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# A TWO-LEVEL FINITE ELEMENT GALERKIN METHOD FOR THE NONSTATIONARY NAVIER-STOKES EQUATIONS I: SPATIAL DISCRETIZATION *1)

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**Abstract**

In this article we consider a two-level finite element Galerkin method using mixed finite elements for the two-dimensional nonstationary incompressible Navier-Stokes equations. The method yields a $H^1$-optimal velocity approximation and a $L^2$-optimal pressure approximation. The two-level finite element Galerkin method involves solving one small, nonlinear Navier-Stokes problem on the coarse mesh with mesh size $H$, one linear Stokes problem on the fine mesh with mesh size $h << H$. The algorithm we study produces an approximate solution with the optimal, asymptotic in $h$, accuracy.

*Key words*: Navier-Stokes equations, Mixed finite element, Error estimate, Finite element method.

## 1. Introduction

Two-level finite element Galerkin method is an efficient numerical method for solving nonlinear partial differential equations, e.g., see Xu [24, 25] for steady semi-linear elliptic equations, Layton [14], Ervin, Layton and Maubach [5], Layton and Lenferink [15] and Layton and Tobiska [16] for the steady Navier-Stokes equations. This method is closely related to the nonlinear Galerkin method [1,10, 17-19, 22] and recently developed in [7,21] to solve the nonstationary Navier-Stokes equations. However,it is well known [1, 10, 17-19] that a defect of the nonlinear Galerkin methods is needed to approximate solution $u_h$ as the large eddy component $y^H$ and the small eddy component $z^h$ and solve the unknown components $y^H$ and $z^h$ simultaneously, that is to solve a coupled nonlinear and linear equations and increase computing price.

In the case of the nonlinear evolution problem, the basic idea of the two-level method is to find an approximation $u_H$ by solving a nonlinear problem on a coarse grid with grid size $H$ and find an approximation $u^h$ by solving a linearized problem about the known approximation $u_H$ on a fine grid with grid size $h$. The semi-discretization in space of the 3D time-dependent Navier-Stokes problem by the two-level method is considered in [7]. Furthermore, the fully discretization in space-time of the 2D and 3D time-dependent Navier-Stokes problem by the two-level method is analyzed in [21], where the local error estimates, stability and convergence are proved, but the global error estimates do not provided. In fact, this scheme is of the global first-order accurate with respect to the time step size $\tau$.

In this report we consider continuity the two-level method used in [21] for the nonstationary, incompressible Navier-Stokes equations and give the error estimates of optimal order for the approximate velocity and pressure. If the equations is discreted by the standard finite element Galerkin method, there will be a large system of nonlinear algebraic equations to be solved.

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To overcome this difficult, we will apply a two-level finite element Galerkin method for solving the nonstationary Navier-Stokes equations in the framework of mixed finite elements. This will yield a small system of nonlinear algebraic equations and a large system of linear algebraic to be solved, i.e., this method can save some computational work. For the standard finite element Galerkin method, the discrete velocity $u_h(\cdot,t)$ and pressure $p_h(\cdot,t)$ are determined in finite element spaces denoted respectively by $X_h$ and $M_h$ which satisfy the so-called inf-sup condition ( see [3,8,11] ). Our two-level finite element Galerkin method consists in

- Finding $(u_H, p_H) \in (X_H, M_H)$ by solving the nonlinear Navier-Stokes problem on the coarse mesh with width $H$;
- Finding $(u^h, p^h) \in (X_h, M_h)$ by solving the linear Stokes problem based on $(u_H, p_H)$ on the fine mesh with width $h << H$.

In this paper, our main results are the following results:

$$\|u^h(t) - u_h(t)\|_{H^1} \le \kappa(t) H^2 \quad \forall t \ge 0, \tag{1.1}$$

$$\|p^h(t) - p_h(t)\|_{L^2} \le \kappa(t) H^2 \quad \forall t > 0\,, \tag{1.2}$$

where $(u_h, p_h)$ is the standard finite element Galerkin approximation based on $(X_h, M_h)$ which satisfies the following error estimates:

$$\|u(t) - u_h(t)\|_{H^1} \le \kappa(t)h, \forall t \ge 0, \tag{1.3}$$

$$\|p(t) - p_h(t)\|_{L^2} \le \kappa(t)h, \forall t > 0. \tag{1.4}$$

These estimates indicate that the two-level finite element Galerkin method gives the same order of approximation as the standard finite element Galerkin method if we choose $H = O(h^{1/2})$. However, in our method, the nonlinearity is only treated on the coarse grid and only the linear problem needs to be solved on the fine grid. Of course, the comparison with the standard finite element Galerkin method, the two-level finite element Galerkin method should be made more precise by studying questions related to time discretization and computational implementation. These will be addressed in the several continuations of this work.

