{3}}) - 1$ is highly oscillatory, but $\cos(1000x^{\frac{1}{3}}) - 1 + \frac{(1000x^{\frac{1}{3}})^2}{2}$ and $\cos(1000x^{\frac{1}{3}}) - 1 + \frac{(1000x^{\frac{1}{3}})^2}{2} - \frac{(1000x^{\frac{1}{3}})^4}{4!}$ are monotonic and smooth.

For $g(x)$ having at most finite stationary points in $[a, b]$, by the intermediate value theorem for derivatives of Darbouxe [10], $g'(x)$ has the same sign between the stationary points and $g(x)$ is monotonic in these subintervals, the $nth(n \geq 2)$ Taylor remainders $T_n(x)$ is not oscillatory functions, either. To calculus the highly oscillatory integrals, we need only lower order Taylor expansions. Here we consider the first and second order Taylor expansions.

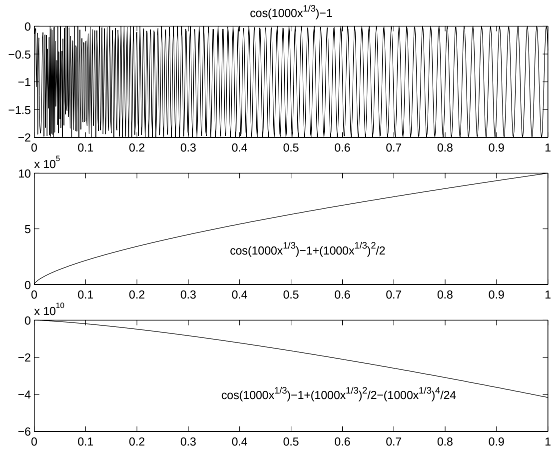

Figure 1: The monotonicity of the Taylor remainders

To compute the integral $\int_0^1 f(x)e^{i\omega g(x)}dx$, for small ω we consider $\int_0^1 f(x)(e^{i\omega g(x)} - 1 - i\omega g(x))dx + \int_0^1 f(x)(1+i\omega g(x))dx$, or $\int_0^1 f(x)(e^{i\omega g(x)} - 1 - i\omega g(x) - (i\omega g(x))^2/2)dx + \int_0^1 f(x)(1 + i\omega g(x) + (i\omega g(x))^2/2)dx$. For example, $\int_0^{2\pi} \log x \sin \omega x dx = \int_0^{2\pi} \log x(\sin \omega x - \omega x)dx + \omega \int_0^{2\pi} x$ log xdx. Compute the integrations in the right hand by the adaptive Simpson's rule in MATLAB6.5 and cite the data in Davis and Robinowitz [1] (P. 155):

ω	Exact	G_{32}	$2\omega \times L_4$	$2\omega \times L_5$	F_7	F_9	F_{11}	$Price$	New
1	-2.4377	-2.4378	-2.4380						-2.4377
2	-1.5572	-1.5570	-1.5880						-1.5572
4	$-.9507$	$-.9507$	$-.9830$						$-.9507$
10	$-.4718$	$-.4721$	$-.4941$	$-.4410$	$-.4567$	$-.4646$	$-.4676$	$-.4617$	$-.4718$
20	$-.2705$	.1237	$-.2853$	$-.2511$	$-.2622$	$-.2666$	$-.2683$	$-.2650$	$-.2705$
30	$-.1939$	-1.1143	$-.2051$	$-.1793$	$-.1880$	$-.1911$	$-.1923$	$-.1899$	$-.1939$

And $\int_0^{2\pi} x \cos 50x \sin \omega x dx = \int_0^{2\pi} x(\cos 50x - 1 + 50^2x^2/2)$ $(\sin\ \omega x - \omega x)dx - \int_0^{2\pi} x(-1 + 50^2x^2/2)(\sin \omega x - \omega x)dx + \omega \int_0^{2\pi} x^2(\cos 50x - 1 + 50^2x^2/2)dx + \omega \int_0^{2\pi} x^2 dx + \omega \int_0^{2\pi} 50^2x^4/2dx$. Compute the integrations in the right hand by the adaptive Simpson's rule in MATLAB6.5 and use the data in Davis and Robinowitz [1] (P. 155):

ω	Exact	G_{32}	$2\omega \times L_4$	F_7	F_{11}	$Price$	New
1	.00251428	2.1561858	.56312612				.00251729
2	.00503460	.87569672	4.7048779				.00504042
4	.01011785	$-.09742242$	.69758325				.01012942
10	.02617994	.10585082	9.3325313	.11049126	.29057620	.14484073	.02620855
20	.05983986	$-.81286374$	.1110445	.09287746	.0534164	.00511070	.05989761
30	.11780972	$-.64534403$	.16391873	.10205656	.11600732	.11013379	.11789654

However for large ω or to get higher accuracy, we define the following notations introduced in [5] and consider the following expansions: Let

$$\sigma_0[f](x) = f(x),$$

$$\sigma_{k+1}[f](x) = \frac{d}{dx}\frac{\sigma_k[f](x)}{g'(x)}, k = 0, 1, 2, \cdots.$$

In case $g'(x) \neq 0$ for all $x \in [a, b]$.

