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ASYMPTOTIC STABILITY OF RUNGE-KUTTA METHODS FOR THE PANTOGRAPH EQUATIONS *1)

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Abstract

This paper considers the asymptotic stability analysis of both exact and numerical solutions of the following neutral delay differential equation with pantograph delay.

$$\begin{cases} x'(t)+Bx(t)+Cx'(qt)+Dx(qt)=0, \qquad t>0, \\ x(0)=x_0, \end{cases}$$

where $B,C,D\in\mathcal{C}^{d\times d}, q\in(0,1)$, and B is regular. After transforming the above equation to non-automatic neutral equation with constant delay, we determine sufficient conditions for the asymptotic stability of the zero solution. Furthermore, we focus on the asymptotic stability behavior of Runge-Kutta method with variable stepsize. It is proved that a L-stable Runge-Kutta method can preserve the above-mentioned stability properties.

Mathematics subject classification: 65L02, 65L05, 65L20.
Key words: Neutral delay differential equations, Pantograph delay, Asymptotic stability, Runge-Kutta methods, L-stable.

1. Introduction

Delay differential equations of neutral type provide a mathematical instrument to applied science[1]. Especially, it exerts important effect on investigating several electromagnetic problems. The general functional differential equation is given by

$$x'(t)=f\left(t,x(t),x'(\alpha(t)),x(\alpha(t))\right).$$

A classical case $\alpha(t)=t-\tau$ of such system has been recently considered by a lot of authors (for example, Kuang et al.[2] and Hu and Mitsui [3]). What's more, another interesting case which is far different from the previous, is the pantograph equation

$$\begin{cases} x'(t)=f(t,x(t),x'(qt),x(qt)), t>0, \\ x(0)=x_0. \end{cases} \tag{1.1}$$

Where f is a given function, $0<q<1$, and $x(t)$ is unknown for $t>0$. There are many applications for (1.1) both in electrodynamics and in the collection of current by the pantograph of an electric locomotive [4, 5]. We are interested in the investigation of the qualitative properties of equation (1.1). To this purpose we restrict ourselves to the special form of equation (1.1) given by

$$\begin{cases} x'(t)+Bx(t)+Cx'(qt)+Dx(qt)=0, t>0, \\ x(0)=x_0, \end{cases} \tag{1.2}$$

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where B, C, D are constant $d \times d$ complex matrices, $0 < q < 1$, and B is regular.

The asymptotic behaviour of the pantograph equation (1.2) has been analyzed, often in a simplified form,by a number of authors.Concerning pantograph equation

$$\begin{cases} x'(t) = ax(t) + bx(qt) + cx'(qt), t > 0, \\ x(0) = x_0, \end{cases} \tag{1.3}$$

where $a, b, c \in \mathcal{C}$ and $0 < q < 1$, an analytical study occurred in the works of Kato and McLeod [6], Carr and Dyson [7], Iserles and Terjéki [8] and Iserles [9], in which Iserles and Terjéki [8] applied a transformation to examine the behavior of the exact solutions, and Iserles [9] applied *Dirichlet series* to equation (1.3) for considering whether the exact solution displayed inside and on its stability boundary. In the present paper, we investigate the stability of exact solution of equation(1.2) by transforming equation(1.2) to a non-automatic neutral equation with constant delay and prove the contractivity and asymptotic stability by norm assessing. Related ideas of reformulating the problem for studying the asymptotic stability of the solutions have been considered in some recent papers such as in [8]. Actually Bellen et al.[10] has been used norm assessing to consider the analytical stability of equation

$$\begin{cases} x'(t) = Lx(t) + M(t)x(t - \tau(t)) + N(t)x'(t - \tau(t)), t > t_0, \\ x(t) = g(t), t \le t_0, \end{cases}$$

but we consider that the coefficient of $x(t)$ is matrix-value function like $M(t)$ and $N(t)$.

On the other hand, the investigations of numerical stability for (1.3) can be found in many papers, such as Buhmann, Iserles and Nørsett [11], Buhmann and Iserles [12, 13] and Liu [14], in which [12] considered a special case when q is a reciprocal of an integer and [14] gave an extension of this analysis to θ-methods by transforming the equation under consideration into a neutral equation with constant time lags. Moreover, Koto [15] dealt with stability of Runge-Kutta methods applied to the equation which is obtained from the equation (1.2) by the same change of the independent variable with [14]

$$y(t) = x(e^t).$$

In 1997, Bellen[16] applied the θ-methods with variable stepsize to (1.3). It is proved that θ-methods are asymptotically stable iff $\theta > 1/2$, which provided the subsequent research with a new idea. In the present paper, the Runge-Kutta methods with variable stepsize is applied to equation(1.2) and asymptotic stability of numerical solution is probed by investigating the Schur polynomial of perturbed equation.