## 2. The Navier-Stokes Equations

Let $\Omega$ be a bounded domain in $R^2$ assumed to have a Lipschitz-continuous boundary $\Gamma$ and to satisfy a further condition stated in (2.5) below. We consider the time dependent Navier-Stokes equations describing the flow of a viscous incompressible fluid confined in $\Omega$:

$$\frac{\partial u}{\partial t} - \nu \Delta u + (u \cdot \nabla)u + \nabla p = f \quad \text{ in } \Omega\,,\, t > 0\,, \tag{2.1}$$

$$\operatorname{div} u = 0 \quad \text{ in } \Omega\,,\, t > 0\,, \tag{2.2}$$

$$u = 0 \quad \text{ on } \Gamma\,,\, t > 0\,, \tag{2.3}$$

$$u(0) = \bar{u}_0 \quad \text{ in } \Omega\,, \tag{2.4}$$

where $u = (u_1, u_2)$ is the velocity, $p$ is the pressure, $f$ represents the density of body forces, $\nu > 0$ is the viscosity and $\bar{u}_0$ is the initial velocity.

In order to introduce a variational formulation, we set

$$X = H_0^1(\Omega)^2\,,\, Y = L^2(\Omega)^2\,,$$

and

$$M = L_0^2(\Omega) = \{q \in L^2(\Omega)\,;\, \int_\Omega q(x)dx = 0\}\,.$$

We denote by $(\cdot,\cdot)$, $|.|$ the inner product and norm on $L^2(\Omega)$ or $L^2(\Omega)^2$. The space $H_0^1(\Omega)$ and $X$ are equipped with their usual scalar product and norm

$$((u,v)) = (\nabla u, \nabla v)\,, \; \|u\| = ((u,u))^{1/2}\,.$$

We define the continuous bilinear forms $a(.,.)$ and $d(.,.)$ on $X \times X$ and $X \times M$ respectively by

$$a(u,v) = \nu((u,v)), \quad \forall u,v \in X\,,$$

and

$$d(v,q) = (q, \operatorname{div} v), \quad \forall v \in X, q \in M\,.$$

Next, we introduce the closed subset $V$ of $X$ given by

$$V = \{v \in X\,;\, d(v,q) = 0, \quad \forall q \in M\} = \{v \in X\,;\, \operatorname{div} v = 0 \quad \text{in } \Omega\}\,,$$

and we denote by $H$ the closure of $V$ in $Y$. One can show (see [3,8,11,13]) that

$$H = \{v \in Y\,;\, \operatorname{div} v = 0 \text{ in } \Omega \text{ and } v \cdot n = 0 \text{ on } \Gamma\}\,,$$

where $n$ denotes the unit outward normal to $\Gamma$. Also, we denote by $A$ the unbounded linear operator on Y given by $Au = -\Delta u$. We assume that $\Omega$ is such that the domain of $A$ is given by

$$D(A) = H^2(\Omega)^2 \cap X\,. \tag{2.5}$$

For instance, (2.5) holds if $\Gamma$ is of class $C^2$ or if $\Omega$ is a convex plane polygonal domain, see [9].

Moreover, we define the trilinear form

$$\begin{aligned} b(u,v,w) &= ((u\cdot\nabla)v, w) + \frac{1}{2}((\operatorname{div} u)v, w) \\ &= \frac{1}{2}((u\cdot\nabla)v, w) - \frac{1}{2}((u\cdot\nabla)w, v), \quad \forall u,v,w \in X\,. \end{aligned}$$

For a given $f \in L^\infty(R^+\,;\,Y)$ and a given $\bar{u}_0 \in H$, the variational formulation of (2.1)-(2.4) reads: find a pair $(u,p)$ with

$$u \in L^\infty(R^+\,;\,H) \cap L^2(0,T\,;\,V)\,, u_t \in L^2(0,T,V')\,, p \in D'(\Omega \times (0,T)), \quad \forall T > 0\,,$$

such that

$$(u_t, v) + a(u,v) + b(u,u,v) - d(v,p) + d(u,q) = (f,v), \quad \forall (v,q) \in (X,M)\,, \tag{2.6}$$

$$u(0) = \bar{u}_0\,. \tag{2.7}$$

It is classical [8, 11, 13, 23] that (2.6)-(2.7) possesses a unique solution $(u,p)$ which satisfies the following regularity results Lemma 2.1 below.

**Lemma 2.1.** *Let $f \in L^\infty(R^+\,;\,Y)$, $f_t \in L^\infty(R^+\,;\,Y)$ and $\bar{u}_0 \in D(A) \cap V$ be given. Then, the solution $(u,p)$ of (2.6)-(2.7) satisfies*

$$|Au(t)| + \|u_t(t)\| + |Au_t(t)| \le \kappa(t) \quad \forall t > 0\,. \tag{2.8}$$

*Here $\kappa(t)$ denotes a generic constant depending on the data $(\Omega, \nu, u_0, f_\infty, t)$ and is continuous with respect to time,*

$$f_\infty = \sup_{t\ge 0}\{|f(t)| + |f_t(t)|\}\,,\; f_t = \frac{df}{dt};$$

*in the later case, such a constant which may stand for different values at its different occurrences.*

Hereafter, we will denote by $c$ a generic constant depending on the data $(\Omega, \nu, f)$ and $c_0, c_1, \cdots$, denote some positive constants depending only on $\Omega$. Finally, we also will use the following Poincare inequality:

$$\lambda_1 |v|^2 \le \|v\|^2, \forall v \in X, \tag{2.9}$$

where $\lambda_1$ is the first eigenvalue of the operator $A$.