Lemma 2.1. *Suppose that $f, g : [a,b] \to \mathbf{R}$ are smooth functions and $g'(x) \neq 0$ for all $x \in [a,b]$. Then*

$$\begin{aligned} I[f] &= \int_a^b f(x)e^{i\omega g(x)}dx \\ &= \sum_{k=1}^n \left\{ \frac{(-1)^{k+1}}{(i\omega)^k}\frac{\sigma_{k-1}[f](b)}{g'(b)}T_{k-1}(b) - \frac{(-1)^{k+1}}{(i\omega)^k}\frac{\sigma_{k-1}[f](a)}{g'(a)}T_{k-1}(a)\right\} \\ &\quad + \frac{(-1)^n}{(i\omega)^n}\int_a^b \sigma_n[f](x)T_{n-1}(x)dx \end{aligned} \tag{2.3}$$

Proof. $I[f] = \int_a^b f(x)e^{i\omega g(x)}dx = \frac{1}{i\omega}\int_a^b \frac{f(x)}{g'(x)}d(e^{i\omega g(x)}-1) = \frac{1}{i\omega}\frac{f(x)}{g'(x)}(e^{i\omega g(x)}-1)|_a^b - \frac{1}{i\omega}\int_a^b \sigma_1[f](x)(e^{i\omega g(x)}-1)dx$. By induction, it is easy to get equation (2.3).

To calculate the integral $\int_a^b \sigma_n[f](x)T_{n-1}(x)dx$, we consider the composite trapezoidal rule, the composite 2-point Gauss-Legendre quadrature and the composite Simpson's rule. Denote the corresponding method by $Q_n^{C.T}[f]$, $Q_n^{C.GS}[f]$ and $Q_n^{C.S}[f]$ and $h = (b-a)/m$, the length of each subinterval.

Firstly, we establish an error analysis for general composite interpolation quadratures with positive weights such as the composite trapezoidal rule, the composite Simpson's rule and the composite Gaussian quadrature, etc.

Lemma 2.2. *Suppose that $u, v : [a,b] \to \mathbf{R}$ are suitably smooth. Then for any partition $\{x_j\}_1^N \subset [a,b]$ and nonnegative sequence $\{a_j\}_1^N$ with $\sum_{j=1}^N a_j = 1$,*

$$\left|\int_a^b u(x)v(x)dx - (b-a)\sum_{j=1}^N a_j u(x_j)v(x_j)\right| \leq \frac{(b-a)^2}{2}\|(uv)'\|_\infty.$$

Proof. Since $\min_{a\le x\le b} u(x)v(x) \leq \sum_{j=1}^N a_j u(x_j)v(x_j) \leq \max_{a\le x\le b} u(x)v(x)$, by the mean value theorem of continuous functions, there exists an $x_0 \in [a,b]$ such that $u(x_0)v(x_0) = \sum_{j=1}^N a_j u(x_j)v(x_j)$. Then

$$\begin{aligned} \left|\int_a^b u(x)v(x)dx - (b-a)\sum_{j=1}^N a_j u(x_j)v(x_j)\right| &= \left|\int_a^b u(x)v(x)dx - \int_a^b u(x_0)v(x_0)dx\right| \\ &\leq \|(uv)'\|_\infty \left|\int_a^b (x-x_0)dx\right| \\ &\leq \frac{(b-a)^2}{2}\|(uv)'\|_\infty. \end{aligned}$$

Theorem 2.3. *Suppose that $f, g : [a,b] \to \mathbf{R}$ are smooth functions and $g'(x) \neq 0$ for all $x \in [a,b]$. Then for $n = 2, 3, \ldots$ and $1 \leq |\omega|$, the error of the composite interpolation quadrature with positive weights for (2.3) is satisfied*

$$|Q_n[f] - I[f]| \leq \frac{(b-a)h}{|\omega|}(\|\sigma_n[f]'\|_\infty + \|\sigma_n[f]\|_\infty\|g'\|_\infty)(e^{\|g\|_\infty}+1), \tag{2.4}$$

where h is the length of each subinterval.