This paper is organized as follows. In Section 2, the sufficient conditions both for the contractivity and for asymptotic stability of the exact solutions are given. In Section 3, we apply Runge-Kutta methods with variable stepsize to equation(1.2) and obtain numerical solutions. In Section 4, the conclusion of numerically asymptotic stability is drawn.

2. Sufficient Conditions of Contractivity and Asymptotic Stability

2.1. Transformation of Equation Form

To eliminate dependence on the derivative of the solution, we transform (1.2) to a neutral equation with constant delay by conversion

$$y(t) = x(e^t),$$

then $y(t)$ satisfies the following initial value problem

$$\begin{aligned} &y'(t) + Be^t y(t) + Cq^{-1}y'(t+\log q) + De^t y(t+\log q) = 0, \qquad t \geq 0, \\ &y(t) = x(e^t), \qquad \log q \leq t \leq 0, \end{aligned} \tag{2.1}$$

where $y(t)$, $\log q \leq t \leq 0$, can be obtained numerically by using Runge-Kutta methods to (1.2). Let $y(t) = g(t)$, $\log q \leq t \leq 0$, $\tau = -\log q > 0$, then (2.1) are written

$$\begin{aligned} &y'(t) + Be^t y(t) + Cq^{-1}y'(t-\tau) + De^t y(t-\tau) = 0, \quad t \geq 0, \\ &y(t) = g(t), \qquad\qquad -\tau \leq t \leq 0, \end{aligned} \tag{2.2}$$

where B,C,$D \in \mathcal{C}^{d\times d}$, B is regular, and $g(t)$ is a C^1-continuous complex-valued function.

Insert a series of points

$$0 = \xi_0 < \xi_1 < \xi_2 < \cdots < \xi_j < \xi_{j+1} < \cdots$$

in interval $[0, +\infty)$, where ξ_{j+1} being the unique solution of $\alpha(\xi) = \xi - \tau = \xi_j$. The following notation is useful to our aims. Let $I_0 := [-\tau(t_0), \xi_0]$ and $I_j := [\xi_{j-1}, \xi_j]$, for $j = 1, 2, 3, \cdots$. It will be shown that our investigation of the behavior of the solutions will be developed across the intervals $\{I_j\}$, by relating the solution in I_j with the one in I_{j-1}.

We propose a reformulation of the considered problem (2.2), which is fundamental for the stability analysis in this paper. First, we rewrite the system (2.2) as follows

$$\begin{cases} y'(t) = -Be^t y(t) + (-De^t + Cq^{-1}Be^t)y(t-\tau) \\ \qquad\qquad -Cq^{-1}(y'(t-\tau) + Be^t y(t-\tau)) & t \geq 0, \\ y(t) = g(t), & -\tau \leq t \leq 0 \end{cases} \tag{2.3}$$

Let

$$\phi(t) = y'(t) + Be^t y(t),$$

then the previous system (2.3) is equivalent to

$$\begin{cases} y'(t) = -Be^t y(t) + \phi(t), t \geq 0, \\ y(0) = g(0), \end{cases} \tag{2.4}$$

where

$$\phi(t) = \begin{cases} -De^t g(t-\tau) - Cq^{-1}g'(t-\tau), & 0 \leq t \leq \xi_1, \\ (-De^t + Cq^{-1}Be^t)y(t-\tau) - Cq^{-1}\phi(t-\tau), & t \geq \xi_1. \end{cases} \tag{2.5}$$

For convenience sometimes we let $g(0) = y_0$.

2.2. Stability Analysis of Exact Solution

In the above approach the neutral system, is transformed into an ordinary differential system and an algebraic recursion. Below we consider the stability of exact solution of (2.1) by investigating (2.4),(2.5). First we prove the following useful lemma.

Lemma 2.1. *Consider the system (2.4), where the components of the forcing term $\phi(t)$ are assumed to be continuous functions. Given an inner product $\langle\cdot,\cdot\rangle$ in C^d and the corresponding norm $\|\cdot\|$, let $\mu[\cdot]$ stands for the logarithmic norm induced by $\langle\cdot,\cdot\rangle$ and $\mu[B] > 0$. Then the following inequality*

$$\|y(t)\| \leq E(0,t)\|y_0\| + (1 - E(0,t)) \sup_{0 \leq x \leq t} \frac{\|\phi(x)\|}{\mu[B]}, \tag{2.6}$$

with

$$E(t_1, t_2) = \exp\left(-\mu[B](\exp t_2 - \exp t_1)\right). \tag{2.7}$$

holds for all $t \geq 0$.

