Journal of Computational Mathematics, Vol.20, No.4, 2002, 349–362.

BIVARIATE FRACTAL INTERPOLATION FUNCTIONS ON RECTANGULAR DOMAINS*1)

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Abstract

Non-tensor product bivariate fractal interpolation functions defined on gridded rectangular domains are constructed. Linear spaces consisting of these functions are introduced. The relevant Lagrange interpolation problem is discussed. A negative result about the existence of affine fractal interpolation functions defined on such domains is obtained.

Key words: Fractal, Bivariate functions, Interpolation.

1. Introduction

In 1986 Barnsley[2] constructed a sort of continuous functions by using certain *Iterated Function Systems* (IFS). Such a function f is defined on a compact interval, which is partitioned into a number of subintervals, and is said to be self-affine since the restriction of f within one of the subintervals is just a composition of a scaling and a translation of f plus an affine function. The graph of f, which interpolates a set of given points, has usually a non-integral fractal dimension and is then called a *Fractal Interpolation Function*, abbreviated FIF. FIF serves a useful tool for constructing, modelling, simulating, and approximating functions which display some sort of self-similarity under magnification and find its applications in several areas such as image compression and wavelet analysis (cf. [1, 3, 4, 5, 8]).

There are two natural ways to extend the idea of FIF to the case of two variables. Geromino *et al.*[6] and Massopust[7] deal with continuous functions with the property of self-affinity defined on the triangulated triangular domains, whose graphs are so called *fractal surfaces*. Unfortunately, the gridded rectangular domain, i.e., the rectangular domain divided into a number of quadrangles, especially rectangles which is most used in the applications of Computer Graphics, is hardly considered. Massopust[8] suggests a construction by trivial taking the tensor product of two univariate FIFs. The drawback of this tensor product scheme is explicit: the derived function is uniquely determined by the its evaluation along a pair of adjacent sides of the rectangular domain, thus it cannot be used to fit in with a set of data more extensively sampled.

In this paper we shall develop the idea of the space of fractal functions introduced by Qian[9] to study the bivariate fractal functions defined on rectangular domains. Such functions are continuous, but not tensor products of univariate FIFs. They can be designated to interpolating given data distributed over the grid points. Their graphs, as those of univariate FIFs, can be generated by IFSs of certain simple forms. We shall show that they are distinct in various aspects from their analogies defined on intervals and triangular domains. For example, it will be proved that the "affine" fractal functions defined on a gridded rectangular domain does, in essence, not exist. For this reason one should be satisfied with the bivariate FIFs generated by IFSs of other forms as simple as possible.

* Received September 14, 1999; Final revised January 12, 2001.
1) Supported in part by the NKBRSF(G1998030600) and in part by the Doctoral Program Foundation of Educational Department of China(1999014115).

This paper is organized as follows. In Section 2 we introduce bivariate FIFs defined on rectangular domains . In Section 3 we discuss the bivariate FIFs generated by affine IFSs and prove that they are usually affine functions. In Section 4 we construct a class of spaces of bivariate FIFs suitable for interpolation and study the structures of such spaces.

2. Linear Spaces of Bivariate Fractal Interpolation Functions

Let $m,n \geq 2$ be two integers . Denote $M = \{1,\cdots,m\}$ and $N = \{1,\cdots,n\}$. Let $-\infty < x_0 < x_1 < \cdots < x_m < \infty$ and $-\infty < y_0 < y_1 < \cdots < y_n < \infty$. Denote by D the rectangular region $[x_0,x_m] \times [y_0,y_n]$ and $D_{i,j} = [x_{i-1},x_i] \times [y_{j-1},y_j]$ for all $i \in M$ and $j \in N$. Denoted by Δ the partition of D given by $D = \bigcup_{i,j} D_{i,j}$. Moreover, let Λ denote a fixed matrix $(s_{i,j})_{m\times n}$ with $-1 < s_{i,j} < 1$. For all $i \in M$ and $j \in N$, define maps $A_i : [x_0,x_m] \longrightarrow [x_{i-1},x_i]$ and $B_j : [y_0,y_n] \longrightarrow [y_{j-1},y_j]$ by

$$A_i(x) = \frac{x_i - x_{i-1}}{x_m - x_0}(x - x_0) + x_{i-1}, \qquad x \in [x_0,x_m]$$

and

$$B_j(y) = \frac{y_j - y_{j-1}}{y_n - y_o}(y - y_o) + y_{j-1}, \qquad y \in [y_0,y_n]$$

respectively.

We denote by $\mathrm{C}(D)$ the linear space of all real-valued continuous functions defined on D and by $\mathrm{Lip}(D)$ the set of all bivariate Lipschitzian functions defined on D. Obviously $\mathrm{Lip}(D)$ is a linear subspace of $\mathrm{C}(D)$. Given a family of functions $\phi_{i,j} \in \mathrm{Lip}(D),\ i \in M,\ j \in N$, define mappings $T_{i,j} : D \times \mathbf{R} \longrightarrow D \times \mathbf{R}$ by

$$T_{i,j}(x,y,z) = (A_i(x), B_j(y), s_{i,j}\,z\, +\, \phi_{i,j}(x,y)),$$

for $(x,y,z) \in D \times \mathbf{R},\ i \in M,\ j \in N$. We now obtain an IFS

$$\{D \times \mathbf{R}\,;\ T_{i,j} :\ i \in M,\ j \in N\}. \tag{2.1}$$

The following is easy to be proved.

Lemma 2.1. *The IFS (2.1) has a unique invariant set.*

Definition 2.1. *An IFS of the form (2.1) is said to be generating if its unique invariant set G is the graph of some $f \in \mathrm{C}(D)$, in which case we also write*

$$\{D \times \mathbf{R}\,;\ T_{i,j} :\ i \in M,\ j \in N\} \to f \tag{2.3}$$

and say that f is generated by the IFS.

One can easily check the following by Definition 2.1.

Proposition 2.1. *The IFS (2.1) is generating if and only if there exists some $f \in \mathrm{C}(D)$ such that*

$$f(A_i(x), B_j(y)) = s_{i,j} f(x,y) + \phi_{i,j}(x,y), \quad (x,y) \in D\,, i \in M,\ j \in N, \tag{2.3}$$

in which case the relation (2.2) holds.

