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DISSIPATIVITY AND EXPONENTIAL STABILITY OF $\theta-$METHODS FOR SINGULARLY PERTURBED DELAY DIFFERENTIAL EQUATIONS WITH A BOUNDED LAG [*1)]

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Abstract

This paper deals with analytic and numerical dissipativity and exponential stability of singularly perturbed delay differential equations with any bounded state-independent lag. Sufficient conditions will be presented to ensure that any solution of the singularly perturbed delay differential equations (DDEs) with a bounded lag is dissipative and exponentially stable uniformly for sufficiently small $\varepsilon > 0$. We will study the numerical solution defined by the linear $\theta-$method and one-leg method and show that they are dissipative and exponentially stable uniformly for sufficiently small $\varepsilon > 0$ if and only if $\theta = 1$.

Key words: Singular perturbation, $\theta-$methods, Dissipativity, Exponential stability.

1. Introduction

Singular perturbation problems(SPPs) form a special class of problems containing a small parameter ε. They are of practical interest in models of instantaneous phenomena and include a subclass of what we frequently thought of as 'stiff' equations. Singularly perturbed delay differential equations of the form

$$\varepsilon y'(t,\varepsilon) = g(t, y(t,\varepsilon), y(t-\tau,\varepsilon)), \qquad 0 \le t \le T, \tag{1}$$

subject to the initial condition

$$y(t,\varepsilon) = \phi(t,\varepsilon), \qquad -\tau \le t \le 0 \tag{2}$$

arise in the study of an "optically bistable device" [7] and in a variety of models for physiological processes or diseases [16]. Such a problem has also appeared to describe the so-called human pupil-light reflex [15]. For example, Ikeda [13] adopted the model

$$\varepsilon y'(t,\varepsilon) = -y(t,\varepsilon) + A^2\left[1 + 2B\cos(y(t-1,\varepsilon))\right]$$

to describe an optically bistable device and showed numerically that instability or chaotic behaviour occurs for small ε and certain values of A, B. This was confirmed experimentally by Gibbs, Hopf, Kaplan and Shoemaker [9].

1.1. A Simple Example

Before we investigate dissipativity and exponential stability of singularly perturbed delay differential equations, we first consider a simple ordinary differential equation in the form

$$\begin{aligned} \varepsilon y'(t) &= \lambda y(t), \quad (\Re\lambda \le 0), \quad t > 0, \\ y(0) &= y_0, \end{aligned} \tag{3}$$

which has the solution

$$y(t) = e^{\frac{\lambda}{\varepsilon}t} y_0.$$

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The most obvious classical difference scheme for solving this problem numerically is $\theta-$method:

$$\varepsilon(y_{n+1}-y_n)=\theta\lambda h y_{n+1}+(1-\theta)\lambda h y_n, \tag{4}$$

where $n\geq 0$. Solving it explicitly, we obtain

$$y_{n+1}=\frac{\varepsilon+(1-\theta)\lambda h}{\varepsilon-\theta\lambda h}y_n, \tag{5}$$

in which $\frac{\varepsilon+(1-\theta)\lambda h}{\varepsilon-\theta\lambda h}$should be an approximation to $e^{\frac{\lambda}{\varepsilon}h}$.

There are several disadvantages of the $\theta-$method. First, the $\theta-$method doesn't possess uniform convergence in ε. Let $\rho=\frac{h}{\varepsilon}$. The general form of the first mesh error is

$$\lim_{h\to 0}|y(h)-y_1|=\left|e^{\lambda\rho}-\left(1+\frac{\lambda\rho}{1-\theta\lambda\rho}\right)\right||y_0|. \tag{6}$$

When $\rho=1, y_0\neq 0$, for example, (6) reads

$$\lim_{h\to 0}|y(h)-y_1|=\left|e^{\lambda}-\left(1+\frac{\lambda}{1-\theta\lambda}\right)\right||y_0|\neq 0, \tag{7}$$

which means nonuniform convergence in ε. In addition, it can be proved that

$$\lim_{\substack{h\to 0\\ \rho\to\infty}}|y(h)-y_1|=0 \tag{8}$$

for any initial value if and only if $\theta=1$.

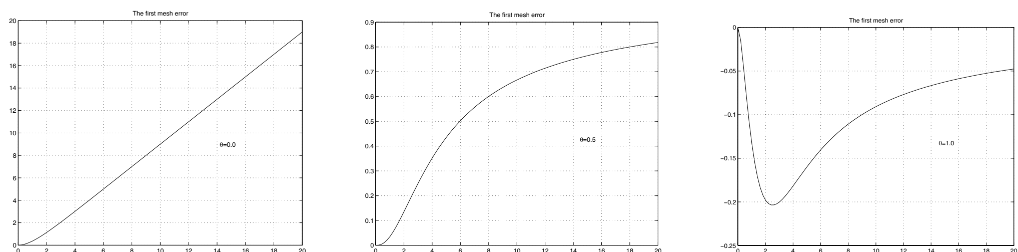

Figure 1: Graph of error function with respect to the ratio ρ of stiff coefficient ε and step-size h.

Figure 1 implies that the error is small only for small ρ when $\theta\neq 1$, while for the backward Euler method (i.e. $\theta=1$), the error is small for small and large ratio ρ and becomes significant when ε and h are of the same order of magnitude.

Second, it is clear that the discrete solution oscillates if $\rho>\frac{1}{-\lambda}$ (where $\lambda\in\mathbb{R}$) except $\theta=1$, because

$$y_n=\left(1+\frac{\lambda\rho}{1-\theta\lambda\rho}\right)^n y_0. \tag{9}$$

These oscillations are spurious since they do not occur in the solution of the continuous problem, and can only be avoided by taking the backward Euler scheme.

Third, the original equation is asymptotically stable and hence numerical approximation should mimic the same property, which requires

$$\left|\frac{\varepsilon+(1-\theta)\lambda h}{\varepsilon-\theta\lambda h}\right|<1. \tag{10}$$

It is well-known that $\theta-$method is A-stable for ODEs if and only if $\theta\in[\frac{1}{2},1]$. Unfortunately, it is easy to verify that (10) is satisfied for any $\lambda h\in\{x:\Re x<0\}$ uniformly in $\varepsilon>0$ if and only if $\theta\in(\frac{1}{2},1]$, which rules out the trapezoidal method since it is not strongly stable at infinity.

We distinguish two cases:

1. $\rho \to 0$. The numerical solution y_n tends to zero extremely slowly even if the exact solution tends to zero rapidly;

2. $\rho \to \infty$. The rate of decrease of numerical solution resembles $\left(\frac{1-\theta}{\theta}\right)^n y_0$ (where $\theta \in (\frac{1}{2}, 1]$).