Proof. By (2.4), we get

$$y(t) = e^{\int_0^t -Be^t dt}\left[y_0 + \int_0^t \phi(x) e^{\int_0^x Be^s ds} dx\right],$$

then

$$\|y(t)\| \leq e^{-\mu[B](e^t - 1)}\left[\|y_0\| + \sup_{0 \leq s \leq t} \frac{\|\phi(s)\|}{\mu[B]e^s} \int_0^t \mu[B] e^s e^{\mu[B](e^s - 1)} ds\right],$$

that is

$$\|y(t)\| \leq e^{-\mu[B](e^t-1)}\|y_0\| + \left(1 - e^{-\mu[B](e^t-1)}\right) \sup_{0 \leq s \leq t} \frac{\|\phi(s)\|}{\mu[B]}, \tag{2.8}$$

by (2.7), it holds that

$$\|y(t)\| \leq E(0,t)\|y_0\| + (1 - E(0,t)) \sup_{0 \leq s \leq t} \frac{\|\phi(s)\|}{\mu[B]}. \tag{2.9}$$

The proof is completed.

By considering $\phi(\xi_n) = \lim_{t \to \xi_n -} \phi(t)$ we can assume the function $\phi(t)$ continuous in the closed interval $I_n = [\xi_{n-1}, \xi_n], n \geq 1$. Therefore, there exists

$$max_{\xi_{n-1} \leq s \leq \xi_n} \|\phi(s)\|$$

for every $n \geq 1$.

The following theorem is devoted to the contractivity properties of the solutions of (2.4), (2.5).

Theorem 2.1. *For the system (2.2), the condition*

$$\frac{\| - D + Cq^{-1}B\|}{\mu[B]q} + \|Cq^{-1}\| \leq 1, \qquad \forall t \geq 0, \tag{2.10}$$

implies that the solution $y(t)$ *satisfies*

$$\|y(t)\| \leq \max\left\{\|g(0)\|, \frac{\mathcal{K}}{\mu[B]}\right\}, \qquad \forall t \geq 0. \tag{2.11}$$

for every initial function $g(t)$ *and norm defined in Lemma 2.1, where* $\mu[B] > 0$ *and*

$$\mathcal{K} = \max_{0 \leq t \leq \xi_1} \| - De^t g(t-\tau) - Cq^{-1} g'(t-\tau)\|. \tag{2.12}$$

Proof. we consider the interval $I_n = [\xi_{n-1}, \xi_n], n \geq 2$. By (2.6), we have, for every $t \in [\xi_{n-1}, \xi_n]$

$$\|y(t)\| \leq E(\xi_{n-1}, t)\|y(\xi_{n-1})\| + (1 - E(\xi_{n-1}, t)) \max_{\xi_{n-1} \leq x \leq t} \frac{\|\phi(x)\|}{\mu[B]e^{\xi_{n-1}}}, \tag{2.13}$$

$\mu[B] > 0$ and (2.7) imply $E(\xi_{n-1}, t) \leq 1$. Inequality (2.13) means that

$$\|y(t)\| \leq \max\left\{\|y(\xi_{n-1})\|, \max_{\xi_{n-1}\leq x\leq t} \frac{\|\phi(x)\|}{\mu[B]e^{\xi_{n-1}}}\right\}, t \in [\xi_{n-1}, \xi_n]. \tag{2.14}$$

For any vector function $\gamma(s)$ and any integer $i \geq 0$, let

$$|||\gamma|||_i := \max_{s\in I_i} \|\gamma(s)\|,$$

then by (2.12) and the condition (2.10), we immediately have

$$|||\phi|||_1 = \mathcal{K},$$

$$\begin{aligned}
|||\phi|||_n &\leq \max_{\xi_{n-1}\leq s\leq \xi_n} \left\{\|Cq^{-1}\phi(s-\tau)\| + \|(-De^s + Cq^{-1}Be^s)y(s-\tau)\|\right\} \\
&\leq \|Cq^{-1}\| \cdot |||\phi|||_{n-1} + \max_{\xi_{n-1}\leq s\leq \xi_n} \left[(1-\|Cq\|^{-1})\mu[B]qe^s\|y(s-\tau)\|\right] \\
&\leq \|Cq^{-1}\| \cdot |||\phi|||_{n-1} + (1-\|Cq\|^{-1}) \cdot \mu[B]e^{\xi_{n-1}}|||y|||_{n-1},
\end{aligned} \tag{2.15}$$

which apply the assumption $q = e^{\log q} = e^{-\tau}$. Since $\|Cq^{-1}\| < 1$, then

$$|||\phi|||_n \leq \max\left\{\mu[B]e^{\xi_{n-1}}|||y|||_{n-1}, |||\phi|||_{n-1}\right\}$$