Theorem 2.1. *IFS (2.1) is generating if and only if there exist functions $p_0, p_1 \in \mathrm{C}[x_0,x_m]$ and $q_0, q_1 \in \mathrm{C}[y_0,y_n]$ such that*

$$\begin{cases} p_0(x_0) = q_0(y_0), & p_0(x_m) = q_1(y_0), \\ p_1(x_0) = q_0(y_n), & p_1(x_m) = q_1(y_n), \end{cases} \tag{2.4}$$

and

$$\phi_{i,j+1}(x,y_0) - \phi_{i,j}(x,y_n) = s_{i,j}p_1(x) - s_{i,j+1}p_0(x), \qquad i \in M,\ 1 \leq j \leq n-1, \tag{2.5}$$

$$\phi_{i+1,j}(x_0,y) - \phi_{i,j}(x_m,y) = s_{i,j}q_1(y) - s_{i+1,j}q_0(y), \qquad 1 \leq i \leq m-1,\ j \in N, \tag{2.6}$$

$$p_0(A_i(x)) = s_{i,1}p_0(x) + \phi_{i,1}(x,y_0), \ \ i \in M, \tag{2.7}$$

$$p_1(A_i(x)) = s_{i,n}p_1(x) + \phi_{i,n}(x,y_n), \ \ i \in M, \tag{2.8}$$

$$q_0(B_j(y)) = s_{1,j}q_0(y) + \phi_{1,j}(x_0,y), \ \ j \in N, \tag{2.9}$$

$$q_1(B_j(y)) = s_{m,j}q_1(y) + \phi_{m,j}(x_m,y), \ \ j \in N. \tag{2.10}$$

Proof. Suppose that for some $f \in \mathrm{C}(D)$ such that (2.2)holds. We set

$$\begin{array}{lll} p_0(x) = f(x,y_0), & p_1(x) = f(x,y_n), & x_0 \le x \le x_m, \\ q_0(y) = f(x_0,y), & q_1(y) = f(x_m,y), & y_0 \le y \le y_n. \end{array}$$

Obviously $p_0, p_1 \in \mathrm{C}[x_0,x_m]$, $q_0, q_1 \in \mathrm{C}[y_0,y_n]$ and (2.4) is satisfied. Note that

$$\begin{array}{l} s_{i,j}p_1(x) + \phi_{i,j}(x,y_n) = s_{i,j}f(x,y_n) + \phi_{i,j}(x,y_n) = f(A_i(x),B_j(y_n)) \\ = f(A_i(x),B_{j+1}(y_0)) = s_{i,j+1}f(x,y_0) + \phi_{i,j+1}(x,y_0) \\ = s_{i,j+1}p_0(x) + \phi_{i,j+1}(x,y_0), \quad i \in M, \ j = 1,\cdots,n-1, \end{array}$$

so that (2.5) holds. Similarly we can deduce (2.6) .

Note that

$$\begin{array}{l} p_0(A_i(x)) = f(A_i(x),y_0) = f(A_i(x),B_1(y_0)) \\ = s_{i.1}f(x,y_0) + \phi_{i,1}(x,y_0) = s_{i,1}p_0(x) + \phi_{i,1}(x,y_0), \quad i \in M. \end{array}$$

Thus (2.7) holds. Analogously, we have (2.8), (2.9) and (2.10).

On the other hand, suppose that there exist $p_0, p_1 \in \mathrm{C}[x_0,x_m]$ and $q_0, q_1 \in \mathrm{C}[y_0,y_n]$ satisfying (2.4) — (2.10). We take

$$\mathrm{C}^*(D) = \left\{ \begin{array}{ll} g \in \mathrm{C}(D): & g(x,y_0) = p_0(x), \ g(x,y_n) = p_1(x), \\ & g(x_0,y) = q_0(y), \ g(x_m,y) = q_1(y) \end{array} \right\}.$$

Clearly $\mathrm{C}^*(D)$ is closed in the complete metric space C(D). Moreover, $\mathrm{C}^*(D)$ is nonempty. To see this, we set

$$h(x,y) = \left(\frac{x_m - x}{x_m - x_0}, \frac{x - x_0}{x_m - x_0}\right)\begin{pmatrix} q_0(y) \\ q_1(y) \end{pmatrix} + (p_0(x), p_1(x))\begin{pmatrix} \frac{y_n - y}{y_n - y_0} \\ \frac{y - y_0}{y_n - y_0} \end{pmatrix}$$

$$-\left(\frac{x_m - x}{x_m - x_0}, \frac{x - x_0}{x_m - x_0}\right)\begin{pmatrix} p_0(x_0) & p_1(x_0) \\ p_0(x_m) & p_1(x_m) \end{pmatrix}\begin{pmatrix} \frac{y_n - y}{y_n - y_0} \\ \frac{y - y_0}{y_n - y_0} \end{pmatrix} (x,y) \in D.$$

It is easy to check that $h \in \mathrm{C}^*(D)$.

We now define the Read-Bajraktarevic operator $\Phi : \mathrm{C}^*(D) \longrightarrow \mathrm{C}(D)$, by

$$\begin{array}{c} (\Phi g)(x,y) = s_{i,j}g(A_i^{-1}(x),B_j^{-1}(y)) + \phi_{i,j}(A_i^{-1}(x),B_j^{-1}(y)), \\ (x,y) \in D_{i,j}, \ i \in M, \ j \in N. \end{array} \tag{2.11}$$

If $x \neq x_i$, $i \in M$ and $y \neq y_j$, $j \in N$, then the value of $h(x,y)$ is uniquely determined by (2.11). Otherwise (x,y) is at the common boundaries of two or more subrectangles, say, $D_{i,j}$ and $D_{i,j+1}$, that is, $x_{i-1} \le x \le x_i$ and $y = y_j$. Noting that $g \in \mathrm{C}^*(D)$), we have two representations of $(\Phi g)(x,y)$,

$$\begin{array}{rcl} (\Phi g)(x,y_j) & = & s_{i,j}g(A_i^{-1}(x),B_j^{-1}(y_i)) + \phi_{i,j}(A_i^{-1}(x),B_j^{-1}(y_j)) \\ & = & s_{i,j}g(A_i^{-1}(x),y_n) + \phi_{i,j}(A_i^{-1}(x),y_n) \\ & = & s_{i,j}p_1(A_i^{-1}(x)) + \phi_{i,j}(A_i^{-1}(x),y_n) \end{array} \tag{2.12}$$

$$\begin{array}{rcl} (\Phi g)(x,y_j) & = & s_{i,j+1}g(A_i^{-1}(x),B_{j+1}^{-1}(y_j)) + \phi_{i,j+1}(A_i^{-1}(x),B_{j+1}^{-1}(y_j)) \\ & = & s_{i,j+1}g(A_i^{-1}(x),y_0) + \phi_{i,j+1}(A_i^{-1}(x),y_0) \\ & = & s_{i,j+1}q_0(A_i^{-1}(x)) + \phi_{i,j+1}(A_i^{-1}(x),y_0). \end{array} \tag{2.14}$$

From (2.5) we can see that the ending sides of (2.12) and (2.13) are equal. Thus the two values of $(\Phi g)(x,y_j)$, given by (2.12) and (2.13) respectively, match up. Analogously, the condition

(2.6) ensures the equivalence of the values of of $(\Phi g)(x,y)$ along net segments $\{(x_i,y): y_{j-1} \le y \le y_j,\ j=1,\cdots,n\}$ Therefore Φg is well defined on whole D.

Φg is continuous because all the functions g, $\phi_{i,j}$, A_i^{-1} and B_j^{-1} are continuous, and then Φ is well defined.