Proof. Let $F(x) = \sigma_n[f](x)$ and $T_{n-1}(x) = U_{n-1}(\omega g(x)) + iV_{n-1}(\omega g(x))$. Then $U_{n-1}(\omega g(x))$ and $V_{n-1}(\omega g(x))$ have same sign and monotonic on $[a,b]$ for all $n \geq 2$ and

$$\frac{1}{(i\omega)^n}\int_a^b F(x)T_{n-1}(x)dx = \frac{1}{(i\omega)^n}\left(\int_a^b F(x)U_{n-1}(\omega g(x))dx + i\int_a^b F(x)V_{n-1}(\omega g(x))dx\right).$$

By Lemma 2.2 on each subinterval, we have that the error of calculating $\frac{1}{(i\omega)^n}\int_a^b F(x)e^{i\omega g(x)}dx$ by the composite interpolation quadrature with positive weights is less than or equal to

$$|Q_n[f]-I[f]| \leq \frac{(b-a)h}{|\omega^n|}(\|F'\|_\infty\|U_{n-1}\|_\infty + \|F\|_\infty\|U'_{n-1}\|_\infty + \|F'\|_\infty\|V_{n-1}\|_\infty + \|F\|_\infty\|V'_{n-1}\|_\infty).$$

Note that $\|U'_{n-1}(wg(x))\|_\infty = |\omega|\|V_{n-1}(\omega g(x))\|_\infty\|g'\|_\infty$ for $n-1$ being even and $\|U'_{n-1}(wg(x))\|_\infty = |\omega|\|V_{n-2}(\omega g(x))\|_\infty\|g'\|_\infty$ for $n-1$ being odd and $|\omega| \geq 1$. It can be verified that

$$\begin{aligned} & |Q_n[f]-I[f]| \\ \leq\ & \frac{(b-a)h}{|\omega|}(\|F'\|_\infty + \|F\|_\infty\|g'\|_\infty)(1+1+\|g\|_\infty + \frac{\|g\|_\infty^2}{2!} + \frac{\|g\|_\infty^3}{3!} + \cdots + \frac{\|g\|_\infty^{n-1}}{(n-1)!}) \\ \leq\ & \frac{(b-a)h}{|\omega|}(\|F'\|_\infty + \|F\|_\infty\|g'\|_\infty)(e^{\|g\|_\infty}+1). \end{aligned}$$

For the composite trapezoidal rule, the composite Simpson's rule and the composite 2-point Gauss-Legendre quadrature, more accurate estimation can be obtained.

Theorem 2.4. *Suppose that $f, g : [a,b] \to \mathbf{R}$ are suitably smooth functions and $g'(x) \neq 0$ for all $x \in [a,b]$. Then for $n = 2,3,4$ and $1 \leq |\omega|$*

$$|Q_n^{C.T}[f]-I[f]| \leq M(f,f',f'',g,g',g'')\frac{(b-a)h^2}{12|\omega|} \tag{2.5}$$

$$\begin{aligned} & |Q_n^{C.S}[f]-I[f]| \\ \leq\ & \frac{|Q_0^{C.S}[\sigma_n[f]]-I[\sigma_n[f]]|}{|\omega|^n} + M_1(f,f',f'',f^{(3)},f^{(4)},g,g',g'',g^{(3)},g^{(4)})\frac{(b-a)h^4}{180|\omega|^{n-2}} \end{aligned} \tag{2.6}$$

$$\begin{aligned} & |Q_n^{C.GS}[f]-I[f]| \\ \leq\ & \frac{|Q_0^{C.GS}[\sigma_n[f]]-I[\sigma_n[f]]|}{|\omega|^n} + M_1(f,f',f'',f^{(3)},f^{(4)},g,g',g'',g^{(3)},g^{(4)})\frac{(b-a)h^4}{1080|\omega|^{n-2}}. \end{aligned} \tag{2.7}$$

where $M(f,f',f'',g,g',g'')$ and $M_1(f,f',f'',f^{(3)},f^{(4)},g,g',g'',g^{(3)},g^{(4)})$ are constants without reference to ω and h.

Proof. Let $F(x) = \sigma_n[f](x)$. Since $\{F(x)e^{i\omega g(x)}\}'' = F''(x)e^{i\omega g(x)} + 2i\omega F'(x)g'(x)e^{i\omega g(x)} - \omega^2 F(x)g''(x)e^{i\omega g(x)} + i\omega F(x)g''(x)e^{i\omega g(x)}$, the error of computing $\frac{1}{(i\omega)^n}\int_a^b F(x)e^{i\omega g(x)}dx$ by the composite trapezoidal rule is less than or equal to $\frac{(b-a)h^2}{12|\omega|^{n-2}}(\|F''\|_\infty + 2\|F'g'\|_\infty + \|Fg'^2\|_\infty + \|Fg''\|_\infty)$. The error of calculating each item $\frac{1}{(i\omega)^n}\int_a^b F(x)(i\omega g(x))^k dx$ by the composite trapezoidal rule is $\frac{(b-a)h^2}{12|\omega|^{n-k}}(\|F''g^k\|_\infty + 2k\|F'g^{k-1}g'\|_\infty + k\|Fg^{k-1}g'^2\|_\infty + \|Fg^{k-1}g''\|_\infty)$.