Now, for all $n \geq 1$ define

$$\alpha_n = \max\left\{|||y|||_n, \frac{|||\phi|||_n}{\mu[B]e^{\xi_{n-1}}}\right\}. \tag{2.16}$$

By (2.14), we get

$$|||y|||_n \leq \max\left\{|||y|||_{n-1}, \frac{|||\phi|||_n}{\mu[B]e^{\xi_{n-1}}}\right\}. \tag{2.17}$$

hence, (2.16) and (2.17) show

$$\alpha_n \leq \max\left\{|||y|||_{n-1}, \frac{|||\phi|||_{n-1}}{\mu[B]e^{\xi_{n-1}}}\right\} \leq \max\left\{|||y|||_{n-1}, \frac{|||\phi|||_{n-1}}{\mu[B]e^{\xi_{n-2}}}\right\} = \alpha_{n-1},$$

then we can obtain non-increasing positive sequence

$$\alpha_1 \geq \alpha_2 \geq \cdots \geq \alpha_n \cdots, n \geq 2.$$

Finally consider the case $n = 1$, for which (2.14) yields

$$|||y|||_1 \leq \max\left\{\|y_0\|, \frac{|||\phi|||_1}{\mu[B]}\right\} = \max\left\{\|g(0)\|, \frac{\mathcal{K}}{\mu[B]}\right\},$$

$$\alpha_1 = \max\left\{|||y|||_1, \frac{|||\phi|||_1}{\mu[B]e^{\xi_0}}\right\},$$

the last two equations imply

$$\alpha_1 \leq \max\left\{\|g(0)\|, \frac{\mathcal{K}}{\mu[B]}\right\}$$

then

$$\|y(t)\| \leq \max\left\{\|g(0)\|, \frac{\mathcal{K}}{\mu[B]}\right\}.$$

is hold $\forall t \in I_n, n \geq 1$.

The following theorem concerns asymptotic stability of the system (2.2).

Theorem 2.2. *For the system (2.2), the conditions*

$$\|Cq^{-1}\| = \zeta < 1, \forall t \geq 0, \tag{2.18}$$

$$\frac{\| - D + Cq^{-1}B\|}{\mu[B]q} \leq k(1 - \|Cq^{-1}\|), \forall t \geq 0, \tag{2.19}$$

for an arbitrary $k < 1$, *imply* $\lim_{t\to\infty} \|y(t)\| = 0$, *for every initial function* $g(t)$.

The proof of this theorem is analogy to the process of proving Theorem 3.2 in [10], hence we omit it here.

In conclusion, though the conversion and reformulation which we applied to the system (1.2) are similar to Iserles and Terjéki [8] and Liu [14], we effectively associate the two different reformulations above-mentioned to obtain a favorable form for analyzing stability and obtain the asymptotic stability for the pantograph system (1.2), that is

$$\lim_{t\to\infty} \|x(t)\| = 0$$

when the conditions of Theorem 2.2 are hold.

3. Runge-Kutta Methods with Variable Stepsize

Bellen et al.[16] described in detail the discretization scheme and constrained global mesh. We show that process here.

Firstly we build a primary mesh based on the following relation,

$$T_k = T_{k-1}/q, \qquad k = 1, 2, \cdots,$$

where $T_0 > 0$ is given. In this way the primary intervals are defined by

$$H_k := T_k - T_{k-1} = \frac{1-q}{q^k} T_0, \qquad k = 1, 2, \cdots. \tag{3.1}$$

Observe that the sequence $\{H_k\}$ increases exponentially, therefore we define the mesh by partition every primary interval into a fixed number m of sub-intervals of the same size. Let $[\cdot]$ denote the integer part, we set

$$h_{n+1} := \frac{H_{[n/m]+1}}{m} = \frac{1}{m} \frac{1-q}{q^{[n/m]+1}} T_0, \qquad n = 0, 1, 2, \cdots, \tag{3.2}$$