Now we come to prove that the range of Φ is contained in $\mathrm{C}^*(D)$. In fact, for each $g \in \mathrm{C}^*(D)$, from condition (2.7) we have

$$\begin{aligned}(\Phi g)(x,y_0) &= s_{i,1}g(A_i^{-1}(x),B_1^{-1}(y_0)) + \phi_{i,1}(A_i^{-1}(x),B_1^{-1}(y_0))\\ &= s_{i,1}g(A_i^{-1}(x),y_0) + \phi_{i,1}(A_i^{-1}(x),y_0)\\ &= s_{i,1}p_0(A_i^{-1}(x)) + \phi_{i,1}(A_i^{-1}(x),y_0)\\ &= p_0(A_i(A_i^{-1}(x))) = p_0(x).\end{aligned}$$

Similarly we can prove

$$(\Phi g)(x,y_n) = p_1(x), \quad (\Phi g)(x_0,y) = q_0(y), \quad (\Phi g)(x_m,y) = q_1(y)$$

by using (2.8), (2.9), and (2.10), respectively. Hence $\Phi g \in \mathrm{C}^*(D)$.

Φ is contractive: for g, $\tilde{g} \in \mathrm{C}^*(D)$, we have

$$\begin{aligned}&\|\Phi g - \Phi\tilde{g}\| = \sup_{(x,y)\in D}\{\mid (\Phi g)(x,y) - (\Phi\tilde{g})(x,y) \mid\}\\ &= \max_{i,j}\sup_{(x,y)\in D_{i,j}} \mid (\Phi g)(x,y) - (\Phi\tilde{g})(x,y) \mid\\ &= \max_{i,j}\sup_{(x,y)\in D_{i,j}} \mid s_{i,j}(g(A_i^{-1}(x),B_j^{-1}(y)) - \tilde{g}(A_i^{-1}(x),B_j^{-1}(y))) \mid\\ &= \max_{i,j} \mid s_{i,j} \mid \sup_{(x,y)\in D_{i,j}} \mid g(x,y) - \tilde{g}(x,y) \mid = s^*\|g-\tilde{g}\|\end{aligned}$$

where $0 \le s^* = \max_{i,j} \mid s_{i,j} \mid < 1$, $\|\cdot\|$ is the norm for $\mathrm{C}(D)$.

To summarize, Φ is a contraction mapping defined on the complete metric space $\mathrm{C}^*(D)$ (viewed as a subspace of $\mathrm{C}(D)$). Thus there exists a unique $f \in \mathrm{C}^*(D)$ such that $\Phi f = f$. Hence $\{D\times\mathbf{R};\ T_{i,j} : i \in M,\ j \in N\} \to f$ by definition.

The following lemma is a simple consequence of Proposition 2.1.

Lemma 2.2. *Let $\alpha,\beta \in \mathbf{R}$. Set*

$$\begin{aligned}&U_{i,j}(x,y,z) = (A_i(x),B_j(y),s_{i,j}z + u_{i,j}(x,y)),\\ &V_{i,j}(x,y,z) = (A_i(x),B_j(y),s_{i,j}z + v_{i,j}(x,y)),\\ &W_{i,j}(x,y,z) = (A_i(x),B_j(y),s_{i,j}z + \alpha u_{i,j}(x,y) + \beta v_{i,j}(x,y)),\end{aligned}$$

where $(x,y,z) \in D\times\mathbf{R}$, $u_{i,j},v_{i,j} \in \mathrm{Lip}(D)$, $i \in M$, $j \in N$. Suppose that there exist $f,g \in \mathrm{C}(D)$ such that

$$\begin{aligned}&\{D\times\mathbf{R};\ U_{i,j} : i \in M,\ j \in N\} \to f,\\ &\{D\times\mathbf{R};\ V_{i,j} : i \in M,\ j \in N\} \to g.\end{aligned}$$

Then

$$\{D\times\mathbf{R};\ W_{i,j} : i \in M,\ j \in N\} \to \alpha f + \beta g.$$

In the remainder of this section, H will denote a fixed linear subspace of $\mathrm{Lip}(D)$.

Proposition 2.2. *Let $F(\Delta_{m,n},\Lambda_{m,n},H)$ denote the set*

$$\left\{\begin{array}{ll} f \in \mathrm{C}(D): & \text{there exist functions } \phi_{i,j} \in H,\ i \in M,\ j \in N \text{such}\\ & \text{that } \{D\times\mathbf{R};\ T_{i,j} : i \in M,\ j \in N\} \to f, \text{ where}\\ & T_{i,j}(x,y,z) = (A_i(x),B_j(y),s_{i,j}z + \phi_{i,j}(x,y)). \end{array}\right\}.$$

Then $F(\Delta_{m,n},\Lambda_{m,n},H)$ is a linear subspace of $\mathrm{C}(D)$.

Proof. It immediately follows from Lemma 2.2.

Definition 2.2. *We call $F(\Delta_{m,n},\Lambda_{m,n},H)$ defined as above the space of bivariate FIFs with respect to the partition $\Delta_{m,n}$, the contraction matrix $\Lambda_{m,n}$ and the bottom space H.*

Obviously Theorem 2.1 can be rewritten as the following form.

Theorem 2.1a. *Let $f \in \mathrm{C}(D)$. Then $f \in F(\Delta_{m,n}, \Lambda_{m,n}, H)$ if and only if there exist $\phi_{i,j} \in H,\ i \in M,\ j \in N$ such that*

$$\begin{aligned}
&\phi_{i,j+1}(x,y_0) - \phi_{i,j}(x,y_n) = s_{i,j} f(x,y_n) - s_{i,j+1} f(x,y_0), \quad i \in M,\ 1 \le j \le n-1,\\
&\phi_{i+1,j}(x_0,y) - \phi_{i,j}(x_m,y) = s_{i+1,j} f(x_m,y) - s_{i,j} f(x_0,y), \quad 1 \le i \le m-1,\ j \in N,\\
&f(A_i(x), y_0) = s_{i,1} f(x,y_0) + \phi_{i,1}(x,y_0), \quad i \in M,\\
&f(A_i(x), y_n) = s_{i,n} f(x,y_n) + \phi_{i,n}(x,y_n), \quad i \in M,\\
&f(x_0, B_j(y)) = s_{1,j} f(x_0,y) + \phi_{1,j}(x_0,y), \quad j \in N,\\
&f(x_m, B_j(y)) = s_{m,j} f(x_m,y) + \phi_{m,j}(x_m,y), \quad j \in N,
\end{aligned}$$

in which case (2.2) holds.

3. Bivariate FIFs Generated by Affine IFSs

In the theory of univariate FIFs so called affine fractal functions, i.e., FIFs generated by affine IFSs, occupy the most important place. However, we shall show that "affine" fractal functions defined on a gridded rectangular domain are practically useless for interpolation.

Throughout this section we denote $L = \{\phi \in \mathrm{C}(D) : \ \phi(x,y) = ax + by + c,\ a, b, c \in \mathbf{R}\}$. Without any loss of generality, we may assume that $D = [0,1] \times [0,1]$. Moreover, we denote by F_0 the set

$$\{f \in F(\Delta_{m,n}, \Lambda_{m,n}, L) : f(0,0) = f(1,0) = f(0,1) = 0\}.$$

Lemma 3.1. *F_0 is linearly isomorphic to the quotient space*

$$F(\Delta_{m,n}, \Lambda_{m,n}, L)/L.$$

Proof. It is easy to see $L \subset F(\Delta_{m,n}, \Lambda_{m,n}, L)$. For each $f \in F_0$, set $Tf = \langle f \rangle$, the equivalent class of f in $F(\Delta_{m,n}, \Lambda_{m,n}, L)$ modulo L. Then T is a desired isomorphism.