Define $M(f,f',f'',g,g',g'')$ be the n times of the maximum of $\|F''\|_\infty + 2\|F'g'\|_\infty + \|Fg'^2\|_\infty + \|Fg''\|_\infty$, $\|F''g^k\|_\infty + 2k\|F'g^{k-1}g'\|_\infty + k\|Fg^{k-1}g'^2\|_\infty + \|Fg^{k-1}g''\|_\infty$, $k = 1,\ldots,n-1$, then

$$|Q_n^{C.T}[f]-I[f]| \leq M(f,f',f'',g,g',g'')\frac{(b-a)h^2}{12|\omega|^{n-2}}.$$

Similarly, by direct computation we can get

$$\begin{aligned} & |Q_n^{C.S}[f]-I[f]| \\ \leq\ & \frac{|Q_0^{C.S}[\sigma_n[f]]-I[\sigma_n[f]]|}{|\omega|^n} + M_1(f,f',f'',f^{(3)},f^{(4)},g,g',g'',g^{(3)},g^{(4)})\frac{(b-a)h^4}{180|\omega|^{n-2}} \end{aligned}$$

$$\begin{aligned} & |Q_n^{C.GS}[f]-I[f]| \\ \leq\ & \frac{|Q_0^{C.GS}[\sigma_n[f]]-I[\sigma_n[f]]|}{|\omega|^n} + M_1(f,f',f'',f^{(3)},f^{(4)},g,g',g'',g^{(3)},g^{(4)})\frac{(b-a)h^4}{1080|\omega|^{n-2}}. \end{aligned}$$

If we use the following estimation about the composite Simpson's rule

$$|\int_a^b s(x)dx - Q^{C.S}[s]| \leq \frac{(b-a)h^3}{196}\|s^{(3)}\|_\infty$$

we can get the estimation for the 2nd remainder of Taylor expansion:

Corollary 2.5. *Suppose that $f, g : [a,b] \to \mathbf{R}$ are smooth functions and $g'(x) \neq 0$ for all $x \in [a,b]$. Then for $n = 3$ and $1 \leq |\omega|$*

$$|Q_n^{C.S}[f] - I[f]| \leq M_2(f, f', f'', f''', g, g', g'', g''')\frac{(b-a)h^3}{196|\omega|} \tag{2.8}$$

where $M_2(f, f', f'', f''', g, g', g'', g''')$ is a constant without reference to ω and h.

In the following we consider some numerical examples (See Figure 2-3, Table 1) based on the following expansions ($n = 2, 3$):

$$\begin{aligned} I(f) &= \textstyle\int_a^b f(x)e^{i\omega g(x)}dx = \frac{1}{i\omega}\int_a^b \frac{f(x)}{g'(x)}d(e^{i\omega g(x)} - 1) \\ &= \frac{1}{i\omega}\frac{f(x)}{g'(x)}(e^{i\omega g(x)} - 1)|_a^b - \frac{1}{i\omega}\textstyle\int_a^b \sigma_1[f](x)(e^{i\omega g(x)} - 1)dx (\Rightarrow Q_1) \\ &= \textstyle\sum_{k=1}^n \left\{ \frac{(-1)^{k+1}}{(i\omega)^k}\frac{\sigma_{k-1}[f](b)}{g'(b)}T_{k-1}(b) - \frac{(-1)^{k+1}}{(i\omega)^k}\frac{\sigma_{k-1}[f](a)}{g'(a)}T_{k-1}(a) \right\} \\ &\quad + \frac{(-1)^n}{(i\omega)^n}\textstyle\int_a^b \sigma_n[f](x)T_{n-1}(x)dx (\Rightarrow Q_n) \end{aligned} \tag{2.9}$$

Example 2.1. Let's consider the numerical quadrature for $\int_0^1 \cos(x)e^{i\omega x}dx$ by the new method (2.9) with $n = 2, 3$ together with the classic methods (see Figure 2), compared with the classical composite trapezoidal rule, composite two-point Gauss-Legendre quadrature. The new methods are quite efficient and the classical methods suffer from difficulty.

Example 2.2. Let's consider the numerical quadrature for $\int_0^1 e^x e^{i\omega(x+\sin(x))}dx$ by the method (2.9) with $n = 2$ together with the composite two-point Gauss-Legendre quadrature compared with the adaptive Simpson quadrature in Matlab6.5 (see Table 1).

Table 1: Numerical quadrature for $\int_0^1 e^x e^{i\omega(x+\sin(x))}dx$

ω	quad(MATLAB6.5)	Exact	$Q_2^{C.GS}$: $h = 0.01$
1	.6495644493+1.3494953139i	.6495644665+1.349495311i	.6495644656+1.349495311i
2	–.6611284693+.6872965838i	–.6611284681+.6872965836i	–.6611284681+.6872965830i
10	–.0601697795–.1149863480i	–.06016978312-.1149863477i	–.06016978341–.1149863480i
10^2	.0164011227+.0114459817i	.01640115897+.01144595130i	.01640116287+.01144595333i
10^3	.0530404104–.1032264282i	$.843801824e-3-.1049835167e-2i$	$.8438135711e-3-.1049923850e-2i$
10^4	.7775528393–.5659099847i	$-.1703373117e-3+.3845531294e-5i$	$-.1703359420e-3+.384591075e-5i$
10^5	–.4489371543+.0153986495i	$-.8406404573e-5-.1051689603e-4i$	$-.8406394455e-5-.1051691982e-4i$
10^6	–.08559189332-.1124372529i	$.1708754572e-5+.5889246310e-7i$	$.1708754574e-5+.5889248448e-7i$