For simplicity(but without loss of generality), we assume $t_0 = T_0 = 1$. Let $l = n \bmod m$, then we define the grid points of H:

$$t_n := T_{[n/m]} + lh_n, \qquad n = 1, 2, \cdots,$$

which yields

$$t_n := q^{-1} t_{n-m}, \qquad n > m. \tag{3.3}$$

Now we consider the application of Runge-Kutta methods to (1.2). We have

$$x_{n+1} = x_n + \sum_{i=1}^{s} \hat{b_i} k_{n,i}, \tag{3.4}$$

where

$$k_{n,i} + h_{n+1}B(x_n + \sum_{j=1}^{s} \hat{a_{ij}} k_{n,j}) + Ck_{n-m,i} + h_{n+1}D(x_{n-m} + \sum_{j=1}^{s} \hat{a_{ij}} k_{n-m,j}) = 0,$$

and by (3.3), the descretization of pantograph delay is given by

$$x_{n-m}+\sum_{j=1}^{s}\hat{a_{ij}}k_{n-m,j}.$$

Let

$$K_n=\left[k_{n,1}^T,k_{n,2}^T,\cdots,k_{n,s}^T\right]^T,$$

$$\widetilde{A}=(\hat{a_{ij}})_{s\times s},$$

$$b=(\hat{b_1},\hat{b_2},\cdots,\hat{b_s})^T,$$

$e=(1,1,\cdots,1)^T$ is a s-dimensional vector.

Applying operator of $\otimes$, we transform (3.4) to

$$x_{n+1}=x_n+(b^T\otimes I_d)K_n,$$

$$\begin{aligned}&(I_s\otimes I_d)K_n+h_{n+1}(e\otimes B)x_n+h_{n+1}(\widetilde{A}\otimes B)K_n+(I_s\otimes C)K_{n-m}\\&+h_{n+1}(e\otimes D)x_{n-m}+h_{n+1}(\widetilde{A}\otimes D)K_{n-m}=0,\end{aligned}\tag{3.5}$$

which yields

$$\begin{aligned}&\begin{pmatrix}I_{sd}+h_{n+1}(\widetilde{A}\otimes B) & 0\\ -b^T\otimes I_d & I_d\end{pmatrix}\begin{pmatrix}K_n\\ x_{n+1}\end{pmatrix}+\begin{pmatrix}0 & h_{n+1}(e\otimes B)\\ 0 & -I_d\end{pmatrix}\begin{pmatrix}K_{n-1}\\ x_n\end{pmatrix}\\&+\begin{pmatrix}h_{n+1}(\widetilde{A}\otimes D)+I_s\otimes C & 0\\ 0 & 0\end{pmatrix}\begin{pmatrix}K_{n-m}\\ x_{n-m+1}\end{pmatrix}+\begin{pmatrix}0 & h_{n+1}(e\otimes D)\\ 0 & 0\end{pmatrix}\begin{pmatrix}K_{n-m-1}\\ x_{n-m}\end{pmatrix}=0.\end{aligned}\tag{3.6}$$

Observe that matrices of equality (3.6) are $(s+1)d\times(s+1)d$ complex matrices. To research the stability of numerical solutions, we transform (3.6) to the following form.

$$\begin{aligned}&\begin{pmatrix}I_{sd}+h_{n+1}(\widetilde{A}\otimes B) & 0\\ -e\otimes b^T\otimes I_d & I_{sd}\end{pmatrix}\begin{pmatrix}K_n\\ e\otimes x_{n+1}\end{pmatrix}+\begin{pmatrix}0 & \frac{1}{s}h_{n+1}(e^T\otimes e\otimes B)\\ 0 & -I_{sd}\end{pmatrix}\begin{pmatrix}K_{n-1}\\ e\otimes x_n\end{pmatrix}\\&+\begin{pmatrix}h_{n+1}(\widetilde{A}\otimes D)+I_s\otimes C & 0\\ 0 & 0\end{pmatrix}\begin{pmatrix}K_{n-m}\\ e\otimes x_{n-m+1}\end{pmatrix}+\begin{pmatrix}0 & \frac{h_{n+1}}{s}(e^T\otimes e\otimes D)\\ 0 & 0\end{pmatrix}\begin{pmatrix}K_{n-m-1}\\ e\otimes x_{n-m}\end{pmatrix}=0.\end{aligned}\tag{3.7}$$

4. Stability Analysis of Numerical Solutions

First we show the definition in [17].