From a numerical analyst's point of view, it is always assumed for stiff problem that $\frac{h}{\varepsilon} \to \infty$ and $h \to 0$ simultaneously. This is the most interesting situation because any effective numerical method for stiff problems should integrate the equation with step-size h such that $\frac{h}{\varepsilon}$ is large. Based on our preceding observations (cf. Figure 1), we conclude that the backward Euler scheme is the only choice, provided we do not strongly require the so-called uniform convergence in ε.

1.2. Outline of the Paper

The dissipativity and stability of DDEs have been studied extensively, but most theory emphasizes linear, constant coefficient equations with a constant lag. In this paper, we will concentrate on the dissipativity and exponential stability of singularly perturbed delay differential equations with any bounded (state-independent) lag (c.f. Zennaro [23]) and their numerics. It is very important for the prescribed numerical method to preserve the dissipativity and stability of the underlying system. In Section 2, we obtain an inequality of Halanay type which plays an essential role in the analysis of dissipativity and exponential stability, and then give sufficient conditions for the dissipativity and exponential stability uniform in sufficiently small ε in Section 3. Section 4 deals with numerical dissipativity and exponential stability of linear $\theta-$method and one-leg $\theta-$method for singularly perturbed DDEs. These results are also applicable to general DDEs with a bounded lag (i.e., $\varepsilon = 1$).

2. A Generalized Halanay Inequality

The following lemma generalizes the famous Halanay inequality (see Halanay [11]) and will play a key role in obtaining our main result of this paper.

Lemma 2.1. *(A generalized Halanay inequality [20]) Suppose*

$$u'(t) \le \gamma(t) - \alpha(t)u(t) + \beta(t) \sup_{t-\tau \le \sigma \le t} u(\sigma)$$

for $t \ge t_0$. Here $\tau \ge 0$ and continuous functions $\gamma(t), \alpha(t), \beta(t)$ satisfy $0 \le \gamma(t) < \gamma^*, \alpha(t) \ge \alpha_0 > 0$, and $0 \le \beta(t) \le q\alpha(t)$ for all $t \ge t_0$ with $0 \le q < 1$. Then

$$u(t) \le \frac{\gamma^*}{(1-q)\alpha_0} + Ge^{-\mu^\star(t-t_0)} \qquad \text{for } t \ge t_0. \tag{11}$$

Here $G = \sup_{t_0-\tau \le t \le t_0} |u(t)|$, and $\mu^\star > 0$ is defined as

$$\mu^\star = \inf_{t \ge t_0} \{\mu(t) : \mu(t) - \alpha(t) + \beta(t)e^{\mu(t)\tau} = 0\}.$$

Proof. Note that the result is trivial if $\tau = 0$. In the following we assume that $\tau > 0$. Denote

$$H(\mu) \equiv \mu - \alpha(t) + \beta(t)e^{\mu\tau}. \tag{12}$$

By assumption $\alpha(t) \ge \alpha_0 > 0, 0 \le \beta(t) \le q\alpha(t)$ for all $t \ge t_0$, then for any given fixed $t \ge t_0$, we see that $H(0) = -\alpha(t) + \beta(t) \le -(1-q)\alpha(t) \le -(1-q)\alpha_0 < 0$, $\lim_{\mu\to\infty} H(\mu) = \infty$, and $H'(\mu) = 1 + \beta(t)\tau e^{\mu\tau} > 0$. Therefore for any $t \ge t_0$ there is a unique positive $\mu(t)$ such that $\mu(t) - \alpha(t) + \beta(t)e^{\mu(t)\tau} = 0$. From the definition, one has $\mu^\star \ge 0$. We have to prove $\mu^\star > 0$. Suppose this is not true. Fix $\tilde{q}$ satisfying $0 \le q < \tilde{q} < 1$ and pick $0 < \epsilon < \min\{(1 - \frac{q}{\tilde{q}})\alpha_0, \frac{1}{\tau}\ln(\frac{1}{\tilde{q}})\}$. Then there is $t^\star \ge t_0$ such that $\tilde{\mu}(t^\star) < \epsilon$ and

$$\tilde{\mu}(t^\star) - \alpha(t^\star) + \beta(t^\star)e^{\tilde{\mu}(t^\star)\tau} = 0.$$

Now we have

$$\begin{array}{rcl} 0 & = & \tilde{\mu}(t^\star) - \alpha(t^\star) + \beta(t^\star)e^{\tilde{\mu}(t^\star)\tau} \\ & < & \epsilon - \alpha(t^\star) + \beta(t^\star)e^{\epsilon\tau} \\ & < & \epsilon - \alpha(t^\star) + \frac{1}{\tilde{q}}\beta(t^\star) \\ & \le & \epsilon - \alpha(t^\star) + \frac{q}{\tilde{q}}\alpha(t^\star) \\ & = & \epsilon - (1-\frac{q}{\tilde{q}})\alpha(t^\star) \\ & \le & \epsilon - (1-\frac{q}{\tilde{q}})\alpha_0 \\ & < & 0, \end{array} \tag{13}$$

which is a contradiction.

Suppose (11) fails. Then there is some $t > t_0$ such that

$$\frac{\gamma^*}{(1-q)\alpha_0} + kGe^{-\mu^*(t-t_0)} < u(t).$$

Set $v(t) = \frac{\gamma^*}{(1-q)\alpha_0} + Ge^{-\mu^\star(t-t_0)}$ and $w(t) = v(t) - u(t)$. Let $\varsigma = \inf\{t \ge t_0 : v(t) - u(t) \le 0\}$. Then we have for some $\varsigma > t_0$ such that $w(\varsigma) = v(\varsigma) - u(\varsigma) = 0$ and

$$w'(\varsigma) = v'(\varsigma) - u'(\varsigma) \le 0. \tag{14}$$

Hence

$$\begin{array}{rcl} w'(\varsigma) & = & v'(\varsigma) - u'(\varsigma) \\ & \ge & -G\mu^\star e^{-\mu^\star(\varsigma-t_0)} - \gamma(\varsigma) - \left[-\alpha(\varsigma)u(\varsigma) + \beta(\varsigma)\sup\limits_{\varsigma-\tau\le\sigma\le\varsigma} u(\sigma)\right] \\ & > & -G\mu^\star e^{-\mu^\star(\varsigma-t_0)} - \gamma^*\left(1 - \dfrac{\alpha(\varsigma)-\beta(\varsigma)}{(1-q)\alpha_0}\right) \\ & & +G\alpha(\varsigma)e^{-\mu^\star(\varsigma-t_0)} - G\beta(\varsigma)e^{-\mu^\star(\varsigma-t_0-\tau)} \\ & \ge & Ge^{-\mu^\star(\varsigma-t_0)}\left[-\mu^\star + \alpha(\varsigma) - \beta(\varsigma)e^{\mu^\star\tau}\right]. \end{array}$$