Remark 2.6. 1. For $Q_0^{C.S}[\sigma_n[f]] - I[\sigma_n[f]]$ and $Q_0^{C.GS}[\sigma_n[f]] - I[\sigma_n[f]]$, note that $|h\sum_{l=1}^m a_l F(x_l)\, e^{i\omega g(x_l)}| \leq (b-a)\|F\|_\infty$ for any nonnegative sequence $\{a_l\}_1^m$ satisfying $\sum_{l=1}^m ha_l = b - a$, then

$$|\int_a^b \sigma_n[f](x)e^{i\omega g(x)} - Q_0^{C.S}[\sigma_n[f]]| \leq 2(b-a)\|\sigma_n[f]\|_\infty,$$

$$|\int_a^b \sigma_n[f](x)e^{i\omega g(x)} - Q_0^{C.GS}[\sigma_n[f]]| \leq 2(b-a)\|\sigma_n[f]\|_\infty.$$

Hence together with (2.5), (2.6) and (2,7), these approaches are suitable for $\omega = O(1)$ and the rate of decay of the error can be increased as frequency grows.

2. Similarly the estimations about the higher order $n(\geq 3)$ Taylor expansion is $|Q_n^{C.T}[f] - I[f]| \leq M(f, f', f'', g, g', g'')\frac{(b-a)h^2}{12|\omega|}$.

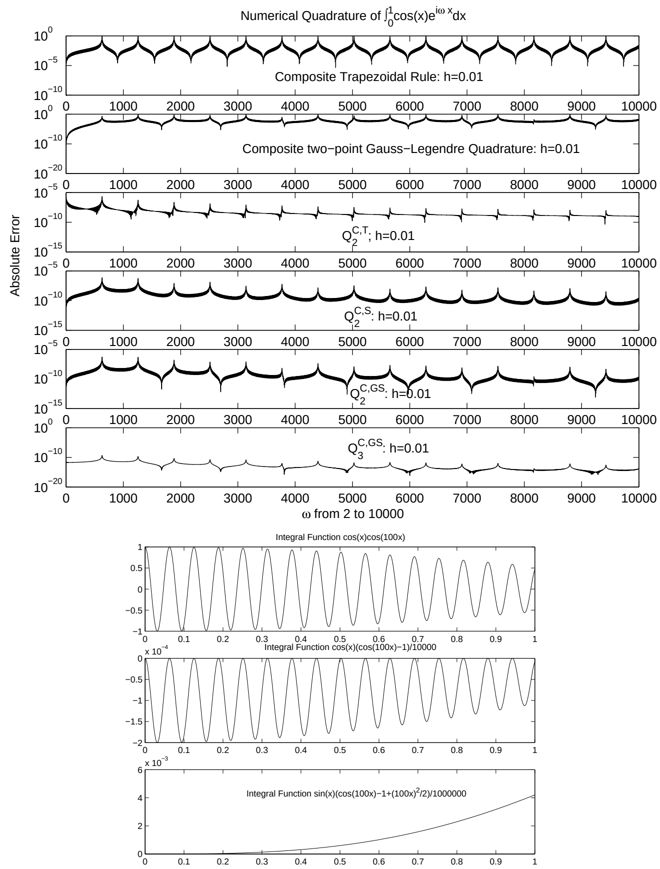

Figure 2: Error analysis of the integration $\int_0^1 \cos(x)e^{i\omega x}dx$. Note that the real part of the integral function in $Q_2^{C.GS}$ is highly oscillatory but keeps the same sign.

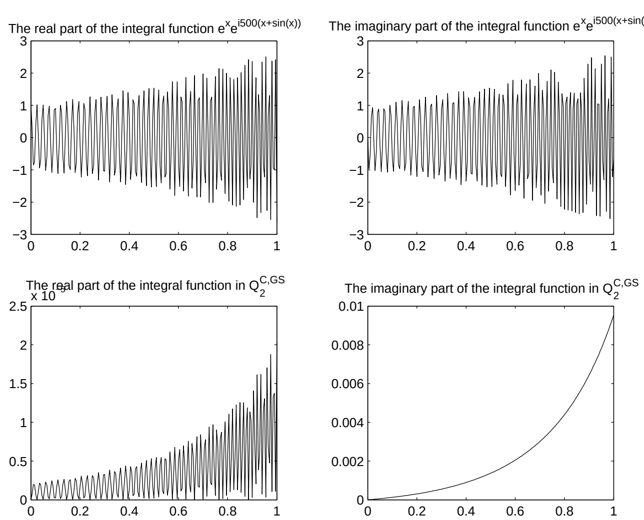

Figure 3: Integrand functions

To get higher order of approach about large ω, we consider the following integral formula:

$$\begin{aligned} I[f] &= \int_a^b f(x)e^{i\omega g(x)}dx \\ &= \sum_{j=1}^s \left\{ \frac{(-1)^{j+1}}{(i\omega)^j}\frac{\sigma_{j-1}[f](b)}{g'(b)} - \frac{(-1)^{j+1}}{(i\omega)^j}\frac{\sigma_{j-1}[f](a)}{g'(a)} \right\} \\ &\quad + \frac{(-1)^s}{(i\omega)^s}\int_a^b \sigma_s[f](x)e^{i\omega g(x)}dx \\ &= \sum_{j=1}^s \left\{ \frac{(-1)^{j+1}}{(i\omega)^j}\frac{\sigma_{j-1}[f](b)}{g'(b)} - \frac{(-1)^{j+1}}{(i\omega)^j}\frac{\sigma_{j-1}[f](a)}{g'(a)} \right\} \\ &\quad + (-1)^s \sum_{k=1}^n \left\{ \frac{(-1)^{k+1}}{(i\omega)^{s+k}}\frac{\sigma_{s+k-1}[f](b)}{g'(b)}T_{k-1}(b) - \frac{(-1)^{k+1}}{(i\omega)^{s+k}}\frac{\sigma_{k-1}[f](a)}{g'(a)}T_{k-1}(a) \right\} \\ &\quad + \frac{(-1)^{s+n}}{(i\omega)^{s+n}}\int_a^b \sigma_{s+n}[f](x)T_{n-1}(x)dx \end{aligned} \tag{2.10}$$

To quadrature the integral $\int_a^b \sigma_{s+n}[f](x)T_{n-1}(x)dx$ we also consider the composite trapezoidal rule, the composite Simpson rule and the composite 2-point Gauss-Legendre quadrature. Denote the corresponding method by $INQ_{n,s}^{C.T}[f]$, $INQ_{n,s}^{C.S}[f]$ and $INQ_{n,s}^{C.GS}[f]$. According to Theorem 2.4, we can get the following

Theorem 2.7. *Suppose that $f, g : [a,b] \to \mathbf{R}$ are smooth functions and $g'(x) \neq 0$ for all $x \in [a,b]$. Then for $s \geq 1$, $n = 2,3$ and $1 \leq |\omega|$*

$$|Q_n^{C.T}[f] - I[f]| \leq M(f,f',f'',g,g',g'')\frac{(b-a)h^2}{12|\omega|^{s+n-2}} \tag{2.11}$$

$$\begin{aligned} &|Q_n^{C.S}[f] - I[f]| \\ \leq\ &\frac{|Q_0^{C.S}[\sigma_n[f]] - I[\sigma_n[f]]|}{|\omega|^{s+n}} + M_1(f,f',f'',f^{(3)},f^{(4)},g,g',g'',g^{(3)},g^{(4)})\frac{(b-a)h^4}{180|\omega|^{s+n-2}} \end{aligned} \tag{2.12}$$

$$\begin{aligned} &|Q_n^{C.GS}[f] - I[f]| \\ \leq\ &\frac{|Q_0^{C.GS}[\sigma_n[f]] - I[\sigma_n[f]]|}{|\omega|^{s+n}} + M_1(f,f',f'',f^{(3)},f^{(4)},g,g',g'',g^{(3)},g^{(4)})\frac{(b-a)h^4}{1080|\omega|^{s+n-2}}. \end{aligned} \tag{2.13}$$

Example 2.3. Let's consider the numerical quadrature for $\int_0^1 \cos(x)e^{i\omega(x+\sin(x))}dx$ by the method (2.10) with $n = 1$ or 2 and $s = 1$ together with the composite two-point Gauss-Legendre quadrature (see Figure 4). The quadrature is more efficient than the corresponding (2.9).

In case $g'(x_0) = 0$ and $g'(x) \neq 0$ for all $x \neq x_0, x \in [a,b]$, without loss of generality, assume $g(x_0) = 0$ since

$$I[f] = e^{ig(x_0)}\int_a^b f(x)e^{i\omega(g(x)-g(x_0))}dx$$

Suppose that there exists an positive integer r such that $g^{(r)}(x_0) \neq 0$. It is not difficult to show that x_0 is a removed discontinuous point of $\sigma_n[f](x)T_{n-1}(x)$ and $\frac{(-1)^{m+1}}{(i\omega)^m}\frac{\sigma_{m-1}[f](x_0)}{g'(x_0)}\ T_{m-1}$ $(x_0) = 0$ for all n. Hence the following formula is also true

$$\begin{aligned} I[f] &= \int_a^b f(x)e^{i\omega g(x)}dx \\ &= \sum_{m=1}^n \left\{ \frac{(-1)^{m+1}}{(i\omega)^m}\frac{\sigma_{m-1}[f](b)}{g'(b)}T_{m-1}(b) - \frac{(-1)^{m+1}}{(i\omega)^m}\frac{\sigma_{m-1}[f](a)}{g'(a)}T_{m-1}(a) \right\} \\ &\quad + \frac{(-1)^n}{(i\omega)^n}\int_a^b \sigma_n[f](x)T_{n-1}(x)dx \end{aligned}$$