Definition 4.1. *A Runge-Kutta method $(\widetilde{A},b,c)$ is called to be L-stable, if it is A-stable, and its stability function given by $R(z)=1+zb^T(I-z\widetilde{A})^{-1}e$ satisfies*

$$\lim_{|z|\to\infty}|R(z)|=0.\tag{4.1}$$

Below we find the stability function of equation

$$x'(t) + Bx(t) = 0, \tag{4.2}$$

which is fundamental for the asymptotic stability analysis. Applying the Runge-Kutta methods with variable stepsize to (4.2), we get

$$x_{n+1} = x_n + \sum_{i=1}^{s} \hat{b}_i k_{n,i}, \tag{4.3}$$

where

$$k_{n,i} + h_{n+1}B(x_n + \sum_{j=1}^{s} \hat{a}_{ij}k_{n,j}) = 0,$$

in analogy to the previous derivation, (4.3) yields

$$e \otimes x_{n+1} = e \otimes x_n + (e \otimes b^T \otimes I_d)K_n,$$

$$K_n + h_{n+1}\frac{1}{s}(e^T \otimes e \otimes B)(e \otimes x_n) + h_{n+1}(\widetilde{A} \otimes B)K_n = 0,$$

then the following equality can be induced.

$$e \otimes x_{n+1} = e \otimes x_n - (e \otimes b^T \otimes I_d)\left(I_{sd} + h_{n+1}(\widetilde{A} \otimes B)\right)^{-1}\frac{1}{s}h_{n+1}(e^T \otimes e \otimes B)(e \otimes x_n),$$

that is

$$e \otimes x_{n+1} = \left[I_{sd} - (e \otimes b^T \otimes I_d)\left(I_{sd} + \widetilde{A} \otimes (h_{n+1}B)\right)^{-1}\frac{1}{s}(e^T \otimes e \otimes (h_{n+1}B))\right](e \otimes x_n),$$

we immediately have

$$R(-h_{n+1}B) = I_{sd} + \left(e \otimes b^T \otimes I_d\right)\left(I_{sd} - \widetilde{A} \otimes (-h_{n+1}B)\right)^{-1}\frac{1}{s}\left(e^T \otimes e \otimes (-h_{n+1}B)\right). \tag{4.4}$$

Assume that the Runge-Kutta method applied to system (1.2) is L-stable, and $\widetilde{A}$ is regular. Let

$$M(h_{n+1}) = \begin{pmatrix} I_{sd} + h_{n+1}(\widetilde{A} \otimes B) & 0 \\ -e \otimes b^T \otimes I_d & I_{sd} \end{pmatrix}, \qquad P(h_{n+1}) = \begin{pmatrix} 0 & \frac{1}{s}h_{n+1}(e^T \otimes e \otimes B) \\ 0 & -I_{sd} \end{pmatrix},$$

$$Q(h_{n+1}) = \begin{pmatrix} h_{n+1}(\widetilde{A} \otimes D) + I_s \otimes C & 0 \\ 0 & 0 \end{pmatrix}, \qquad R(h_{n+1}) = \begin{pmatrix} 0 & \frac{h_{n+1}}{s}(e^T \otimes e \otimes D) \\ 0 & 0 \end{pmatrix}.$$

Then it holds that

$$M^{-1}(h_{n+1}) = \begin{pmatrix} (I_{sd} + h_{n+1}(\widetilde{A} \otimes B))^{-1} & 0 \\ (e \otimes b^T \otimes I_d)(I_{sd} + h_{n+1}(\widetilde{A} \otimes B))^{-1} & I_{sd} \end{pmatrix}.$$

Hence we have

$$M^{-1}(h_{n+1})P(h_{n+1}) = \begin{pmatrix} 0 & \frac{h_{n+1}}{s}(I_{sd} + h_{n+1}(\widetilde{A} \otimes B))^{-1}(e^T \otimes e \otimes B) \\ 0 & -I_{sd} + (e \otimes b^T \otimes I_d)(I_{sd} + h_{n+1}(\widetilde{A} \otimes B))^{-1}\frac{h_{n+1}}{s}(e^T \otimes e \otimes B) \end{pmatrix},$$

$$M^{-1}(h_{n+1})Q(h_{n+1}) = \begin{pmatrix} (I_{sd}+h_{n+1}(\widetilde{A}\otimes B))^{-1}(h_{n+1}(\widetilde{A}\otimes D)+I_s\otimes C) & 0 \\ (e^T\otimes b^T\otimes I_d)(I_{sd}+h_{n+1}(\widetilde{A}\otimes B))^{-1}(h_{n+1}(\widetilde{A}\otimes D)+I_s\otimes C) & 0 \end{pmatrix},$$

$$M^{-1}(h_{n+1})R(h_{n+1}) = \begin{pmatrix} 0 & \frac{h_{n+1}}{s}(I_{sd}+h_{n+1}(\widetilde{A}\otimes B))^{-1}(e^T\otimes e\otimes D) \\ 0 & (e\otimes b^T\otimes I_d)(I_{sd}+h_{n+1}(\widetilde{A}\otimes B))^{-1}\frac{h_{n+1}}{s}(e^T\otimes e\otimes D) \end{pmatrix}. \quad (4.5)$$