Let $\mu(\varsigma)$ satisfy $\mu(\varsigma) - \alpha(\varsigma) + \beta(\varsigma)e^{\mu(\varsigma)\tau} = 0$. Then according to the definition of $\mu^\star$, it follows that

$$\begin{array}{rcl} -\mu^\star + \alpha(\varsigma) - \beta(\varsigma)e^{\mu^\star\tau} & = & [-\mu^\star + \alpha(\varsigma) - \beta(\varsigma)e^{\mu^\star\tau}] + [\mu(\varsigma) - \alpha(\varsigma) + \beta(\varsigma)e^{\mu(\varsigma)\tau}] \\ & = & (\mu(\varsigma) - \mu^\star) + \beta(\varsigma)[e^{\mu(\varsigma)\tau} - e^{\mu^\star\tau}] \\ & \ge & 0. \end{array}$$

Therefore

$$\begin{array}{rcl} w'(\varsigma) & = & v'(\varsigma) - u'(\varsigma) \\ & > & Ge^{-\mu^\star(\varsigma-t_0)}\left[-\mu^\star + \alpha(\varsigma) - \beta(\varsigma)e^{\mu^\star\tau}\right] \\ & \ge & 0. \end{array}$$

This contradicts (14) and hence

$$u(t) \le \frac{\gamma^*}{(1-q)\alpha_0} + Ge^{-\mu^\star(t-t_0)} \qquad \text{for } t \ge t_0.$$

This completes the proof.

Remark. In the case $\gamma(t) \equiv 0, \alpha(t) \equiv \alpha > \beta(t) \equiv \beta$ then the result is a reformulation of Halanay's statement in Halanay [11].

3. Dissipativity and Exponential Stability

In this section we will apply the generalized Halanay inequality to study the following dissipativity and exponential stability for the nonlinear singularly perturbed delay differential

system.
Consider

$$\varepsilon y'(t,\varepsilon) = f(t, y(t,\varepsilon), y(t-\tau(t),\varepsilon)), \qquad t \geq t_0, \tag{15}$$

with initial function

$$y(t,\varepsilon) = \phi(t), \qquad t \leq t_0, \tag{16}$$

where $f : \mathbb{R}^+ \equiv [t_0,\infty) \times \mathbb{C}^s \times \mathbb{C}^s \mapsto \mathbb{C}^s$, and $y(t,\varepsilon) : \mathbb{R}^+ \times \mathbb{R}^+ \mapsto \mathbb{C}^s$.

3.1. Dissipativity

Many interesting problems arising in physics and engineering are modelled by dissipative dynamical systems. They are characterized by possessing a bounded positively invariant absorbing set in which all trajectories enter in a finite time and thereafter remain.

Definition 3.1. *A singularly perturbed delay differential equation (15) is said to be dissipative uniformly in ε if there is a bounded, positively invariant set $B \subseteq \mathbb{C}^s$ with the property that for any bounded set $E \subseteq \mathbb{C}^s$, there exists $t^* = t^*(B,E) \geq t_0$, which is independent of ε, such that $y(t,\varepsilon) \in E$ for all $t > t^*$ and for any initial functions $\phi(t) \in B$. The set B is called an absorbing set.*

We have the following theorem.

Theorem 3.2. *Consider*

$$\begin{aligned} \varepsilon y'(t,\varepsilon) &= f(t, y(t,\varepsilon), y(t-\tau(t),\varepsilon)), \qquad t \geq t_0, \\ y(t,\varepsilon) &= \phi(t), \qquad t_0 - \tau^* \leq t \leq t_0, \end{aligned} \tag{17}$$

where f is sufficiently differentiable, $0 \leq \tau(t) \leq \tau^\star$, where τ^* is a constant, and the initial function $\phi(t)$ is continuous for $t_0 - \tau^\star \leq t \leq t_0$. Suppose

$$\begin{aligned} \Re\langle f(t,y,u), y\rangle &\leq \gamma(t) - \alpha(t) \parallel y \parallel^2 + \beta(t)\|u\|^2, \\ &\quad \forall t \in \mathbb{R}^+, \forall u, y \in \mathbb{C}^s, \end{aligned} \tag{18}$$

and $\alpha(t), \beta(t), \gamma(t)$ are continuous functions and satisfy for $t \geq t_0$

$$0 \leq \gamma(t) < \gamma^*, \quad \alpha(t) \geq \alpha_0 > 0, \quad 0 \leq \beta(t) \leq q\alpha(t), \quad 0 \leq q < 1, \tag{19}$$

where $\|\cdot\|$ is the induced norm of the inner product $\langle u, v\rangle = v^H u$, and α_0, γ^*, q are constants. Then (15) has a unique solution, and there exists a small $\varepsilon_0 > 0$ such that (15) is dissipative uniformly for sufficiently small $\varepsilon \in (0, \varepsilon_0]$ with an absorbing set $B = B\left(0, \sqrt{\frac{\gamma^*}{(1-q)\alpha_0} + \delta}\right)$ for any given $\delta > 0$.

Proof. According to the definition of the norm, we have

$$\begin{aligned} \frac{1}{2}\varepsilon \frac{d}{dt}\left(\parallel y(t,\varepsilon) \parallel^2\right) &= \Re\langle \varepsilon y'(t,\varepsilon), y(t,\varepsilon)\rangle \\ &= \Re\langle f(t, y(t,\varepsilon), y(t-\tau(t),\varepsilon)), y(t,\varepsilon)\rangle \\ &\leq \gamma(t) + \alpha(t) \parallel y(t,\varepsilon) \parallel^2 + \beta(t) \sup_{t-\tau^\star \leq \sigma \leq t} \|y(\sigma,\varepsilon)\| \end{aligned}$$

Denote $V(t,\varepsilon) = \|y(t,\varepsilon)\|^2$. It follows with $\varepsilon > 0$ that

$$V'(t,\varepsilon) \leq \frac{2\alpha(t)}{\varepsilon} V(t,\varepsilon) + \frac{2\beta(t)}{\varepsilon} \sup_{t-\tau^\star \leq \sigma \leq t} V(\sigma,\varepsilon), \qquad t \geq t_0.$$

Application of the inequality in Lemma 2.1 to the above equation yields

$$V(t,\varepsilon) \leq \frac{\gamma^*}{(1-q)\alpha_0} + G e^{-\mu^\star(\varepsilon)(t-t_0)}, \qquad t \geq t_0. \tag{20}$$