The following is a simple consequence of Theorem 2.1a.

Lemma 3.2. *Let $f \in F_0$. Let IFS $\{D \times \mathbf{R} : T_{i,j} : i \in M,\ j \in N\}$ have the form*

$$T_{i,j}(x,y,z) = (A_i(x),\ \ B_j(y),\ \ s_{i,j} z + a_{i,j} x + b_{i,j} y + c_{i,j}), \tag{3.1}$$

where $a_{i,j}, b_{i,j}, c_{i,j} \in \mathbf{R}$. Then $\{D \times \mathbf{R}; T_{i,j} : i \in M,\ j \in N\} \to f$ if and only if the real coefficients $a_{i,j}, b_{i,j}, c_{i,j}$ satisfy the following equations:

$$(a_{i,j+1} - a_{i,j})x = s_{i,j} f(x,1) - s_{i,j+1} f(x,0), \quad i \in M,\ \ j = 1, \cdots, n-1, \tag{3.2}$$

$$c_{i,j+1} = b_{i,j} + c_{i,j}, \quad i \in M,\ \ j = 1, \cdots, n-1, \tag{3.3}$$

$$(b_{i+1,j} - b_{i,j})y = s_{i,j} f(1,y) - s_{i+1,j} f(0,y), \quad i = 1, \cdots, m-1,\ j \in N, \tag{3.4}$$

$$c_{i+1,j} = a_{i,j} + c_{i,j}, \quad i = 1, \cdots, m-1,\ j \in N, \tag{3.5}$$

$$f(A_i(x), 0) = s_{i,1} f(x,0) + a_{i,1} x + c_{i,1}, \quad i \in M, \tag{3.6}$$

$$f(A_i(x), 1) = s_{i,n} f(x,1) + a_{i,n} x + b_{i,n} + c_{i,n}, \quad i \in M, \tag{3.7}$$

$$f(0, B_j(y)) = s_{1,j} f(0,y) + b_{1,j} y + c_{1,j}, \quad j \in N, \tag{3.8}$$

$$f(1, B_j(y)) = s_{m,j} f(1,y) + b_{m,j} y + a_{m,j} + c_{m,j}, \quad j \in N. \tag{3.9}$$

In the following Lemma 3.4 — Lemma 3.6, we always assume that $f \in F_0, \{D \times \mathbf{R}; T_{i,j} : i \in M,\ j \in N\} \to f$,where $T_{i,j}$ are defined by (3.1), and denote

$$\begin{cases} \theta = f(1,1) \\ \xi_i = f(x_i, 0), \quad i = 0, 1, \cdots, m \\ \eta_j = f(0, y_j), \quad j = 0, 1, \cdots, n. \end{cases} \tag{3.10}$$

A simple computation shows the following by applying Lemma 3.3.

Lemma 3.3. *The coefficients $a_{i,j}$, $b_{i,j}$, $c_{i,j}$ satisfy*

$$a_{i,1} = \xi_i - \xi_{i-1}, \qquad i \in M, \tag{3.11}$$

$$a_{i,j} = (s_{i,1} + \cdots + s_{i,j-1})\theta + \xi_i - \xi_{i-1} \quad i \in M, \;\; j = 2, \cdots, n \tag{3.12}$$

$$b_{1,j} = \eta_j - \eta_{j-1}, \qquad j \in N \tag{3.13}$$

$$b_{i,j} = (s_{1,j} + \cdots + s_{i-1,j})\theta + \eta_j - \eta_{j-1}, \quad i = 2, \cdots, m, \;\; j \in N, \tag{3.14}$$

$$c_{i,1} \;=\; \xi_{i-1}, \qquad i \in M, \tag{3.15}$$

$$c_{1,j} \;=\; \eta_{j-1}, \qquad j \in N, \tag{3.16}$$

$$c_{i,j} = \left(\sum_{\alpha=1}^{i-1} \sum_{\beta=1}^{j-1} s_{\alpha,\beta}\theta \right) + \xi_{i-1} + \eta_{j-1}, \quad i = 2, \cdots, m, \;\; j = 2, \cdots, n. \tag{3.17}$$

By this and applying Lemma 3.2 again we have

Lemma 3.4. *If $\sum_{i=1}^{m} \sum_{j=1}^{n} s_{i,j} \neq 1$, then $\theta = f(0,0) = 0$.*

Definition 3.1 *The contraction matrix $\Lambda_{m,n} = (s_{i,j})_{m \times n}$ is called row trivial if*

$$\begin{vmatrix} s_{i,j} & s_{i+1,j} \\ s_{k,h} & s_{k+1,h} \end{vmatrix} = 0$$

for any $1 \leq i, k \leq m-1$ and $j, h \in N$. $\Lambda_{m,n}$ is called row nontrivial if it is not row trivial.

Respectively, $\Lambda_{m,n}$ is called column trivial if

$$\begin{vmatrix} s_{i,j} & s_{i,j+1} \\ s_{k,h} & s_{k,h+1} \end{vmatrix} = 0$$

for any $i, k \in M$ and $1 \leq j, h \leq n-1$. $\Lambda_{m,n}$ is called column nontrivial if it is not column trivial.

$\Lambda_{m,n}$ is called nontrivial if it is neither row trivial nor column trivial.

The following is deduced from Lemma 3.2 and Lemma 3.3.

Lemma 3.5.

(a) *Suppose that $\Lambda_{m,n}$ is row nontrivial. Then*

$$f(0,y) \equiv 0, \;\; f(1,y) = \theta y, \quad 0 \leq y \leq 1. \tag{3.18}$$

Furthermore, if $\theta \neq 0$, then

$$y_j - y_{j-1} = s_{1,j} + \cdots + s_{m,j}, \quad j \in N. \tag{3.19}$$

(b) *Suppose that $\Lambda_{m,n}$ is row nontrivial. Then*

$$f(x,0) \equiv 0, \quad f(x,1) = \theta x, \quad 0 \leq x \leq 1. \tag{3.20}$$

Furthermore, if $\theta \neq 0$, then

$$x_i - x_{i-1} = s_{i,1} + \cdots + s_{i,n}, \quad i \in M. \tag{3.21}$$

Definition 3.2. *$\Lambda_{m,n}$ and $\Delta_{m,n}$ are said to be synchronous if they satisfy*

$$s_{i,1} + \cdots + s_{i,n} = x_i - x_{i-1}, \quad i \in M,$$

$$s_{1,j} + \cdots + s_{m,j} = y_j - y_{j-1}, \quad j \in N.$$

Theorem 3.1.