## 3. Finite Element Galerkin Approximation

From now on, $h$ will be a real positive parameter tending to 0. We let $\tau_h(\Omega)$ be a uniformly regular mesh of $\Omega$ made of $n$-simplices $K$ with mesh size $h$. We construct velocity-pressure finite element spaces $(X_h, M_h) \subset (X, M)$ based upon the mesh $\tau_h(\Omega)$ and define the subspace $V_h$ of $X_h$ given by

$$V_h = \{v_h \in X_h\,;\, d(v_h, q_h) = 0, \quad \forall q_h \in M_h\}\,. \tag{3.1}$$

Let $P_h : Y \to X_h$ and $\rho_h : M \to M_h$ denote the $L^2$-orthogonal projections defined respectively by

$$(P_h v, v_h) = (v, v_h), \quad \forall v \in Y\,,\, v_h \in X_h\,,$$

and

$$(\rho_h q, q_h) = (q, q_h), \quad \forall q \in M\,,\, q_h \in M_h\,.$$

We assume that the couple $(X_h, M_h)$ satisfies the following approximation properties:

for each $v \in D(A)$ and $q \in H^1(\Omega) \cap M$, there exist approximations $I_h v \in X_h$ and $J_h q \in M_h$ such that

$$\|v - I_h v\| \le ch|Av|\,,\ |q - J_h q| \le ch\|q\|_1\,, \tag{3.2}$$

together with the inverse inequality

$$\|v_h\| \le ch^{-1}|v_h|, \quad \forall v_h \in X_h\,, \tag{3.3}$$

and the so-called inf-sup inequality: for each $q_h \in M_h$, there exists $v_h \in X_h, v_h \neq 0$ such that

$$d(v_h, q_h) \ge \bar{\beta}|q_h|\|v_h\|\,, \tag{3.4}$$

where $\bar{\beta} > 0$ is a constant independent of $h$, where $\|.\|_1$ denotes the usual norm of the Sobolev space $H^1(\Omega)$.

The following properties which are classical consequences of (3.2)-(3.4) (see [1, 3, 8]) will be very useful

$$\|P_h v\| \le c\|v\|, \quad \forall v \in X\,, \tag{3.5}$$

$$|v - P_h v| \le ch\|v\|, \quad \forall v \in X\,, \tag{3.6}$$

$$|v - P_h v| + h\|v - P_h v\| \le ch^2|Av|, \quad \forall v \in D(A). \tag{3.7}$$

It is well known that $b(\cdot,\cdot,\cdot)$ satisfies the following properties (see [1,8, 11, 16, 18]):

$$b(u, v, w) = -b(u, w, v)\,, \tag{3.8}$$

$$\begin{aligned}|b(u, v, w)| &\le c_0|u|^{1/2}\|u\|^{1/2}\|v\||w|^{1/2}\|w\|^{1/2}\\ &+ c_0\|u\||v|^{1/2}\|v\|^{1/2}|w|^{1/2}\|w\|^{1/2}, \forall u, v, w \in X,\end{aligned} \tag{3.9}$$

$$|b(u, v, w)| \le c_0\|u\|\|v\|\|w\|, \forall u, v, w \in X. \tag{3.10}$$

The Galerkin approximation of (2.6)-(2.7) based on $(X_h, M_h)$ reads: find $(u_h, p_h) \in H^1(0, T;$ $X_h) \times L^2(0, T\,;\, M_h)\,,\, \forall T > 0\,,$ such that

$$\begin{aligned}&(u_{h,t}, v_h) + a(u_h, v_h) + b(u_h, u_h, v_h) - d(v_h, p_h) + d(u_h, q_h)\\ &= (f, v_h), \quad \forall (v_h, q_h) \in (X_h, M_h)\,,\end{aligned} \tag{3.11}$$

$$u_h(0) = P_h \bar{u}_0\,. \tag{3.12}$$

The following error estimates are classical (see [1, 3,11,20]).

**Theorem 3.1.** *Under the assumptions (3.2)-(3.4), let* $f \in L^\infty(R^+; Y)$ , $f_t \in L^\infty(R^+; Y)$ *and* $\bar{u}_0 \in D(A) \cap V$ *be given. Then, (3.11)-(3.12) possesses a unique solution* $(u_h, p_h)$ *and the following error estimates hold:*

$$|u(t) - u_h(t)| + |u_t(t) - u_{h,t}(t)| + h\|u(t) - u_h(t)\| \le \kappa(t)h^2, \quad \forall t \ge 0\,, \tag{3.13}$$

$$|p(t) - p_h(t)| \le \kappa(t)h, \quad \forall t > 0\,. \tag{3.14}$$

We conclude this section by giving some examples of subspaces $X_h$ and $M_h$ such that the assumptions (3.2)-(3.4) are satisfied. Let $\Omega$ be a polygonal domain and let $\{\tau_h\}, h > 0$, be a uniformly regular family of triangulations of $\Omega$ made of $n$-simplices $K$ with diameters bounded by $h$. For any integer $l$, we denote by $P_l(K)$ the space of polynomials of degree less than or equal to $l$ on $K$.