Here

$$\mu^\star(\varepsilon) = \inf_{t\geq t_0}\{\mu(t) : \mu(t) - \frac{2\alpha(t)}{\varepsilon} + \frac{2\beta(t)}{\varepsilon}e^{\mu(t)\tau^\star} = 0\}, \tag{21}$$

and $G \geq 0$ only depends on the initial condition $\|\phi(t)\|$.
For any fixed $t \geq t_0$, let $\mu(t,\varepsilon)$ be defined as the unique positive zero of

$$\mu - \frac{2\alpha(t)}{\varepsilon} + \frac{2\beta(t)}{\varepsilon}e^{\mu\tau^\star} = 0. \tag{22}$$

It can be proven that $\mu(t,\varepsilon_1) \geq \mu(t,\varepsilon_2)$ whenever $\varepsilon_2 \geq \varepsilon_1 > 0$. This indicates that $\mu^\star(\varepsilon_1) \geq \mu^\star(\varepsilon_2)$ and thus we proved that $\mu^\star(\varepsilon)$ is monotonically decreasing with respect to the variable ε. Hence we deduce that there exists a small $\varepsilon_0 > 0$ such that the solution $y(t,\varepsilon)$ bounded uniformly for sufficiently small $\varepsilon \in (0,\varepsilon_0]$ as

$$\|y(t,\varepsilon)\|^2 \leq \frac{\gamma^*}{(1-q)\alpha_0} + Ge^{-\mu^*(\varepsilon_0)(t-t_0)}.$$

Hence the solution can not blow up and the system has a uniquely defined solution for all $t \geq t_0$. It is very easy to verify that for any given $\delta > 0$, B is an absorbing set. This completes the proof.

Remark. From the generalized Halanay inequality and the proof above, it is obvious that the result of this theorem still holds when we set $\varepsilon = 1$.

3.2. Exponential Stability

We start with the following definition for exponential stability.

Definition 3.3. *The solution $y(t,\varepsilon)$ of Equation (15) is said to be ν-exponentially stable uniformly for sufficiently small ε if it is asymptotically stable and there exist finite constants $K > 0, \nu > 0$ and $\delta > 0$, which are independent of $\varepsilon \in (0,\varepsilon_0]$ for some ε_0, such that $\|y(t,\varepsilon) - z(t,\varepsilon)\| \leq Ke^{-\nu(t-t_0)}$ for $t \geq t_0$ and for any initial perturbation satisfying $\sup_{s\in[t_0-\tau,t_0]}\|\phi(s)-\psi(s)\| < \delta$. Here $z(t,\varepsilon)$ is the solution of Equation (15) corresponding to the initial function ψ.*

Theorem 3.4. *Consider*

$$\begin{aligned} \varepsilon y'(t,\varepsilon) &= f(t, y(t,\varepsilon), y(t-\tau(t),\varepsilon)), \qquad t \geq t_0, \\ y(t,\varepsilon) &= \phi_1(t), \qquad t \leq t_0, \end{aligned} \tag{23}$$

and

$$\begin{aligned} \varepsilon z'(t,\varepsilon) &= f(t, z(t,\varepsilon), z(t-\tau(t),\varepsilon)), \qquad t \geq t_0, \\ z(t,\varepsilon) &= \phi_2(t), \qquad t \leq t_0, \end{aligned} \tag{24}$$

where f is sufficiently differentiable with respect to both the last two variables, $0 \leq \tau(t) \leq \tau^\star$, where τ^* is a constant, and the initial functions $\phi_1(t)$ and $\phi_2(t)$ are continuous for $t_0 - \tau^\star \leq t \leq t_0$. Suppose

$$\Re\langle f(t,y_1,u) - f(t,y_2,u), y_1 - y_2\rangle \leq \eta(t) \| y_1 - y_2 \|^2, \quad \forall t \in \mathbb{R}^+, \forall u, y_1, y_2 \in \mathbb{C}^s, \tag{25}$$

$$\| f(t,y,u_1) - f(t,y,u_2) \| \leq \zeta(t) \| u_1 - u_2 \|, \quad \forall t \in \mathbb{R}^+, \forall y, u_1, u_2 \in \mathbb{C}^s, \tag{26}$$

and $\eta(t), \zeta(t)$ are continuous and satisfy for $t \geq t_0$

$$\eta(t) \leq -\eta_0 < 0, \qquad 0 \leq \zeta(t) \leq -q\eta(t), \qquad 0 \leq q < 1, \tag{27}$$

where η_0 and q are constants. If (23) and (24) each has a unique solution, then there exists a small $\varepsilon_0 > 0$ such that the solution of (15) is exponentially stable uniformly for sufficiently small $\varepsilon \in (0,\varepsilon_0]$.

Proof. The theorem can be proved analogously to Theorem 3.2 by setting $\gamma^* > 0$ and then take the limit with respect to γ^*.

4. Linear $\theta-$Method

In this section we shall examine the numerical dissipativity and exponential stability of the linear $\theta-$method. The linear $\theta-$method applied to (15) with constant step size $h \geq \varepsilon > 0$ gives rise to

$$\begin{aligned} \varepsilon y_{n+1}^{\varepsilon} &= \varepsilon y_n^{\varepsilon} + h\theta f(t_{n+1}, y_{n+1}^{\varepsilon}, y_h^{\varepsilon}(t_{n+1} - \tau(t_{n+1}))) \\ &\quad + h(1-\theta) f(t_n, y_n^{\varepsilon}, y_h^{\varepsilon}(t_n - \tau(t_n))), \end{aligned} \tag{28}$$

where $y_h^{\varepsilon}(t) = \phi_1(t)$ for $t \leq t_0$, and $y_h^{\varepsilon}(t)$ with $t \geq t_0$ are defined by a piecewise linear interpolation procedure, for instance,

$$\begin{aligned} y_h^{\varepsilon}(t) &= \frac{t - nh}{h} y_{n+1}^{\varepsilon} + \frac{(n+1)h - t}{h} y_n^{\varepsilon}, \\ &\qquad \text{for } nh \leq t \leq (n+1)h, n = 0, 1, 2, \ldots. \end{aligned}$$

4.1. Numerical dissipativity

Let y_n^{ε} denote an approximate solution $y_n^{\varepsilon}(\phi)$ to (15) computed using a prescribed numerical method and a given initial function $\phi(t)$ for $t \leq t_0$. Corresponding to the previous dissipativity property of the analytic solution, we therefore introduce the following analogous definition for an approximate solution.