(a) *If $\Lambda_{m,n}$ is nontrivial, then*

$$\mathrm{dim} F_0 = \begin{cases} 1, & \text{if } \Lambda_{m,n} \text{ and } \Delta_{m,n} \text{ are synchronous,} \\ 0, & \text{otherwise.} \end{cases}$$

(b) *If* $\Lambda_{m,n}$ *is row nontrivial but column trivial and* $s_{i,j} = s_{i,1},\ i \in M,\ j \in N$, *then*

$$\dim F_0 = \begin{cases} m, & \text{if } \Lambda_{m,n} \text{ and } \Delta_{m,n} \text{ are synchronous,} \\ m-1, & \text{otherwise.} \end{cases}$$

(c) *If* $\Lambda_{m,n}$ *is row nontrivial but column trivial and there exist real numbers* $q \neq 0, 1$, *such that* $s_{i,j} = q^{j-1} s_{i,1},\ i \in M,\ j \in N$, *then*

$$\dim F_0 = \begin{cases} 1, & \text{if } \Lambda_{m,n} \text{ and } \Delta_{m,n} \text{ are synchronous,} \\ 0, & \text{otherwise.} \end{cases}$$

(d) *If* $\Lambda_{m,n}$ *is row nontrivial but column trivial and* $s_{i,j} = 0$ *for all* $i \in M$ *and* $j \geq 2$, *then* $\dim F_0 = 0$.

(e) *If* $\Lambda_{m,n}$ *is row nontrivial but column trivial and* $s_{i,j} = 0$ *for all* $i \in M$ *and* $j \leq n-1$, *then* $\dim F_0 = 0$.

(f) *If* $\Lambda_{m,n}$ *is row trivial but column nontrivial and* $s_{i,j} = s_{1,j},\ i \in M,\ j \in N$, *then*

$$\dim F_0 = \begin{cases} n, & \text{if } \Lambda_{m,n} \text{ and } \Delta_{m,n} \text{ are synchronous,} \\ n-1, & \text{otherwise.} \end{cases}$$

(g) *If* $\Lambda_{m,n}$ *is row trivial but column nontrivial and there exist real numbers* $p \neq 0, 1$, *such that* $s_{i,j} = p^{i-1} s_{1,j},\ i \in M,\ j \in N$, *then*

$$\dim F_0 = \begin{cases} 1, & \text{if } \Lambda_{m,n} \text{ and } \Delta_{m,n} \text{ are synchronous,} \\ 0, & \text{otherwise.} \end{cases}$$

(h) *If* $\Lambda_{m,n}$ *is row trivial but column nontrivial and* $s_{i,j} = 0$ *for all* $i \geq 2$ *and* $j \in N$, *then* $\dim F_0 = 0$.

(i) *If* $\Lambda_{m,n}$ *is row nontrivial but column trivial and* $s_{i,j} = 0$ *for all* $i \leq m-1$ *and* $j \in N$, *then* $\dim F_0 = 0$.

(j) *If all* $s_{i,j} = 0,\ \ i \in M,\ j \in N$, *then* $\dim F_0 = m + n - 2$.

(k) *If one and only one element in* $\{s_{1,1}, s_{1,n}, s_{m,1}, s_{m,n}\}$ *is nonzero and all other elements* $s_{i,j}$ *of* $\Lambda_{m,n}$ *vanish, then* $\dim F_0 = 0$.

(l) *If either all the elements on the 1st row of* $\Lambda_{m,n}$ *or all those on the* m*th row are nonzero, and the elements in other rows are all zeros, then*

$$\dim F_0 = \begin{cases} m-1, & \text{if all nonzero elements of } \Lambda_{m,n} \text{ are equal,} \\ 0, & \text{otherwise.} \end{cases}$$

(m) *If either all the elements on the 1st column of* $\Lambda_{m,n}$ *or all those on the* n*th column are nonzero, and the elements in other columns are all zeros, then*

$$\dim F_0 = \begin{cases} n-1, & \text{if all nonzero elements of } \Lambda_{m,n} \text{ are equal,} \\ 0, & \text{otherwise.} \end{cases}$$

(n) *If* $\Lambda_{m,n}$ *satisfies none of the above cases (a) — (m), then there exist real numbers* $s, p, q \neq 0$, *such that* $s_{i,j} = p^i q^j s,\ i \in M,\ j \in N$ *and*

$$\dim F_0 = \begin{cases} m+n-2, & \text{if } p = q = 1, \\ m-1, & \text{if } p \neq 1,\ q = 1, \\ n-1, & \text{if } p = 1,\ q \neq 1, \\ 1, & \text{if } p \neq 1,\ q \neq 1, p^{m-1} \neq 1,\ q^{n-1} \neq 1, \text{ and} \\ & \Lambda_{m,n} \text{ and } \Delta_{m,n} \text{ are synchronous,} \\ 0, & \text{if } p \neq 1,\ q \neq 1, p^{m-1} \neq 1,\ q^{n-1} \neq 1, \text{ and} \\ & \Lambda_{m,n} \text{ and } \Delta_{m,n} \text{ are not synchronous,} \\ n-1, & \text{if } p = -1,\ m \text{ odd},\ q^{n-1} \neq 1, \\ m-1, & \text{if } q = -1,\ n \text{ odd},\ p^{m-1} \neq 1, \\ 0, & \text{if } p = q = -1,\ m \text{ and } n \text{ odd.} \end{cases}$$

Proof. Let $f \in F_0$. Let $\{D \times \mathbf{R}; T_{i,j} : i \in M, j \in N\} \to f$, where $T_{i,j}$ are defined as (3.1), and we denote $\xi_i = f(x_i, 0)$, $i = 0, 1, \cdots, m$, $\eta_j = f(0, y_j)$, $j = 0, 1, \cdots, n$, $\theta = f(1,1)$.

(a) By Lemma 3.5, in this case we have

$$f(x,0) \equiv 0,\ f(x,1) = \theta x,\ f(0,y) \equiv 0,\ f(1,y) = \theta y.$$

From Lemma 3.4,

$$a_{i,1} = c_{i,1} = b_{1,j} = c_{1,j} = 0, \quad i \in M,\ j \in N, \tag{3.22}$$

$$a_{i,j} = (s_{i,1} + \cdots + s_{i,j-1})\theta, \quad i \in M, \quad j = 2, \cdots, n, \tag{3.23}$$

$$b_{i,j} = (s_{1,j} + \cdots + s_{i-1,j})\theta, \quad i = 2, \cdots, m, \quad j \in N, \tag{3.24}$$

$$c_{i,j} = \sum_{\alpha=1}^{i-1} \sum_{\beta=1}^{j-1} s_{\alpha,\beta}\theta, \quad i = 2, \cdots, m, \ \ j = 2, \cdots, n. \tag{3.25}$$

If $\Lambda_{m,n}$ and $\Delta_{m,n}$ are not synchronous, then by Lemma 3.5, $\theta = 0$, in which case all $a_{i,j}, b_{i,j}, c_{i,j}$ are equal to zero. Thus $f = 0$, i.e., $F_0 = \{0\}$. Clearly $\dim F_0 = 0$.

If $\Lambda_{m,n}$ and $\Delta_{m,n}$ are synchronous, then it is easy to check that for any given $\theta \in \mathbf{R}$, the coefficients $a_{i,j}, b_{i,j}, c_{i,j}$ determined by (3.22) — (3.25) must satisfy (3.2) — (3.9), which means that f can be arbitrarily evaluated at the point $(1,1)$ and then f itself is uniquely determined. Hence $\dim F_0 = 1$.