**Example 3.1 (Girault-Raviart[8]).** *We set*

$$X_h = \{v_h \in C^0(\overline{\Omega})^2 \cap X\,;\, v_h|_K \in P_2(K)^2\,, \quad \forall K \in \tau_h\}\,,$$

$$M_h = \{q_h \in M\,;\, q_h|_K \in P_0(K)\,, \quad \forall K \in \tau_h\}\,.$$

**Example 3.2 (Bercovier-Pironneau[2]).** *We consider the triangulation* $\tau_{h/2}$ *obtained by dividing each triangle of* $\tau_h$ *in four triangles (by joining the mid-sides). We set*

$$X_h = \{v_h \in C^0(\overline{\Omega})^2 \cap X\,;\, v_h|_K \in P_1(K)^2\,, \quad \forall K \in \tau_{h/2}\}\,,$$

$$M_h = \{q_h \in C^0(\overline{\Omega}) \cap M\,;\, q_h|_K \in P_1(K)\,, \quad \forall K \in \tau_h\}\,.$$

## 4. Two-Level Finite Element Galerkin Method

In this section, we choose a coarse mesh width $H$ and a fine mesh width $h$ with $H >> h > 0$, and construct associated conforming finite element spaces $(X_H, M_H)$ and $(X_h, M_h)$, where $(X_H, M_H) \subset (X_h, M_h)$. Now find $(u^h, p^h)$ as follows:

- **Step I: Solve nonlinear problem on coarse mesh**

  Find $(u_H, p_H) \in (X_H, M_H)$ such that, for all $(v, q) \in (X_H, M_H)$

$$(u_{H_t}, v) + a(u_H, v) + b(u_H, u_H, v) - d(v, p_H) + d(u_H, q) = (f, v), \tag{4.1}$$

$$u_H(0) = P_H\bar{u}_0\,. \tag{4.2}$$

- **Step II: Update on fine mesh with linear Stokes problem**

  Find $(u^h, p^h) \in (X_h, M_h)$ such that, for all $(v, q) \in (X_h, M_h)$

$$(u^h_t, v) + a(u^h, v) + b(u_H, u_H, v) - d(v, p^h) + d(u^h, q) = (f, v), \tag{4.3}$$

$$u^h(0) = P_h\bar{u}_0. \tag{4.4}$$

**Remark 4.1.** According to the results stated in section 1, the two-level finite element Galerkin method and the standard finite element Galerkin method have the convergence rates of same order. But, the two-level finite element Galerkin method is more simple than the standard finite element Galerkin method. If fact, the two-level finite element Galerkin method only consists in dealing with the nonlinearity on coarse mesh and the linearity on fine mesh and coarse mesh; while the standard Galerkin method need to deal with the nonlinearity on fine mesh. Due to

$h << H$, our method can save much more computational time than the standard finite element Galerkin method.

The existence and uniqueness of a solution $(u^h, p^h)$ of the two-level finite element Galerkin approximate problem (4.1)-(4.4) will be proved in the next theorem.

**Theorem 4.1.** *Under the assumptions (3.2)-(3.4), let* $f \in L^\infty(R^+; Y)$ , $f_t \in L^\infty(R^+; Y)$ *and* $\bar{u}_0 \in H^2(\Omega)^2 \cap V$ *be given. Then, for* $h < H \leq 1$, *the problem (4.1)-(4.4) possesses a unique solution* $(u^h, p^h)$ *defined for* $t \geq 0$ *with*

$$u^h \in C^\infty(0, T; X_h)\,,\ p^h \in C^\infty(0, T; M_h)\ \textit{for all}\ T > 0.$$

*Proof.* From Theorem 3.1, problem (4.1)-(4.2) admits a unique solution $(u_H, p_H) \in (X_H, M_H)$. Moreover, we can prove that $u_H$ satisfies the following priori estimates:

$$|u_H(t)|^2 + \nu \int_0^t e^{-\nu\lambda_1(t-s)} \|u_H(s)\|^2 ds \leq M_0^2, \tag{4.5}$$

where $M_0$ is a positive constant.

Next, we will prove that problem (4.3)-(4.4) admits a unique solution $(u^h, p^h) \in (X_h, M_h)$. We associate problem (4.3)-(4.4) to the following problem: find $u^h \in V_h$ such that

$$(u_t^h, v) + a(u^h, v) + b(u_H, u_H, v) = (f, v), \quad \forall v \in V_h\,, \tag{4.6}$$

$$u^h(0) = P_h u_0\,. \tag{4.7}$$

Clearly, if $(u^h, p^h)$ is a solution of (4.3)-(4.4), then $u^h$ is a solution of (4.6)-(4.7).