Definition 4.1. *A numerical method is said to be dissipative uniformly for sufficiently small ε if there exists positive constant r such that, for any initial function $\phi(t)$, there is a positive integer n_0 such that the resulting numerical solution $\|y_n^{\varepsilon}\| \leq r$. Here r and n_0 are independent of $\varepsilon \in (0, \varepsilon_0]$ for some ε_0,*

Theorem 4.2. *Assume that the function f in equations (15) satisfies (19) and $0 \leq \tau(t) < \tau^{\star}$. Then the linear $\theta-$method with $h \geq \varepsilon > 0$ is dissipative uniformly in $\varepsilon \in (0, \varepsilon_0]$ if and only if $\theta = 1$.*

Proof. For $\theta \in [0, 1)$ and $1 > \varepsilon = \varepsilon_1 > 0$, where ε_1 is any given positive number, consider the following special DDE

$$\begin{aligned} \varepsilon_1 y'(t, \varepsilon_1) &= -\chi(t) y(t, \varepsilon_1) - \theta \chi(t) y(t-1, \varepsilon_1), \quad t \geq 0, \\ y(t, \varepsilon_1) &= \phi(t), \quad t \leq 0, \end{aligned}$$

where $\chi(t) \geq \frac{\varepsilon_1(1-\theta)}{4}$ is a real continuous function, and $\phi(t)$ is continuous. Then the condition (19) is satisfied. Set $a(t) = \frac{\chi(t)}{\varepsilon_1}$. Let step size $h = 1$. Then (28) reads

$$y_{n+1}^{\varepsilon_1} = \frac{1 - (1-\theta)a(t_n) - \theta^2 a(t_{n+1})}{1 + \theta a(t_{n+1})} y_n^{\varepsilon_1} - \frac{\theta(1-\theta)a(t_n)}{1 + \theta a(t_{n+1})} y_{n-1}^{\varepsilon_1}, \tag{29}$$

where $n = 0, 1, 2, \cdots$.

Now we choose a sequence $\{a(t_n)\}$ such that it is 2-periodic with $a(t_0) = a(t_2) = \cdots = e = \frac{1-\theta}{4}$, $a(t_1) = a(t_3) = \cdots = f = \frac{4}{1-\theta}$. Define $Y_k^{\varepsilon_1} = (y_k^{\varepsilon_1}, y_{k-1}^{\varepsilon_1})^T$. Hence Equation (29) is equivalent to

$$Y_{n+1}^{\varepsilon_1} = A_n Y_n^{\varepsilon_1},$$

where $n = 0, 1, ...,$ and

$$A_n = \begin{bmatrix} \frac{1-(1-\theta)a(t_n)-\theta^2 a(t_{n+1})}{1+\theta a(t_{n+1})} & \frac{-\theta(1-\theta)a(t_n)}{1+\theta a(t_{n+1})} \\ 1 & 0 \end{bmatrix}$$

The periodicity of the sequence $\{a(t_n)\}$ yields

$$Y_{n+2}^{\varepsilon_1} = BY_n^{\varepsilon_1},$$

where $B = A_{n+1}A_n$, $n = 0, 2, 4, \ldots$.

We obtain

$$B = \begin{bmatrix} c_1 d_1 + c_2 & c_1 d_2 \\ d_1 & d_2 \end{bmatrix}$$

where

$$\begin{aligned} c_1 &= \frac{1-(1-\theta)e-\theta^2 f}{1+\theta f}, \quad c_2 = \frac{-\theta(1-\theta)e}{1+\theta f}, \\ d_1 &= \frac{1-(1-\theta)f-\theta^2 e}{1+\theta e}, \quad d_2 = \frac{-\theta(1-\theta)f}{1+\theta e}. \end{aligned}$$

Thus

$$\begin{aligned} \det(\lambda I - B) &= \lambda^2 - (d_2 + c_1 d_1 + c_2)\lambda + d_2 c_2 \\ &= \lambda^2 - \frac{1-e-f+(1-\theta)^2+\theta^4}{(1+\theta e)(1+\theta f)}\lambda + \frac{1+\theta f+\theta e+\theta^2+\theta^2(1-\theta)^2}{(1+\theta e)(1+\theta f)}. \end{aligned}$$

Since

$$\left| \frac{1+\theta e+\theta f+\theta^2+\theta^2(1-\theta)^2}{(1+\theta e)(1+\theta f)} \right| > 1,$$

the spectral radius of the iteration matrix B is greater than 1. Hence the numerical solution generated by the linear $\theta-$method blows up for some initial functions and hence the linear $\theta-$method is not dissipative for $\theta \in [0, 1)$.

For any fixed step size $h > 0$ and $\theta = 1$, (28) reads

$$\varepsilon y_{n+1}^{\varepsilon} = \varepsilon y_n^{\varepsilon} + hf(t_{n+1}, y_{n+1}^{\varepsilon}, y_h^{\varepsilon}(t_{n+1} - \tau(t_{n+1}))). \tag{30}$$

Now $\tau^\star$ can be written as $\tau^\star = mh - \delta h, \delta \in [0, 1)$, and m is a nonnegative integer. From (30), it follows that

$$\begin{aligned} \|y_{n+1}^{\varepsilon}\|^2 &= \|y_n^{\varepsilon}\|^2 + \frac{2h}{\varepsilon}\Re\langle y_{n+1}^{\varepsilon}, f(t_{n+1}, y_{n+1}^{\varepsilon}, y_h^{\varepsilon}(t_{n+1} - \tau(t_{n+1})))\rangle \\ &\quad -\frac{2h}{\varepsilon}\langle f(t_{n+1}, y_{n+1}^{\varepsilon}, y_h^{\varepsilon}(t_{n+1} - \tau(t_{n+1}))), f(t_{n+1}, y_{n+1}^{\varepsilon}, y_h^{\varepsilon}(t_{n+1} - \tau(t_{n+1})))\rangle \\ &\le \|y_n^{\varepsilon}\|^2 + \frac{2h}{\varepsilon}\left(\gamma(t_{n+1}) - \alpha(t_{n+1})\|y_{n+1}^{\varepsilon}\|^2 + \beta(t_{n+1})\|y_h^{\varepsilon}(t_{n+1} - \tau(t_{n+1}))\|^2\right). \end{aligned}$$

Therefore

$$\|y_{n+1}^{\varepsilon}\|^2 \le \frac{2h\gamma(t_{n+1})}{\varepsilon + 2h\alpha(t_{n+1})} + \frac{\varepsilon}{\varepsilon + 2h\alpha(t_{n+1})}\|y_n^{\varepsilon}\|^2 + \frac{2h\beta(t_{n+1})}{\varepsilon + 2h\alpha(t_{n+1})}\|y_h^{\varepsilon}(t_{n+1} - \tau(t_{n+1}))\|^2.$$