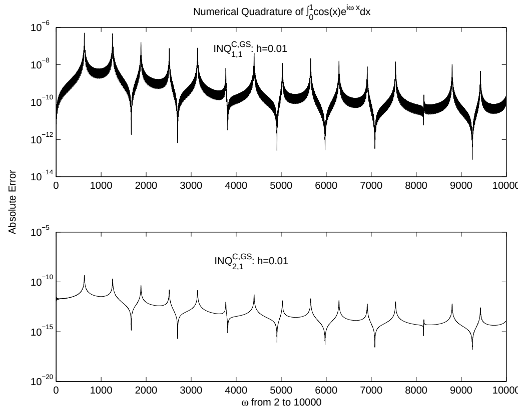

Figure 4: Numerical quadrature of integral $\int_0^1 \cos(x)e^{i\omega x}dx$ by (2.10)

Numerical examples show that in this case the method is also accurate for the number of composite subintervals is larger than $10\omega^{\frac{1}{2}}$ and can be efficiently used for Bessel transforms (cf. [11]).

Example 2.4. Let's consider the numerical quadrature for $\int_0^1 e^{i\omega x^2}dx$, $\int_0^1 e^{i\omega\cos(x)}dx$ and $\int_0^{2\pi} log(x)\sin(\omega x)dx$ by the method (2.9) with $n = 1, 2$ or 3, together with the composite two-point Gauss-Legendre quadrature (see Table 2-4).

Table 2: The Real Part of Numerical Quadrature for $\int_0^1 e^{i\omega x^2}dx$

ω	Exact	$Q_1^{C,GS}: h = 10^{-4}$	$Q_1^{C,GS}: h = 10^{-5}$
1	.9045242379	.9045242379	.9045242379
10^2	.06011251848	.06011251848	.06011251848
10^4	.006251292600	.006251292338	.006251292348
10^5	.001981842195	.001981930823	.001981842418
10^6	.0006264818245	.0006266791662	.0006264817632
10^8	.00006267045435	.00006294843361	.00006269030065

Table 3: The Real Part of Numerical Quadrature for $\int_0^1 e^{i\omega cos(x)}dx$

ω	Exact	$Q_1^{C,GS}: h = 10^{-3}$	$Q_1^{C,GS}: h = 10^{-4}$
1	.6597810536	.6597810536	.6597810536
10	$-.3019277972$	$-.3019277972$	$-.3019277972$
10^3	.03899526001	.03899526012	.03899526001
10^4	$-.01108796294$	$-.01108812902$	$-.01108796293$
10^5	$-.002711212278$	$-.002702214698$	$-.002711249511$
10^6	.0005211644843	.000532892649	.0005210915902

Table 4: The Numerical Quadrature for $\int_0^{2\pi} log(x)\sin(\omega x)dx$

ω	1	10	100	1000	10000
$Q^{C.GS}_{2(h=0.001)}$	-2.43765339	-0.47179307	-0.07020264	-0.00926222	-0.00107962
$Q^{C.GS}_{3(h=0.001)}$	-2.43765339	-0.47179307	-0.07020265	-0.00931860	-0.00116613
Exact	-2.43765339	-0.47179307	-0.07020266	-0.00932285	-0.00116254

3. Quadrature of Integral $\int_a^b f(x)e^{ig(\omega,x)}dx$

Suppose that $f, g(\omega, x)$ are suitably smooth and $\lim_{\omega\to\infty} g'_x(\omega, x) = \infty$ for all x in $[a, b]$.

Let

$$\sigma_0[f](x) = f(x),$$

$$\sigma_{k+1}[f](x) = \frac{d}{dx}\frac{\sigma_k[f](x)}{g'_x(\omega,x)}, k = 0, 1, 2, \cdots,$$

and the nth order Taylor polynomials of $e^{ig(\omega,x)}$

$$F_0 = 1, \quad F_n(ig(\omega,x)) = 1 + ig(\omega,x) + \frac{(ig(\omega,x))^2}{2!} + \frac{(ig(\omega,x))^3}{3!} + \cdots + \frac{(ig(\omega,x))^n}{n!}$$

and

$$T_n(x) = e^{ig(\omega,x)} - F_n(ig(\omega,x)).$$

Notice that $T_n(x)$ is not highly oscillatory. Let

$$\sigma_0[f](x) = f(x),$$

$$\sigma_{k+1}[f](x) = \frac{d}{dx}\frac{\sigma_k[f](x)}{g'_x(\omega,x)}, k = 0, 1, 2, \cdots.$$