Now, let us denote by $N$ the dimension of $V_h$. We consider a basis $\{e_1, \cdots, e_N\}$ of $V_h$ such that $\{e_1, \cdots, e_N\}$ is a basis of $V_h$ with respect to the scalar product $(\cdot, \cdot)$. We look for a solution of (4.6)-(4.7) of the form

$$u^h(t) = \sum_{j=1}^N g_{j,h}(t) e_j\,.$$

Then (4.6)-(4.7) re-writes

$$\frac{d}{dt} g_{i,h} + \sum_{j=1}^N g_{j,h}(t) a(e_j, e_i) = (f, e_i) - b(u_H, u_H, e_i), 1 \leq i \leq N\,,$$

$$g_{i,h}(0) = g_{i,h,0} \text{ if } P_h u_0 = \sum_{j=1}^N g_{j,h,0} e_j\,.$$

Therefore, the system (4.6)-(4.7) is equivalent to an ODE for the $g_{j,h}$, $1 \leq j \leq N$. The existence and uniqueness of a solution of this problem defined on a maximal interval $[0, T_h)$ follow promptly from standard theorems on the Cauchy problem for ODEs. Also, this solution is of class $C^\infty$ on its domain of definition.

We aim now to show that $T_h = \infty$, thanks to some a priori estimates.

By taking $v = u^h$ in (4.6), we obtain

$$(u_t^h, u^h) + \nu \|u^h\|^2 + b(u_H, u_H, u^h) = (f, u^h)\,.$$

Hence, by using Poincare inequality (2.9), (3.8)-(3.9) and (4.5), we see that

$$\frac{1}{2}\frac{d}{dt}|u^h|^2 + \nu\|u^h\|^2 + b(u_H, u_H, u^h) \leq |f||u^h| \leq \frac{\nu}{8}\|u^h\|^2 + \frac{2}{\lambda_1 \nu}|f|^2\,,$$

$$
\begin{aligned}
|b(u_H,u_H,u^h)| &\le c_0|u_H|\|u_H\|\|u^h\| + c_0|u_H|^{1/2}\|u_H\|^{3/2}|u^h|^{1/2}\|u^h\|^{1/2} \\
&\le c_0\Big(1+\frac{c}{H^{1/2}\lambda_1^{1/4}}\Big)|u_H|\|u_H\|\|u^h\| \\
&\le \frac{\nu}{8}\|u^h\|^2 + c(1+H^{-1})\|u_H\|^2.
\end{aligned}
$$

Therefore, we obtain

$$
\frac{d}{dt}|u^h|^2 + \nu\lambda_1|u^h|^2 + \frac{1}{2}\nu\|u^h\|^2 \le c|f|^2 + c(1+H^{-1})\|u_H\|^2\,, \tag{4.8}
$$

which yields

$$
\begin{aligned}
&\frac{d}{dt}(e^{\nu\lambda_1 t}|u^h|^2) + \frac{1}{2}\nu e^{\nu\lambda_1 t}\|u^h\|^2 \\
&\qquad\qquad \le c(1+H^{-1})e^{\nu\lambda_1 t}(|f|^2 + \|u_H\|^2).
\end{aligned} \tag{4.9}
$$

By integrating (4.9) and using (4.5), we get that for $t \ge 0$,

$$
|u^h(t)|^2 + \frac{1}{2}\nu\int_0^t e^{-\nu\lambda_1(t-s)}\|u^h\|^2 ds \le e^{-\nu\lambda_1 t}|P_h\bar{u}_0|^2 + c(1+H^{-1}). \tag{4.10}
$$

This estimate guarantees that the solution of (4.6)-(4.7) can not blow up in finite time, so that $T_h = \infty$.

Once $u^h$ is obtained as the solution of (4.6)-(4.7), there remains to solve: find $p^h$ such that

$$
d(v,p^h) = (u_t^h, v) + a(u^h, v) + b(u_H, u_H, v) - (f, v), \quad \forall v \in X_h\,. \tag{4.11}
$$

Here, the right-hand side of (4.11) is a functional on $X_h$ which, due to the definition of $u^h$, vanishes on $V_h$. It is classical that the inf-sup condition (3.4) guarantees that (4.11) is uniquely solvable in the space $M_h$. This concludes the proof of Theorem 4.1.

## 5. Error Estimates

In this section, we aim to derive error estimates for the two-level FE Galerkin method presented in section 4. First, we need to introduce the discrete analogue $A_h : X_h \to X_h$ of the Laplace operator given by

$$
(A_h u_h, v_h) = ((u_h, v_h)), \quad \forall u_h\,,\, v_h \in X_h\,.
$$

We will need the following estimates for the trilinear form $b$.

**Lemma 5.1.** *The trilinear form $b$ satisfies the following estimate:*

$$
|b(u_{h_1}, v_{h_2}, w_{h_3})| + |b(v_{h_2}, u_{h_1}, w_{h_3})| \le c_1\|u_{h_1}\|^{1/2}|A_{h_1}u_{h_1}|^{1/2}\|v_{h_2}\||w_{h_3}|\,, \tag{5.1}
$$

$$
|b(u_{h_1}, v_{h_2}, w_{h_3})| + |b(u_{h_1}, w_{h_3}, v_{h_2})| \le c_1|u_{h_1}|\|v_{h_2}\||A_{h_3}w_{h_3}|^{1/2}\|w_{h_3}\|^{1/2}, \tag{5.2}
$$