Below we consider the difference equation (3.7). Let

$$X_{n+1} = \begin{pmatrix} K_n \\ e\otimes x_{n+1} \end{pmatrix},$$

by which, we transform (3.7) to

$$\begin{aligned} &X_{n+1} + M^{-1}(h_{n+1})P(h_{n+1})X_n + M^{-1}(h_{n+1})Q(h_{n+1})X_{n-m+1} \\ &+M^{-1}(h_{n+1})R(h_{n+1})X_{n-m} = 0 \end{aligned} \quad (4.6)$$

By (3.2) we conclude that the stepsize h_{n+1} yields $h_{n+1}\to\infty$. Let

$$P = lim_{h_{n+1}\to\infty}M^{-1}(h_{n+1})P(h_{n+1}),$$

$$Q = lim_{h_{n+1}\to\infty}M^{-1}(h_{n+1})Q(h_{n+1}),$$

$$R = lim_{h_{n+1}\to\infty}M^{-1}(h_{n+1})R(h_{n+1}),$$

then by the Definition 4.1 and the equality(4.4), we obtain

$$P = \begin{pmatrix} 0 & \frac{1}{S}e^T\otimes(\widetilde{A}^{-1}e)\otimes I_d \\ 0 & 0 \end{pmatrix}, \qquad Q = \begin{pmatrix} I_s\otimes(B^{-1}D) & 0 \\ e\otimes b^T\otimes(B^{-1}D) & 0 \end{pmatrix},$$

$$R = \begin{pmatrix} 0 & \frac{1}{S}e^T\otimes(\widetilde{A}^{-1}e)\otimes(B^{-1}D) \\ 0 & \frac{1}{S}ee^T\otimes(b^T\widetilde{A}^{-1}e)\otimes(B^{-1}D) \end{pmatrix}, \quad (4.7)$$

where P,Q,R are $2sd\times 2sd$ matrices and zeros in matrices denote 0_{sd}. Thus we obtain the perturbation equation of the difference equation (4.6)

$$X_{n+1} + PX_n + QX_{n-m+1} + RX_{n-m} = 0 \quad (4.8)$$

Let

$$X_{n+1} = z^n x, \quad (4.9)$$

where $x = (\xi^{(1)},\xi^{(2)},\cdots,\xi^{(2sd)})^T$ is a $2sd$-dimensional complex vector, z is a complex variable. By the assumption (4.9), equation (4.8) induces

$$(z^{m+1}I_{2sd} + Pz^m + Qz + R) = 0. \quad (4.10)$$

Finally, we given the main theorem of this paper.

Theorem 4.1. *The L-stable Runge-Kutta method $(\widetilde{A},b,c)$ which is applied to the system (1.2) is asymptotically stable if $\widetilde{A}$ is regular, and the condition*

$$\|B^{-1}D\| < 1$$

holds.

Proof. To prove the theorem, it is important to show the implication

$$\det\left(z^{m+1}I_{2sd} + Pz^m + Qz + R\right) = 0 \qquad \Rightarrow \qquad |z| < 1.$$

We have

$$\begin{aligned}
&\det\left(z^{m+1}I_{2sd} + Pz^m + Qz + R\right)\\
&= \det\left(z^m(zI_{2sd} + P) + Qz + R\right) \qquad\qquad (4.11)\\
&= \det\left[z^m I_{2sd} + (Qz + R)(zI_{2sd} + P)^{-1}\right](zI_{2sd} + P),
\end{aligned}$$

where $(zI_{2sd} + P)$ is regular, whose inverse is given by

$$(zI_{2sd} + P)^{-1} = \begin{pmatrix} z^{-1}I_{sd} & -\frac{z^{-2}}{s}e^T \otimes (\widetilde{A}^{-1}e) \otimes I_d \\ 0 & z^{-1}I_{sd} \end{pmatrix},$$