(b) The row triviality implies that there exists some $i_1 \in M$ such that $s_{i_1,1} \neq 0$. By (3.2), $s_{i_1,1}(f(x,1) - f(x,0)) = (a_{i_1,2} - a_{i_1,1})x$. Thus $f(x,1) - f(x,0)$ is a linear function. By setting $x = 0$ and $x = 1$ respectively, we can see that $f(x,0) - f(x,1) = \theta x$. Subtracting (3.6) from (3.7), we have

$$\begin{aligned} \theta A_i(x) &= f(A_i(x), 1) - f(A_i(x), 0) \\ &= s_{i,1}(f(x,1) - f(x,0)) + (a_{i,n} - a_{i,1})x + b_{i,n} + c_{i,n} - c_{i,1} \\ &= s_{i,1}\theta x + (a_{i,n} - a_{i,1})x + b_{i,n} + c_{i,n} - c_{i,1}, \quad i \in M. \end{aligned}$$

From (3.11) and (3.12),

$$a_{i,n} - a_{i,1} = (s_{i,1} + \cdots + s_{i,n-1})\theta, \quad i \in M.$$

Combine these results and compare the coefficients,

$$(x_i - x_{i-1})\theta = s_{i,1}\theta + (s_{i,1} + \cdots + s_{i,n-1})\theta = (s_{i,1} + \cdots + s_{i,n})\theta, \quad i \in M.$$

Applying Lemma 3.5, we know that if $\Lambda_{m,n}$ and $\Delta_{m,n}$ are not synchronous, then $\theta = 0$. In this case, noting that $\xi_0 = \xi_m = \eta_j = 0$, for any $(\xi_1, \cdots, \xi_{m-1}) \in \mathbf{R}^{m-1}$, the coefficients

$$a_{i,j} = \xi_i - \xi_{i-1},\ b_{i,j} = 0,\ c_{i,j} = \xi_{i-1}, \quad i \in M,\ j \in N$$

determined by Lemma 3.3 satisfy (3.2) — (3.9). Therefore f can be arbitrarily evaluated at the points $(x_i, 0)$, $i = 1, \cdots, m-1$ and then f is uniquely determined by these values. Hence $\dim F_0 = m - 1$.

If $\Lambda_{m,n}$ and $\Delta_{m,n}$ are synchronous, then for any $(\xi_1, \cdots, \xi_{m-1}, \theta) \in \mathbf{R}^m$, the coefficients $a_{i,j}$, $b_{i,j}$, $c_{i,j}$ determined by Lemma 3.3 satisfy (3.2) — (3.9). Therefore $\dim F_0 = m$.

(c) Set $s_{i_1,1} \neq 0$. Then from (3.2) we have

$$s_{i_1,1}(f(x,1) - qf(x,0)) = s_{i_1,1}f(x,1) - s_{i_1,2}f(x,0) = (a_{i_1,2} - a_{i_1,1})x.$$

Thus $f(x,1) - qf(x,0)$ is linear. In this case $f(x,1) - qf(x,0) = \theta x$. From (3.7) and (3.6),

$$\begin{aligned} \theta A_{i_1}(x) = f(A_{i_1}(x), 1) - qf(A_{i_1}(x), 0) &= s_{i_1,n}f(x,1) - qs_{i_1,1}f(x,0)) \\ &+ (a_{i_1,n} - qa_{i_1,1})x + b_{i_1,n} + c_{i_1,n} - qc_{i_1,1}, \end{aligned}$$

or

$$q^{n-1}s_{i_1,1}f(x,1) - qs_{i_1,1}f(x,0) = \theta A_{i_1}(x) + (qa_{i_1,1} - a_{i_1,n})x$$

$$+qc_{i_1,1}-b_{i_1,n}-c_{i_1,n}, \quad i\in M. \tag{3.26}$$

The right side above is a linear function of x. Combining this with $f(x,1)-qf(x,0)=\theta x$, note that

$$\begin{vmatrix} q^{n-1}s_{i_1,1} & -qs_{i_1,1} \\ 1 & -q \end{vmatrix} = qs_{i_1,1}(1-q^{n-1}).$$

Thus if $q^{n-1}\neq 1$, then both $f(x,0)$ and $f(x,1)$ are linear, in which case $f(x,0)\equiv 0,\ f(x,1)=\theta x$. Applying this in (3.26) and comparing the coefficients, we have

$$(x_i-x_{i-1})\theta=(s_{i,1}+\cdots+s_{i,n})\theta, \quad i\in M.$$

From Lemma 3.5 we know that if $\Lambda_{m,n}$ and $\Delta_{m,n}$ are not synchronous, then $\theta=0$. Thus $f=0$ and $F_0=0$. If $\Lambda_{m,n}$ and $\Delta_{m,n}$ are synchronous, then we can again verify that for any $\theta\in\mathbf{R}$, the coefficients $a_{i,j},\, b_{i,j},\, c_{i,j}$ determined by Lemma 3.3 satisfy (3.2) — (3.9). Therefore $\dim F_0=1$.

If $q^{n-1}=1$, then $q=-1$ and n is odd for $q\neq 1$. In this case $\sum_{i=1}^m s_{i,j}=-\sum_{i=1}^m s_{i,j+1},\ 1\leq j\leq n-1$. Obviously for some j, $y_j-y_{j-1}\neq\sum_{i=1}^m s_{i,j}$. By Lemma 3.6, it means $\theta=0$. Thus $f(x,1)+f(x,0)\equiv 0$. Note that in this case $s_{i,n}=s_{i,1}$. Add (3.6) to (3.7),

$$(a_{i,1}+a_{i,n})x+c_{i,1}+b_{i,n}+c_{i,n}=0, \quad i\in M. \tag{3.27}$$

But from Lemma 3.3, $a_{i,1}=a_{i,n}=\xi_i-\xi_{i-1},\ \ b_{i,j}=0,\ \ c_{i,j}=\xi_{i-1},\ \ i\in M,\ j\in N$. Combining these results and setting $x=1$, we have $\xi_i=0,\ i\in M$, which yields all $a_{i,j}=b_{i.j}=c_{i,j}=0$. Thus $f\equiv 0$ and then $F_0=\{0\}$.

(d) In this case $s_{1,n}+\cdots+s_{m,n}=0\neq y_n-y_{n-1}$. By Lemma 3.5,

$$f(0,y)=f(1,y)\equiv 0.$$

Set $s_{i_1,1}\neq 0$. From (3.2),

$$s_{i_1,1}f(x,1)=(a_{i_1,2}-a_{i_1,1})x.$$

Thus $f(x,1)$ is linear and $f(x,1)=\theta x\equiv 0$. Combining this result with (3.7), we have $a_{i,n}=0,\ i\in M$. By (3.12), $\xi_i-\xi_{i-1}=a_{i,n}=0$ or $\xi_i=\xi_{i-1},\ i\in M$. Thus $\xi_i=\xi_0=0,\ i\in M$. According to Lemma 3.3, all $a_{i,j},\ b_{i,j},\ c_{i,j}$ vanish. Hence $f=0$ and $F_0=\{0\}$.

(e) Analogous to (d).

(f) — (i) Symmetric to (b) — (e) respectively.

(j) By Lemma 3.4 we have $\theta=0$. For any $(\xi_1,\cdots,\xi_{m-1},\eta_1,\cdots,\eta_{n-1})\in\mathbf{R}^{m+n-2}$, the coefficients $a_{i,j},\, b_{i,j},\, c_{i,j}$ determined by Lemma 3.3 satisfy (3.2) — (3.9).