*for any $u_{h_1} \in X_{h_1}, v_{h_2} \in X_{h_2}, w_{h_3} \in X_{h_3}$, where $X_{h_1}, X_{h_2}$ and $X_{h_3}$ are three finite element spaces corresponding to grid parameters $h_1, h_2$ and $h_3$, respectively.*

*Proof.* To prove (5.1), we will need the discrete analogues of several Sobolev inequalities borrowed from Heywood-Rannacher [11], namely for any $h > 0$,

$$
\|\phi_h\|_{L^6} \le c\|\phi_h\|, \quad \forall \phi_h \in X_h, \tag{5.3}
$$

$$
\|\phi_h\|_{L^\infty} + \|\nabla\phi_h\|_{L^3} \le c\|\phi_h\|^{1/2}\|A_h\phi_h\|^{1/2}, \quad \forall \phi_h \in X_h. \tag{5.4}
$$

Moreover, we note that for any $u_{h_1} \in X_{h_1}, v_{h_2} \in X_{h_2}$ and $w_{h_3} \in X_{h_3}$

$$
|b(u_{h_1}, v_{h_2}, w_{h_3})| \le c\|u_{h_1}\|_{L^\infty}\|v_{h_2}\||w_{h_3}| + c\|\nabla u_{h_1}\|_{L^3}\|v_{h_2}\|_{L^6}|w_{h_3}|,
$$

$$|b(v_{h_2},u_{h_1},w_{h_3})|\le c\|v_{h_2}\|_{L^6}\|\nabla u_{h_1}\|_{L^3}|w_{h_3}|+c\|v_{h_2}\|\|u_{h_1}\|_{L^\infty}|w_{h_3}|,$$
$$|b(u_{h_1},v_{h_2},w_{h_3})|\le c|u_{h_1}|\|v_{h_2}\|\|w_{h_3}\|_{L^\infty}+c|u_{h_1}|\|\nabla w_{h_3}\|_{L^3}\|v_{h_2}\|_{L^6},$$
$$|b(u_{h_1},w_{h_3},v_{h_2})|\le c|u_{h_1}|\|\nabla w_{h_3}\|_{L^3}\|v_{h_2}\|_{L^6}+c|u_{h_1}|\|v_{h_2}\|\|w_{h_3}\|_{L^\infty},$$

which and (5.3)-(5.4) imply (5.1)-(5.2).

Moreover, the following estimates are borrowed from Ait Ou Ammi and Marion [1] .

**Lemma 5.2.** *Under the assumptions of Theorem 4.2, the solution $(u_h,p_h)$ of (3.11)-(3.12) satisfies*

$$|u_h(t)|+\|u_h(t)\|+\|u_{h,t}(t)\|+|A_hu_h(t)|+|A_hu_{h,t}|\le\kappa(t),\quad \forall t\ge 0\,, \tag{5.5}$$

$$\|(I-P_H)u_h\|\le\kappa(t)H. \tag{5.6}$$

**Theorem 5.3.** *Under the assumptions of Theorem 4.1, the solution $(u^h,p^h)$ of problem (4.1)-(4.4) satisfies*

$$\|u_h(t)-u^h(t)\|\le\kappa(t)H^2,\forall t\ge 0\,, \tag{5.7}$$

$$|p_h(t)-p^h(t)|\le\sigma(t)^{-1/2}\kappa(t)H^2,\forall t>0\,. \tag{5.8}$$

*where $\sigma(t)=\min\{1,t\}$.*

**Remark 5.1.** By combining (3.13)-(3.14) and (5.7)-(5.8), we obtain that

$$|u(t)-u^h(t)|+h\|u(t)-u^h(t)\|\le\kappa(t)\{h^2+H^4\},\forall t\ge 0, \tag{5.9}$$

$$|p(t)-p^h(t)|\le\sigma(t)^{-1/2}\kappa(t)\{h+H^2\}\quad \forall t>0, \tag{5.10}$$

which is an asymptotic error estimate for the two-level finite element Galerkin method. It indicates that the scheme provides the same order of approximation as the standard finite element Galerkin method if we choose $H=O(h^{1/2})$.

*Proof.* The proof of the Theorem 5.3 will consist of several steps. We set

$$E=u_h-u^h,\eta=p_h-p^h,\text{ where } E(0)=0.$$

$(a)$ Estimates of the velocity $(I)$

The next lemma gives estimates on $E$.