Similar to Lemma 2.1, we have

Lemma 3.1. *Suppose that $f(x), g(\omega,x)$ are smooth functions and $g'_x(\omega,x) \neq 0$. Then*

$$\begin{aligned} I[f] &= \int_a^b f(x)e^{ig(\omega,x)}dx \\ &= \sum_{m=1}^n \left\{ \frac{(-1)^{m+1}}{i^m}\frac{\sigma_{m-1}[f](b)}{g'_x(\omega,b)}T_{m-1}(b) - \frac{(-1)^{m+1}}{i^m}\frac{\sigma_{m-1}[f](a)}{g'_x(\omega,a)}T_{m-1}(a)\right\} \\ &\quad -\frac{1}{i^n}\int_a^b \sigma_n[f](x)T_{n-1}(x)dx \end{aligned} \tag{3.1}$$

To calculate the integral $\int_a^b \sigma_n[f](x)T_{n-1}(x)dx$ we also consider the composite 2-point Gauss-Legendre quadrature based on the lower Taylor expansions (3.1) or (2.10). Denote the corresponding method by $GQ_n^{C.GS}[f]$ and $GINQ_{n,s}^{C.GS}[f]$ respectively.

Example 3.1. We give a numerical experiment to calculate the integra $I = \int_0^1 e^{i(\omega x+\omega^2x^2)}dx$. The adaptive Simpson's quadrature and the composite two-point Gauss-Legendre quadrature for $\int_0^1 e^{i(\omega x+\omega^2x^2)}dx$ are unstable and completely failure when $\omega \geq 30$ and $\omega \geq 1000$ respectively. And the asymptotic method in [5] with the 2nd expansion is also completely failure.

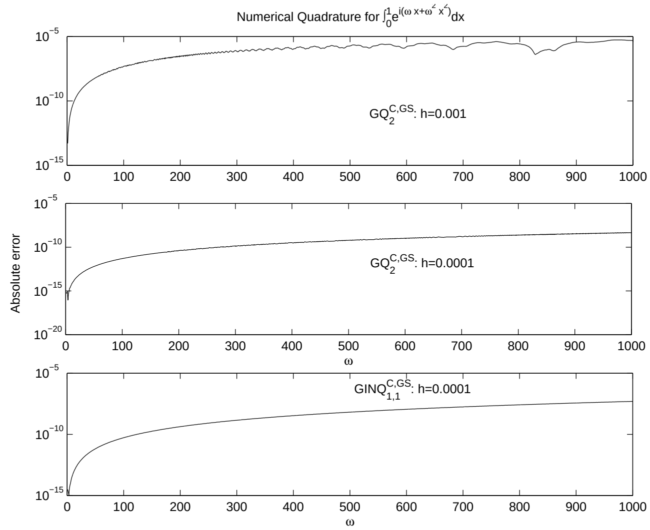

Figure 5: Error analysis of the integration $\int_0^1 e^{i(\omega x+\omega x^2)}dx$ by (3.1) with $n = 2$ or (2.10) with $n = s = 1$ with the composite two-point Gauss-Legendre quadrature, and by (2.10) with $n = 1$ and $s = 1$ with the composite two-point Gauss-Legendre quadrature.

Table 5: The Numerical Quadrature for $\int_0^1 e^{i(\omega x+\omega^2 x^2)}dx$

ω	quad(in MATLAB6.5)	Exact	$Q_2^{C.GS}: h = 0.001$	$Q_2^{C.GS}: h = 0.0001$
1	$.5720708 + .6143220i$	$.5720708 + .6143220i$	$.5720708 + .6143220i$	$.5720708 + .6143220i$
10	$.0268622 + .0582458i$	$.0268623 + .0582457i$	$.0268623 + .0582457i$	$.0268623 + .0582457i$
20	$.0125165 + .0260591i$	$.0125166 + .0260592i$	$.0125166 + .0260592i$	$.0125166 + .0260592i$
30	$-.0203178 - .0101731i$	$.0090652 + .0172849i$	$.0090652 + .0172849i$	$.0090652 + .0172849i$
50	$-.0253604 - .0101731i$	$.0052465 + .0105862i$	$.0052465 + .0105862i$	$.0052465 + .0105862i$
10^2	$-.0656340 - .1018820i$	$.0027160 + .0053973i$	$.0027160 + .0053973i$	$.0027160 + .0053973i$
10^3	$-.0846111 + .1686435$	$.0002708 + .0005345i$	$.0002665 + .0005321i$	$.0002708 + .0005345i$

ω	Exact	$GINQ_{1,1}^{C.GS}$:h=10^{-4}	Gauss:$h = 10^{-4}$	Asymptotic method:s=2
1	$.5720708 + .6143220i$	$.5720708 + .6143220i$	$.5720708 + .6143220i$	$2.3339248 + 1.0713607i$
10	$.0268623 + .0582457i$	$.0268623 + .0582457i$	$.0268623 + .0582457i$	$.1998109 + .1047582i$
10^2	$.0027160 + .0053973i$	$.0027160 + .0053973i$	$.0027160 + .0053971i$	$.0200109 + .0100485i$
10^3	$.0002708 + .0005345i$	$.0002708 + .0005345i$	$.0034855 + .0035744i$	$.0020003 + .0009996i$

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