From the interpolation procedure, we have, if $\tau(t_{n+1}) \ge h$,

$$\|y_{n+1}^{\varepsilon}\|^2 \le \frac{2h\gamma(t_{n+1})}{\varepsilon + 2h\alpha(t_{n+1})} + \frac{\varepsilon}{\varepsilon + 2h\alpha(t_{n+1})}\|y_n^{\varepsilon}\|^2 + \frac{2h\beta(t_{n+1})}{\varepsilon + 2h\alpha(t_{n+1})}\sup_{1\le s\le m}\|y_{n+1-s}^{\varepsilon}\|^2$$

or, if $\tau(t_{n+1}) = (1 - \delta_{n+1})h, \delta_{n+1} \in (0, 1)$,

$$\|y_{n+1}^{\varepsilon}\|^2 \le \frac{2h\gamma(t_{n+1})}{\varepsilon + 2h\alpha(t_{n+1}) - 2h\delta_{n+1}\beta(t_{n+1})} + \frac{\varepsilon + 2(1-\delta_{n+1})h\beta(t_{n+1})}{\varepsilon + 2h\alpha(t_{n+1}) - 2\delta_{n+1}h\beta(t_{n+1})}\|y_n^{\varepsilon}\|^2.$$

It is true that the following inequality

$$\|y_{n+1}^{\varepsilon}\|^2 \le \frac{\gamma^*}{(1-q)\alpha_0} + \frac{\varepsilon}{\varepsilon + 2h\alpha(t_{n+1})}\|y_n^{\varepsilon}\|^2 + \frac{h\beta(t_{n+1})}{\varepsilon + 2h\alpha(t_{n+1})}\sup_{1\le s\le m}\|y_{n+1-s}^{\varepsilon}\|^2 \tag{31}$$

holds.
Set $A=\frac{\gamma^*}{(1-q)\alpha_0}$ and $B=\frac{1}{1+2\alpha_0}$. When $n=0$, (31) reads

$$\|y_1^\varepsilon\|^2 \le A+\frac{\varepsilon}{\varepsilon+2h\alpha(t_1)}\|y_0^\varepsilon\|^2+\frac{2h\beta(t_1)}{\varepsilon+2h\alpha(t_1)}\sup_{1\le s\le m}\|y_{1-s}^\varepsilon\|^2 \le A+M(1+P(\varepsilon)h),$$

where

$$\begin{aligned} P(\varepsilon) &= \frac{-2(1-q)h\alpha_0}{\varepsilon+2h\alpha_0}<0, \\ M &= \max_{-m\le s\le 0}\|y_s^\varepsilon\|^2. \end{aligned}$$

One easily shows that for any $n\le m-1$

$$\|y_{n+1}^\varepsilon\|^2 \le \sum_{i=0}^{n} AB^i+M(1+P(\varepsilon)h).$$

Applying the technique of induction with respect to n, we can conclude that

$$\|y_n^\varepsilon\|^2 \le \sum_{i=0}^{n-1} AB^i+M(1+P(\varepsilon)h)^{r+1} \le A\sum_{i=0}^{n} B^i+M\exp(P(\varepsilon)(r+1)h)$$

for $rm<n\le(r+1)m, r=0,1,2,\ldots$. Choosing $\widehat{\mu}(\varepsilon)=-\frac{P(\varepsilon)}{m}>0$ indicates that

$$\|y_{n+1}^\varepsilon\|^2 \le \frac{A}{1-B}+\exp(-\widehat{\mu}(\varepsilon)(t_n-t_0)).$$

Since $h\ge\varepsilon>0$, we observe that

$$\widehat{\mu}(\varepsilon)=-\frac{P(\varepsilon)}{m}=\frac{2(1-q)h\alpha_0}{m\varepsilon+2mh\alpha_0}\ge\frac{2(1-q)\alpha_0}{\tau^\star(1+2\alpha_0)}>0$$

and hence prove the backward Euler method (i.e. $\theta=1$) is dissipative uniformly in ε.

4.2. Numerical Stability

Let y_n^ε denote an approximate solution $y_n^\varepsilon(\phi)$ to (15) computed using a prescribed numerical method and a given initial function $\phi(t)$ for $t\le t_0$. We can then write $\delta y_n^\varepsilon := y_n^\varepsilon(\phi+\delta\phi)-y_n^\varepsilon(\phi)$. Corresponding to the exponential stability property of the analytic solution, we introduce the following analogous definition for an approximate solution.

Definition 4.3. *(Exponentially stable uniformly for small ε) A numerical solution $\{y_n^\varepsilon\}$ computed by a prescribed numerical method is called exponentially stable for sufficiently small ε if there exist positive constants $\widetilde{G}$ and $\widetilde{\mu}$, which are independent of $\varepsilon\in(0,\varepsilon_0]$ for some ε_0, such that $\|\delta y_n^\varepsilon\|\le\widetilde{G}\exp(-\widetilde{\mu}(t_n-t_0))$ (corresponding to two initial functions $\phi,\phi+\delta\phi$ defined for $t\le t_0$).* We also call the prescribed numerical method exponentially stable if the resulting numerical solution is exponentially stable, under the assumed conditions, when the corresponding analytic solution is exponentially stable.

Definition 4.4. *Assume that the function f in equations (23) and (24) (s=1) satisfies (27) and $0\le\tau(t)<\tau^\star$. Then the resulting method with $h\ge\varepsilon>0$ is exponentially stable uniformly in $\varepsilon\in(0,\varepsilon_0]$ if and only if $\theta=1$.*

Proof. The linear $\theta-$method applied to (23) and (24) with step size $h\ge\varepsilon>0$ gives rise to

$$\begin{aligned} \varepsilon y_{n+1}^\varepsilon &= \varepsilon y_n^\varepsilon+h\theta f(t_{n+1},y_{n+1}^\varepsilon,y_h^\varepsilon(t_{n+1}-\tau(t_{n+1}))) \\ &\quad +h(1-\theta)f(t_n,y_n^\varepsilon,y_h^\varepsilon(t_n-\tau(t_n))) \end{aligned} \tag{32}$$

and

$$\begin{aligned} \varepsilon z_{n+1}^\varepsilon &= \varepsilon z_n^\varepsilon+h\theta f(t_{n+1},z_{n+1}^\varepsilon,z_h^\varepsilon(t_{n+1}-\tau(t_{n+1}))) \\ &\quad +h(1-\theta)f(t_n,z_n^\varepsilon,z_h^\varepsilon(t_n-\tau(t_n))), \end{aligned} \tag{33}$$

respectively, where $y_h^\varepsilon(t)=\phi_1(t), z_h^\varepsilon(t)=\phi_2(t)$ for $t\le t_0$, and $y_h^\varepsilon(t)$ and $z_h^\varepsilon(t)$ with $t\ge t_0$ are defined by a piecewise linear interpolation procedure. Tian and Kuang [19] have proved that the linear $\theta-$method is not asymptotically stable for $\theta \in [0,1)$. So it remains to prove the backward Euler method is exponentially stable uniformly in ε.