**Lemma 5.4.** *Under the assumptions of Theorem 5.3, the following estimates hold for $t\ge 0$,*

$$|E(t)|^2\le\kappa(t)H^4\,, \tag{5.11}$$

$$\int_0^t\|E(s)\|^2ds\le\kappa(t)H^4\,. \tag{5.12}$$

*Proof.* We combine (3.11)-(3.12) with (4.5)-(4.6) to obtain

$$\begin{aligned}(\frac{d}{dt}E,v)+a(E,v)+b(u_h-u_H,u_h,v)+b(u_H,u_h-u_H,v)\\ -d(v,\eta)+d(E,q)=0,\forall(v,q)\in(X_h,M_h).\end{aligned} \tag{5.13}$$

By taking $v=E,q=\eta$ in (5.13), we have

$$\frac12\frac{d}{dt}|E|^2+\nu\|E\|^2+b(u_h-u_H,u_h,E)+b(u_H,u_h-u_H,E)=0\,. \tag{5.14}$$

Next, we estimate the trilinear terms in (5.14). Thanks to (3.8), (5.1)-(5.2) and Theorem 3.1, we find that

$$\begin{aligned}|b(u_h-u_H,u_h,E)|&\le c_1|u_h-u_H|\|u_h\|^{1/2}|A_hu_h|^{1/2}\|E\|\\ &\le\frac{\nu}{8}\|E\|^2+\frac{2}{\nu}c_1^2\|u_h\||A_hu_h||u_h-u_H|^2,\end{aligned} \tag{5.15}$$

$$|b(u_H, u_h - u_H, E)| \leq \frac{\nu}{8}\|E\|^2 + \frac{2}{\nu}c_1^2\|u_H\||A_H u_H||u_h - u_H|^2, \tag{5.16}$$

$$|u_h - u_H| \leq |u - u_h| + |u - u_H| \leq \kappa(t)H^2. \tag{5.17}$$

Combining (5.14) with (5.15)-(5.17) and applying Lemma 5.2 yield

$$\frac{d}{dt}|E|^2 + \nu\|E\|^2 \leq \kappa(t)H^4\,. \tag{5.18}$$

Integrating (5.18) from 0 to $t$, we derive (5.11) and (5.12).

(b) Estimates of the velocity (II)

**Lemma 5.5.** *Under the assumptions of Theorem 4.1, the following estimates hold for* $t \geq 0$,

$$\|E(t)\|^2 \leq \kappa(t)H^4\,, \tag{5.19}$$

$$\int_{t_0}^{t} |E_t|^2 ds \leq \kappa(t)H^4\,. \tag{5.20}$$

*Proof.* From (5.13), we derive

$$d(E, q) = 0, d(E_t, q) = 0, \forall q \in M_h.$$

Thus, by taking $v = E_t, q = \eta$ in (5.13), we find that

$$|E_t|^2 + \frac{\nu}{2}\frac{d}{dt}\|E\|^2 + b(u_h - u_H, u_h, E_t) + b(u_H, u_h - u_H, E_t) = 0. \tag{5.21}$$

Using the identity

$$b(u_h - u_H, u_h, E_t) = \frac{d}{dt}b(u_h - u_H, u_h, E) - b(u_h - u_H, u_{h,t}, E) - b(u_{h,t} - u_{H,t}, u_h, E),$$

$$b(u_H, u_h - u_H, E_t) = \frac{d}{dt}b(u_H, u_h - u_H, E) - b(u_{H,t}, u_h - u_H, E) - b(u_H, u_{h,t} - u_{H,t}, E),$$

we re-write (5.21) as follows:

$$\begin{aligned}|E_t|^2 &+ \frac{\nu}{2}\frac{d}{dt}\|E\|^2 + \frac{d}{dt}b(u_h, u_h - u_H, E) + \frac{d}{dt}b(u_H, u_h - u_H, E) \\ &= b(u_{h,t} - u_{H,t}, u_h, E) + b(u_h - u_H, u_{h,t}, E) \\ &+ b(u_{H,t}, u_h - u_H, E) + b(u_H, u_{h,t} - u_{H,t}, E).\end{aligned} \tag{5.22}$$

We aim to estimate the trilinear terms in (5.23). Thanks to (3.8), (5.1)-(5.2), Theorem 3.1 and Lemma 5.2, we find that

$$\begin{aligned}|b(u_{h,t} &- u_{H,t}, u_h, E)| + |b(u_H, u_{h,t} - u_{H,t}, E)| \\ &\leq c_1(\|u_H\|^{1/2}|A_H u_H|^{1/2} + \|u_h\|^{1/2}|A_h u_h|^{1/2})|u_{h,t} - u_{H,t}|\|E\| \\ &\leq \frac{\nu}{8}\|E\|^2 + \kappa(t)H^4\,,\end{aligned} \tag{5.23}$$

$$|b(u_h - u_H, u_{h,t}, E) + b(u_{H,t}, u_h - u_H, E)| \leq \frac{\nu}{8}\|E\|^2 + \kappa(t)H^4. \tag{5.24}$$

Combining (5.22) with (5.23)-(5.24) yields

$$\begin{aligned}|E_t|^2 + \nu\frac{d}{dt}\|E\|^2 &\leq -2\frac{d}{dt}b(u_h - u_H, u_h, E) \\ &- 2\frac{d}{dt}b(u_H, u_h - u_H, E) + \frac{\nu}{4}\|E\|^2 + \kappa(t)H^4\,.\end{aligned} \tag{5.25}$$

By integrating (5.25) between 0 and $t$, we obtain that

$$\int_0^t |E_s(s)|^2 ds + \nu \|E(t)\|^2 \leq 2|b(u_h - u_H, u_h, E)|$$
$$+ 2|b(u_H, u_h - u_H, E)| + \int_0^t \|E(s)\|^2 ds + \kappa(t) H^4 \,. \tag{5.26}$$

Moreover, by using some trilinear estimates as (5.23)-(5.24), we derive that

$$|b(u_h - u_H, u_h, E) + b(u_H, u_h - u_H, E)| \leq \frac{\nu}{2}\|E(t)\|^2 + \kappa(t) H^4, \tag{5.27}$$

which and (5.26) give

$$\int_0^t |E_s(s)|^2 ds + \frac{\nu}{2}\|E(t)\|^2 \leq \frac{\nu}{2}\int_0^t \|E(s)\|^2 ds + \kappa(t) H^4. \tag{5.28}$$

This yields readily (5.19)-(5.20) by applying the Gronwall Lemma.