(k) In this case again we have $\theta=0$. By symmetry we may assume $s_{1,1}\neq 0$ and all other $s_{i,j}=0$. From (3.2) and (3.4), $s_{1,1}f(x,1)=(a_{1,2}-a_{1,1})x,\ \ s_{1,1}f(1,y)=(b_{2,1}-b_{1,1})y$. Thus $f(x,1)=f(1,y)\equiv 0$. By Lemma 3.3 and from (3.7) and (3.9) we can conclude that

$$\xi_i=\xi_0=0, \quad \eta_j=\eta_0=0, \quad i\in M,\ j\in N.$$

Again we obtain all $a_{i,j}=b_{i,j}=c_{i,j}=0$ and then $f=0$, which implies $F_0=\{0\}$.

(l) At first we assume that all $s_{1,j},\ j\in N$, do not vanish. From the column triviality we have $s_{1,j}=q^{j-1}s_{1,1},\ j\in N$, where $q\neq 0$. By (3.2), $s_{1,1}(f(x,1)-qf(x,0))=(a_{1,2}-a_{1,1})x$, from which it follows that $f(x,1)-qf(x,0)=\theta x$. Combining this with (3.8) and (3.9), we have

$$\theta A_i(x)=s_{i,n}f(x,1)-qs_{i,1}f(x,0)+(a_{i,n}-qa_{i,1})x+b_{i,n}+c_{i,n}-qc_{i,1},\ \ i\in M.$$

Comparing the coefficients and applying Lemma 3.4, we have

$$(q-1)\xi_i=(s_{1,1}+\cdots+s_{1,n}-x_i)\theta, \quad i\in M. \tag{3.28}$$

From (3.4), $s_{1,1}f(1,y)=(b_{2,1}-b_{1,1})y$. Thus $f(1,y)=\theta y$. Combining with (3.9), we have $\theta B_j(y)=b_{m,j}y+a_{m,j}+c_{m,,j},\ j\in N$. By comparing the coefficients,

$$\eta_j=\theta(y_j-s_{1,1}-\cdots-s_{1,j}), \quad j\in N. \tag{3.29}$$

If $q = 1$, then we have $\theta = 0$ by (3.28). Thus $\eta_j = 0,\ j \in N$, by (3.29). In this case it is easy to verify that for any $(\xi_1, \cdots, \xi_{m-1}) \in \mathbf{R}^{m-1}$, the coefficients

$$a_{i,j} = \xi_i - \xi_{i-1},\quad b_{i,j} = 0,\quad c_{i,j} = \xi_{i-1},\quad i \in M,\ j \in N$$

determined by Lemma 3.4 satisfy (3.2) — (3.9). Then f is uniquely determined by these values. Hence $\dim F_0 = m - 1$.

If $q \neq 1$, then all $\xi_i,\ \eta_j$ are determined by θ from (3.28) and (3.29). If we have further $s_{1,1} + \cdots + s_{1,n} \neq 1$, then $\theta = 0$ by Lemma 3.4. Thus all $\xi_i = \eta_j = 0$ and then $f = 0$, which means that $F_0 = \{0\}$. If $s_{1,1} + \cdots + s_{1,n} = 1$, we consider q^{n-1}. If $q^{n-1} \neq 1$, then from $f(x,1) - qf(x,0) = \theta x$ and

$$q^{n-2} f(x,1) - f(x,0) = (qs_{1,1})_{-1}(\theta A_1(x) + (qa_{1,1} - a_{1,n})x + qc_{1,1} - b_{1,n} - c_{1,n},$$

we have $f(x,0) \equiv 0$. Thus $\xi_i = 0,\ i \in M$. From (3.28) we know $\theta = 0$. Hence all $\eta_j = 0,\ j \in N$ and then we must have $f = 0$. Thus $F_0 = \{0\}$. If $q^{n-1} = 1$, but $q \neq 1$, then $q = -1$ and n is odd, in which case

$$s_{1,1} + \cdots + s_{1,n} = \sum_{j=1}^{n} (-1)^{j-1} s_{1,j} = s_{1,1} < 1.$$

According to Lemma 3.4, we still have $\theta = 0$. Then all $\xi_i,\ \eta_j$ vanish. It yields $f = 0$. Thus $F_0 = \{0\}$.

If all $s_{m,j},\ j \in N$, do not vanish, then from (3.4) we have $s_{m,1} f(0,y) = (b_{m,1} - b_{m-1,1})y$. Thus $f(0,y)$ is linear and then must vanish. Hence $\eta_j = 0,\ j \in N$. Applying the column triviality as above, we can obtain similar results as desired.

(m) Symmetric to (l).

(n) In this case we can easily deduce that $f(x,1) - qf(x,0) = \theta x$ (cf. the proof of (l)) and $f(1,y) - pf(0,y) = \theta y$. Then from (3.6) — (3.9) we obtain

$$\theta A_i(x) = p^i q^n s f(x,1) - p^i q^2 f(x,0) + (a_{i,n} - qa_{i,1})x + b_{i,n} + c_{i,n} - qc_{i,1},\quad i \in M, \tag{3.30}$$

$$\theta B_j(y) = p^m q^j s f(1,y) - p^2 q^j f(0,y) + (b_{m,j} - pb_{1,j})y + a_{m,j} + c_{m,j} - qc_{1,j},\quad j \in N. \tag{3.31}$$

Comparing the coefficients, we have

$$(1-q)\xi_i = \left(x_i - \sum_{\alpha=1}^{i} \sum_{\beta=1}^{n} s_{\alpha,\beta} \right) \theta,\quad i \in M, \tag{3.32}$$

$$(1-p)\eta_j = \left(y_j - \sum_{\alpha=1}^{m} \sum_{\beta=1}^{j} s_{\alpha,\beta} \right) \theta,\quad j \in N. \tag{3.33}$$

If $p = q = 1$, then $\theta = 0$. In this case we can easily check that for any

$$(\xi_1, \cdots, \xi_{m-1}, \eta_1, \cdots, \eta_{n-1}) \in \mathbf{R}^{m-n-2},$$

the coefficients

$$a_{i,j} = \xi_i - \xi_{i-1},\ \ b_{i,j} = 0,\ \ c_{i,j} = \xi_{i-1},\ \ i \in M,\ j \in N$$

determined by Lemma 3.4 satisfy (3.2) — (3.9), where $f(x,0) = f(x,1)$ is uniquely determined by $f(x_i, 0) = \xi_i,\ \ i = 0, 1, \cdots, m$, and $f(0,y) = f(1,y)$ is determined by $f(0, y_j) = \eta_j,\ \ j = 0, 1, \cdots, n$. Hence $\dim F_0 = m - n - 2$.