For any fixed step size $h>0$, (32) and (33) read

$$\begin{aligned} \varepsilon y_{n+1}^{\varepsilon} &= \varepsilon y_n^{\varepsilon} + hf(t_{n+1}, y_{n+1}^{\varepsilon}, y_h^{\varepsilon}(t_{n+1}-\tau(t_{n+1}))), \quad (34)\\ \varepsilon z_{n+1}^{\varepsilon} &= \varepsilon z_n^{\varepsilon} + hf(t_{n+1}, z_{n+1}^{\varepsilon}, z_h^{\varepsilon}(t_{n+1}-\tau(t_{n+1}))). \quad (35) \end{aligned}$$

Now $\tau^\star$ can be written as $\tau^\star = mh - \delta h, \delta \in [0,1)$, and m is a nonnegative integer. Subtracting (35) from (34) gives

$$\begin{aligned} & \varepsilon(y_{n+1}^{\varepsilon} - z_{n+1}^{\varepsilon}) - \varepsilon(y_n^{\varepsilon} - z_n^{\varepsilon}) \\ = \; & h\left[f(t_{n+1}, y_{n+1}^{\varepsilon}, y_h^{\varepsilon}(t_{n+1}-\tau(t_{n+1}))) - f(t_{n+1}, z_{n+1}^{\varepsilon}, z_h^{\varepsilon}(t_{n+1}-\tau(t_{n+1})))\right] \\ = \; & h\left[f(t_{n+1}, y_{n+1}^{\varepsilon}, y_h^{\varepsilon}(t_{n+1}-\tau(t_{n+1}))) - f(t_{n+1}, z_{n+1}^{\varepsilon}, y_h^{\varepsilon}(t_{n+1}-\tau(t_{n+1})))\right] \quad (36)\\ & + h\left[f(t_{n+1}, z_{n+1}^{\varepsilon}, y_h^{\varepsilon}(t_{n+1}-\tau(t_{n+1}))) - f(t_{n+1}, z_{n+1}^{\varepsilon}, z_h^{\varepsilon}(t_{n+1}-\tau(t_{n+1})))\right] \\ = \; & hd_{n+1}\left[y_{n+1}^{\varepsilon} - z_{n+1}^{\varepsilon}\right] + hl_{n+1}\left[y_h^{\varepsilon}(t_{n+1}-\tau(t_{n+1})) - z_h^{\varepsilon}(t_{n+1}-\tau(t_{n+1}))\right], \end{aligned}$$

where

$$\begin{aligned} & f(t_{n+1}, y_{n+1}^{\varepsilon}, y_h^{\varepsilon}(\tau(t_{n+1}))) - f(t_{n+1}, z_{n+1}^{\varepsilon}, y_h^{\varepsilon}(\tau(t_{n+1}))) = d_{n+1}(y_{n+1}^{\varepsilon} - z_{n+1}^{\varepsilon}), \\ & f(t_{n+1}, z_{n+1}^{\varepsilon}, y_h^{\varepsilon}(t_{n+1}-\tau(t_{n+1}))) - f(t_{n+1}, z_{n+1}^{\varepsilon}, z_h^{\varepsilon}(t_{n+1}-\tau(t_{n+1}))) \\ & \quad = l_{n+1}\left[y_h^{\varepsilon}(t_{n+1}-\tau(t_{n+1})) - z_h^{\varepsilon}(t_{n+1}-\tau(t_{n+1}))\right]. \end{aligned}$$

Condition (27) implies that $\Re(d_{n+1}) \le \eta(t_{n+1})$ and $|l_{n+1}| \le \zeta(t_{n+1})$. From (36), it follows that

$$e_{n+1}^{\varepsilon} \le \left|\frac{\varepsilon}{\varepsilon - hd_{n+1}}\right| e_n^{\varepsilon} + \left|\frac{h\zeta(t_{n+1})}{\varepsilon - hd_{n+1}}\right| \sup_{1\le s\le m} e_{n+1-s}^{\varepsilon}, \text{ if } \tau(t_{n+1}) \ge h,$$

or

$$e_{n+1}^{\varepsilon} \le \left|\frac{\varepsilon + (1-\delta_{n+1})hl_{n+1}}{\varepsilon - hd_{n+1} - \delta_{n+1}hl_{n+1}}\right| e_n^{\varepsilon}, \quad \text{if } \tau(t_{n+1}) = (1-\delta_{n+1})h,$$

where $e_{n+1}^{\varepsilon} = |y_{n+1}^{\varepsilon} - z_{n+1}^{\varepsilon}|$, and $\delta_{n+1} \in (0,1)$. It is true that the following inequality

$$e_{n+1}^{\varepsilon} \le \left|\frac{\varepsilon}{\varepsilon - h\eta(t_{n+1})}\right| e_n^{\varepsilon} + \left|\frac{h\zeta(t_{n+1})}{\varepsilon - h\eta(t_{n+1})}\right| \sup_{1\le s\le m} e_{n+1-s}^{\varepsilon} \quad (37)$$

holds.

Keep the condition (27) in mind and proceed step by step. When $n=0$, (37) reads

$$e_1^{\varepsilon} \le \left|\frac{\varepsilon}{\varepsilon - h\eta(t_1)}\right| e_0^{\varepsilon} + \left|\frac{h\zeta(t_1)}{\varepsilon - h\eta(t_1)}\right| \sup_{1\le s\le m} e_{1-s}^{\varepsilon} \le M(1+P(\varepsilon)h),$$

where

$$\begin{aligned} P(\varepsilon) &= \frac{-(1-q)\eta_0}{\varepsilon + h\eta_0} < 0, \\ M &= \max_{-m\le s\le 0} e_s^{\varepsilon}. \end{aligned}$$

One easily shows that for any $n \le m-1$

$$e_{n+1}^{\varepsilon} \le M(1+P(\varepsilon)h).$$

Applying the technique of induction with respect to n, we can conclude that

$$e_n^{\varepsilon} \le M(1+P(\varepsilon)h)^{r+1} \le M\exp(P(\varepsilon)(r+1)h)$$

for $rm < n \le (r+1)m, r = 0,1,2,\ldots$. Choosing $\widehat{\mu}(\varepsilon) = -\frac{P(\varepsilon)}{m} > 0$ indicates that $e_{n+1}^{\varepsilon} \le M\exp(-\widehat{\mu}(\varepsilon)(t_n - t_0))$. Since $h \ge \varepsilon > 0$, we observe that

$$\widehat{\mu}(\varepsilon) = -\frac{P(\varepsilon)}{m} = \frac{(1-q)\eta_0}{m\varepsilon + mh\eta_0} \ge \frac{(1-q)\eta_0}{\tau^\star(1+\eta_0)} > 0$$

and hence prove the Backward Euler method is exponentially stable uniformly in ε.