(c) Estimate of the pressure

We aim now to derive (5.2). For this purpose, we will need a further estimate on the time derivative of $E$.

**Lemma 5.6.** *Under the assumptions of Theorem 4.1, the following estimate holds for $t > 0$,*

$$|E_t(t)|^2 \leq t^{-1}\kappa(t) H^4 \,. \tag{5.29}$$

*Proof.* Differentiating (5.13) with respect to time gives that

$$\begin{aligned}
&(E_{tt}, v) + a(E_t, v) - d(v, \eta_t) + d(E_t, q) \\
&\quad + b(u_{h,t} - u_{H,t}, u_h, v) + b(u_h - u_H, u_{h,t}, v) \\
&\quad + b(u_{H,t}, u_h - u_H, v) + b(u_H, u_{h,t} - u_{H,t}, v) \\
&\quad = 0, \forall (v, q) \in (X_h, M_h).
\end{aligned} \tag{5.30}$$

By taking $v = E_t, q = \eta_t$ in (5.30), we obtain

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}|E_t|^2 + \nu \|E_t\|^2 \\
= &\; - b(u_{h,t} - u_{H,t}, u_h, E_t) - b(u_{H,t}, u_h - u_H, E_t) \\
&\; - b(u_h - u_H, u_{h,t}, E_t) - b(u_H, u_{h,t} - u_{H,t}, E_t).
\end{aligned} \tag{5.31}$$

We aim to bound the trilinear terms in the right-hand side of (5.31). First, according to (3.8), (5.1)-(5.2) and Theorem 3.1 and Lemma 5.2, we have

$$\begin{aligned}
&|b(u_{h,t} - u_{H,t}, u_h, E_t) + b(u_H, u_{h,t} - u_{H,t}, E_t)| \\
\leq &\; c_1 |u_{h,t} - u_{H,t}| (\|u_h\|^{1/2} |A_h u_h|^{1/2} + \|u_H\|^{1/2} |A_H u_H|^{1/2}) \|E_t\| \\
\leq &\; \frac{\nu}{8}\|E_t\|^2 + \kappa(t) H^4 \,,
\end{aligned} \tag{5.32}$$

$$|b(u_{H,t}, u_h - u_H, E_t) + b(u_h - u_H, u_{h,t}, E_t)| \leq \frac{\nu}{8}\|E_t\|^2 + \kappa(t) H^4 \,. \tag{5.33}$$

Combining (5.31) with (5.32)-(5.33) enables us to say that

$$\frac{d}{dt}|E_t|^2 + \nu \|E_t\|^2 \leq \kappa(t) H^4 \,. \tag{5.34}$$

Multiplying (6.37) by $t$, we see that

$$\frac{d}{dt}(t|E_t|^2) \leq |E_t|^2 + \kappa(t)\tau H^4 \,. \tag{5.35}$$

By integrating this inequality, we find that

$$t|E_t(t)|^2 \leq \int_0^t |E_s(s)|^2 ds + \kappa(t)H^4 \,, \tag{5.36}$$

which and Lemma 5.5 provides (5.29). #

We are now ready to conclude the proof of Theorem 5.3 by deriving the estimate (5.8). The inf-sup condition (3.4) guarantees that

$$\bar{\beta}|p_h - p^h| \leq \sup_{v \in X_h} \frac{d(v, p_h - p^h)}{\|v\|} \,, \tag{5.37}$$

where, due to (5.13),

$$d(v, p_h - p^h) = (E_t, v) + a(E, v) + b(u_h - u_H, u_h, v) + b(u_H, u_h - u_H, v), \forall v \in X_h \,. \tag{5.38}$$

Let us derive some estimates of the terms in the right-hand side of (5.38). According to (3.8), (5.1)-(5.2), Theorem 3.1 and Lemma 5.2, we see that

$$|(E_t, v)| \leq |E_t||v| \leq \lambda_1^{-1/2}|E_t|\|v\| \,, \tag{5.39}$$

$$|a(E, v)| \leq \nu\|E\|\|v\| \,, \tag{5.40}$$

$$|b(u_h - u_H, u_h, E) + b(u_H, u_h - u_H, E)| \leq \kappa(t)|u_h - u_H|\|E\|. \tag{5.41}$$

Combining (5.37) with (5.38)-(5.41) yields

$$\bar{\beta}|p_h - p^h| \leq \kappa(t)(|E_t| + \|E\|),$$

which and Lemma 5.5 and Lemma 5.6 give (5.8).

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