which induces

$$\begin{aligned}
(Qz + R)(zI_{2sd} + P)^{-1} &= \begin{pmatrix} I_s \otimes (B^{-1}D)z & \frac{1}{s}e^T \otimes (\widetilde{A}^{-1}e) \otimes (B^{-1}D) \\ e \otimes b^T \otimes (B^{-1}D)z & \frac{1}{s}ee^T \otimes (b^T\widetilde{A}^{-1}e) \otimes (B^{-1}D) \end{pmatrix} \cdot (zI_{2sd} + P)^{-1}\\
&= \begin{pmatrix} I_s \otimes (B^{-1}D) & 0 \\ e \otimes b^T \otimes (B^{-1}D) & 0 \end{pmatrix},
\end{aligned}$$

by assumption of theorem and (4.7), the last equality implies

$$\rho(-P) = 0 < 1, \qquad \rho\left(-(Qz + R)(zI_{2sd} + P)^{-1}\right) \le \|B^{-1}D\| < 1. \qquad (4.12)$$

Rewrite (4.11) to the following form:

$$\det\left(zI_{2sd} - (-P)\right) \cdot \det\left[z^m I_{2sd} - \left(-(Qz + R)(zI_{2sd} + P)^{-1}\right)\right] = 0,$$

then by (4.12), it is obvious that

$$|z| < 1.$$

Now we consider the relation between the difference equation(4.6) and its perturbation equation (4.8). Equation(4.6) denotes

$$\mathcal{R}\big\{\{X_i\}_{i=0}^{\infty}; n\big\} = F_n, \qquad n \ge 0,$$

where

$$\mathcal{R}\big\{\{X_i\}_{i=0}^{\infty}; n\big\} = X_{n+1} + PX_n + QX_{n-m+1} + RX_{n-m},$$

$$\begin{aligned}
F_n = {}& (P - M^{-1}(h_{n+1})P(h_{n+1}))X_n + (Q - M^{-1}(h_{n+1})Q(h_{n+1}))X_{n-m+1}\\
&+ (R - M^{-1}(h_{n+1})R(h_{n+1}))X_{n-m}.
\end{aligned}$$

Let

$$(P - M^{-1}(h_{n+1})P(h_{n+1})) = \begin{pmatrix} P_1 & P_2 \\ P_3 & P_4 \end{pmatrix}, \text{ where } P_1, P_2, P_3, P_4 \text{ are } sd \times sd \text{ matrices .}$$

Obviously,

$$P_1 = 0_{sd}, \qquad P_3 = 0_{sd},$$

$$P_2 = \frac{1}{s} e^T \otimes \widetilde{A}^{-1} e \otimes I_d - \frac{h_{n+1}}{s} \Big(I_{sd} + h_{n+1} (\widetilde{A} \otimes B) \Big)^{-1} (e^T \otimes e \otimes B)$$

$$= h_{n+1}^{-1} \frac{1}{s} \Big[h_{n+1}^{-1} I_{sd} + (\widetilde{A} \otimes B) \Big]^{-1} (e^T \otimes \widetilde{A}^{-1} e \otimes I_d).$$

Moreover, P_4 is the stability function of Runge-Kutta method, then $P_4 = O(h_{n+1})G$, where G is a $sd \times sd$ matrix, and there exist a positive number M_G such that $\|G\| \le M_G$. Immediately, we have the estimation

$$\|P - M^{-1}(h_{n+1}) P(h_{n+1})\| \le h_{n+1}^{-1} M_P,$$

for some $M_P > 0$. Analogously, we have

$$\|Q - M^{-1}(h_{n+1}) Q(h_{n+1})\| \le h_{n+1}^{-1} M_Q,$$

$$\|R - M^{-1}(h_{n+1}) R(h_{n+1})\| \le h_{n+1}^{-1} M_R,$$

where $M_Q, M_R > 0$. Let $M = \max \big\{ M_P, M_Q, M_R \big\}$, then we get

$$\|F_n\| \le h_{n+1}^{-1} M \big(\|X_n\| + \|X_{n-m+1}\| + \|X_{n-m}\| \big), \qquad n > 0. \tag{4.13}$$

By reference [18] and inequality (4.13), the difference equation (4.6) preserve the asymptotic stability of its perturbation equation (4.8). Then the above theorem is proved.

The following theorem is obvious.

Theorem 4.2. *Suppose that conditions (2.18) and (2.19) holds.Then the L-stability Runge-Kutta method* $(\widetilde{A}, b, c)$ *applied to the system (1.2) is asymptotically stable, where* $\widetilde{A}$ *is regular. That is, the solution of (3.7) satisfies*

$$\lim_{n \to \infty} x_n = 0.$$

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