5. One-leg $\theta-$Method

We new investigate the numerical dissipativity and exponential stability of the one-leg $\theta-$method. The one-leg $\theta-$method applied to (15) with constant step size $h \geq \varepsilon > 0$ gives rise to

$$\varepsilon y^{\varepsilon}_{n+1} = \varepsilon y^{\varepsilon}_{n} + hf(t_n + \theta h, \theta y^{\varepsilon}_{n+1} + (1-\theta) y^{\varepsilon}_{n}, y^{\varepsilon}_{h}(t_n + \theta h - \tau(t_n + \theta h))) \tag{38}$$

where $y^{\varepsilon}_{h}(t) = \phi_1(t)$ for $t \leq t_0$, and $y^{\varepsilon}_{h}(t)$ with $t \geq t_0$ are defined in the same way by the piecewise linear interpolation.

Theorem 5.1. *Dissipativity Assume that the function f in equations (15) satisfies (19) and $0 \leq \tau(t) < \tau^{\star}$. Then the one-leg $\theta-$method with $h \geq \varepsilon > 0$ is dissipative uniformly in $\varepsilon \in (0, \varepsilon_0]$ if and only if $\theta = 1$.*

Proof. For $\theta \in [0, 1)$, consider the following special DDE

$$\begin{aligned} \varepsilon y'(t,\varepsilon) &= -ay(t,\varepsilon) - \theta a y(t-\theta,\varepsilon), \quad t \geq 0, \\ y(t,\epsilon) &= \phi(t), \quad t \leq 0, \end{aligned}$$

where $a > 0$ is a real constant, and $\phi(t)$ is continuous. Then the condition (19) is satisfied.

Let step size $h = 1$. Then (38) reads

$$\varepsilon y^{\varepsilon}_{n+1} = \varepsilon y^{\varepsilon}_{n} - a(\theta y^{\varepsilon}_{n+1} + (1-\theta) y^{\varepsilon}_{n}) - \theta a y^{\varepsilon}_{n} \tag{39}$$

where $n = 0, 1, 2, \cdots$. Hence we have

$$y^{\varepsilon}_{n+1} = \frac{\varepsilon - a}{\varepsilon + \theta a} \tag{40}$$

In the case $\theta = 0$, we have $\lim_{\varepsilon \to 0^+} \left|\frac{\varepsilon - a}{\varepsilon}\right| = \infty$, otherwise $\lim_{\varepsilon \to 0^+} \left|\frac{\varepsilon - a}{\varepsilon + \theta a}\right| = \frac{1}{\theta} > 1$. Therefore the numerical solution blows up for some initial function $\phi(t)$ and the one-leg $\theta-$method is not dissipative for $\theta \in [0, 1)$.

For any fixed step size $h > 0$ and $\theta = 1$, the one-leg method (38) and (28) coincides and hence the method is dissipative uniformly in ε. (28)

Theorem 5.2. *(Exponential stability) Assume that the function f in equations (23) and (24) (s=1) satisfies (27) and $0 \leq \tau(t) < \tau^{\star}$. Then the resulting one-leg $\theta-$method with $h \geq \varepsilon > 0$ is exponentially stable uniformly in $\varepsilon \in (0, \varepsilon_0]$ if and only if $\theta = 1$.*

Proof. The stability can be proved in a similar way as the dissipativity.

References

[1] K. Allen and S. McKee, Fixed discretization methods for delay differential equations, *Comp. & Math. with Appl.*, **7** (1981), 413-423.

[2] C. T. H. Baker and A. Tang, Generalized Halanay inequalities for Volterra functional differential equations and the discrete versions, Volterra Centennial Meeting, Arlington, June, 1996.

[3] A. Bellen, Contractivity of continuous Runge-Kutta methods for delay differential equations, *Appl. Numer.Math.* **24** (1997), 219-231.

[4] R. Bellman and K. L. Cooke, Differential-Difference Equations, Academic Press, New York, 1963.

[5] K. L. Cooke, The condition of regular degeneration for singularly perturbed linear differential-difference equations, *J. Diff. Eqns.*, **1** (1965), 39-94.

[6] K. Dekker and J.G. Verwer, Stability of Runge-Kutta Methods for Stiff Nonlinear Differential Equations, CWI Monographs 2, North-Holland, Amsterdam, 1984.

[7] M. W. Derstine, H. M. Gibbs, F. A. Hopf and D. L. Kaplan, Bifurcation gap in a hybrid optical system, *Phys. Rev.*, A, **26** (1982), 3720-3722.

[8] L. E. El'sgol'c, *Qualitative Methods in Mathematical Analysis*, American Mathematical Society, Providence, Rhode Island, 1964.

[9] H. M. Gibbs, F. A. Hopf, D. L. Kaplan and R. L. Shoemaker, Observation of chaos in optical bistability, *Phys. Rev. Lett.*, **46** (1981), 474-477.

[10] E. Hairer, C. Lubich and M. Roche, Error of Runge-Kutta methods for stiff problems studied via differential algebraic equations, *BIT*, **28** (1988), 678-700.

[11] A. Halanay, Differential Equations, Stability, Oscillations, Time Lags, Academic Press, New York, 1966.

[12] J. K. Hale and S. M. Verduyn Lunel, Introduction to Functional Differential Equations, Springer-Verlag, New York, 1993.

[13] K. Ikeda, Multiple-valued stationary state and its instability of the transmitted light by a ring cavity system, *Opt. Comm.*, **30** (1979), 257.

[14] M.Z. Liu and M.N. Spijker, The stability of the $\theta-$methods in the numerical solution of delay differential equations, *IMA J. Numer. Anal.* **10** (1990), 31-48.

[15] A. Longtin, and J. Milton, Complex oscillations in the human pupil light reflex with mixed and delayed feedback, *Math. Biosciences*, **90** (1988), 183-199.

[16] M. C. Mackey, and L. Glass, Oscillation and chaos in physiological control systems, *Science*, **197** (1977), 287-289.

[17] D. R. Smith, Singular Perturbation Theory, Cambridge University Press, 1985.

[18] H. Tian, Numerical Treatment of Singularly Perturbed Delay Differential Equations, Ph.D. thesis, University of Manchester, 2000.

[19] H. Tian and J. Kuang, The stability analysis of $\theta-$methods for delay differential equations, *J. Comp. Math.*, **14** (1996), 203-212.

[20] H. Tian, Numerical and analytic dissipativity of delay differential equations. *Submited*, 2001.

[21] L. Torelli, A sufficient condition for GPN-stability for delay differential equations, *Numer. Math.*, **25** (1991), 311–320.

[22] M. Wazewska-Czyzewska and A. Lasota, Mathematical models of the red cell system, *Mat. Stos.*, **6** (1976), 25-40.

[23] M. Zennaro, Asymptotic stability analysis of Runge-Kutta methods for nonlinear systems of delay differential equations, *Numer. Math.*, **77** (1997), 549-563.