If $p = 1,\ q \neq 1$, then $\theta = 0,\ \xi_i = 0,\ i \in M$. For any $(\eta_1, \cdots, \eta_{n-1}) \in \mathbf{R}^{n-1}$, it is easy to verify that the coefficients

$$a_{i,j} = 0,\ \ b_{i,j} = \eta_j - \eta_{j-1},\ \ c_{i,j} = \eta_{j-1},\ \ i \in M,\ j \in N$$

satisfy (3.2) — (3.9), where $f(x,0) = f(x,1) \equiv 0$, and $f(0,y) = f(1,y)$ is determined by $f(0,y_j) = \eta_j, \quad j = 0,1,\cdots,n$. Thus $\dim F_0 = n-1$.

If $p \neq 1, \ q = 1$, then we have $\dim F_0 = n-1$ similarly.

If $p \neq 1, \ q \neq 1$, then from (3.32) and (3.33) we can see that all $\xi_i, \ \eta_j, \ i \in M, \ j \in N$, are determined by the value of θ. If $(p^{m-1}-1)(q^{n-1}-1) \neq 0$, then, combining (3.30) with $f(x,1) - qf(x,0) = \theta x$, we have $f(x,0) \equiv 0$. Thus all $\xi_i = 0$. According to the same reason we have $\eta_j = 0$. If $\Lambda_{m,n}$ and $\Delta_{m,n}$ are synchronous, then we can verify that

$$a_{i,j} = p^i \frac{q^j - q}{q-1} s\theta, \quad b_{i,j} = \frac{p^i - p}{p-1} q^j s\theta,$$
$$c_{i,j} = \frac{(p^i-p)(q^j-q)}{(p-1)(q-1)} s\theta, \quad i \in M, \ j \in N,$$

satisfy (3.2) — (3.9), where $f(x,0) = f(0,y) \equiv 0, \ f(x,1) = \theta x, \ f(1,y) = \theta y$ and θ is an arbitrary real number. Obviously that means $\dim F_0 = 1$. If $\Lambda_{m,n}$ and $\Delta_{m,n}$ are not synchronous, then by (3.32) and (3.33) we have $\theta = 0$, which implies $a_{i,j} = b_{i,j} = c_{i,j} = 0, \ i \in M, \ j \in N$. Hence $f = 0, \ F_0 = \{0\}$. If $p^{m-1} = 1$ but $q^{n-1} \neq 1$, then we must have $f(x,0) \equiv 0, \ f(x,1) = \theta x, \ p = -1$, and m odd. By (3.7),

$$\theta A_i(x) = f(A_i(x),1) = s_{i,n}\theta x + a_{i,n}x + b_{i,n} + c_{i,n}, \quad i \in M.$$

In this case $a_{i,n} = (s_{i,1} + \cdots + s_{i,n-1})\theta, \ i \in M$. Combining this with the preceding equation and comparing the coefficients, we have

$$(x_i - x_{i-1})\theta = (s_{i,1} + \cdots + s_{i,n})\theta = (-1)^i (q^{n+1} - q)s\theta/(q-1), \ i \in M,$$

which yields $\theta = 0$. In this case, we can easily check that for any $(\eta_1, \cdots, \eta_{n-1}) \in \mathbf{R}^{n-1}$, the coefficients

$$a_{i,j} = 0, \quad b_{i,j} = \eta_j - \eta_{j-1}, \quad c_{i,j} = \eta_{j-1}, \quad i \in M, \ j \in N$$

satisfy (3.2) — (3.9), where $f(x,0) = f(x,1) \equiv 0, \ f(0,y) = f(1,y)$ is uniquely determined by $f(0,y_j) = \eta_j, \ j = 0,1,\cdots,n$. Thus $\dim F_0 = n-1$. Symmetrically, if $p^{m-1} \neq 1$ but $q^{n-1} = 1$, then $\dim F_0 = m-1$. If $p^{m-1} = q^{n-1} = 1$, then we must have $p = q = -1, \ m$ and n odd. This yields $\sum_{i=1}^m \sum_{j=1}^n s_{i,j} = s_{1,1} < 1$. Thus according to Lemma 3.3 we can obtain

$$a_{i,j} = \xi_i - \xi_{i-1}, \quad b_{i,j} = \eta_j - \eta_{j-1}, \quad c_{i,j} = \xi_i + \eta_j \quad i \in M, \ j \in N.$$

Note that in this case $f(x,1) + f(x,0) = \theta x \equiv 0$. Adding (3.6) to (3.7), we have

$$\begin{aligned}
0 &= f(A_i(x),1) + f(A_i(x),0) \\
&= s_{1,n} f(x,1) + s_{i,1} f(x,0) + (a_{i,n} + a_{i,1})x + b_{i,n} + c_{i,n} + c_{i,1} \\
&= s_{i,1}(f(x,1) + f(x,0)) + 2(\xi_i - \xi_{i-1})x + \xi_{i-1} \\
&= 2(\xi_i - \xi_{i-1})x + \xi_{i-1}, \quad i \in M.
\end{aligned}$$

This yields $\xi_i = 0, \ i \in M$. Symmetrically $\eta_j = 0, \ j \in N$. Thus we again obtain $f = 0$ and then $F_0 = \{0\}$.

4. Space $F(\Delta_{m,n}, \lambda, \mathbf{K})$

Throughout this section, we suppose that $\Lambda_{m,n}$ has the following special form: $s_{i,j} = \lambda, \ i \in M, \ j \in N$, where $-1 < \lambda < 1$. We denote $\mathbf{K}$ by

$$\{\phi \in \mathrm{C}(D) : \phi(x,y) = axy + bx + cy + d, \ (a,b,c,d) \in \mathbf{R}^4\}.$$

Obviously $\mathbf{K}$ is a linear subspace of $\mathrm{C}(D)$, $\dim \mathbf{K} = 4$. We abbreviate $F(\Delta_{m,n}, \Lambda_{m,n}, \mathbf{K})$ to $F(\Delta_{m,n}, \lambda, \mathbf{K})$. Without any loss of generality we still suppose that

$$x_0 = y_0 = 0, \ x_m = y_n = 1.$$

Lemma 4.1. *Let* $a_{i,j}, b_{i,j}, c_{i,j}, d_{i,j} \in \mathbf{R}$, $i \in M$, $j \in N$. *Let IFS* $\{D \times \mathbf{R}; W_{i,j} : i \in M,\ j \in N\}$ *be defined by*

$$W_{i,j}(x,y,z) = (A_i(x), B_j(y), \lambda z + a_{i,j}xy + b_{i,j}x + c_{i,j}y + d_{i,j}) \tag{4.1}$$

for $(x,y,z) \in D \times \mathbf{R}$, $i \in M$, $j \in N$. *Then IFS* $\{D \times \mathbf{R}; W_{i,j} : i \in M,\ j \in N\}$ *is generating if and only if the coefficients* $a_{i,j}, b_{i,j}, c_{i,j}, d_{i,j}$ *satisfy the following* $3mn-3$ *linearly independent equations:*

$$a_{i,j} + c_{i,j} - c_{i+1,j} = a_{1,1} + c_{1,1} - c_{2,1}, \quad i = 1, \cdots, m-1,\ j \in N, \tag{4.2}$$

$$a_{m,j} + b_{m,j} - b_{m,j+1} = a_{1,1} + b_{1,1} - b_{1,2}, \quad j = 1, \cdots, n-1, \tag{4.3}$$

$$a_{m,n} + b_{m,n} - b_{m,1} = (a_{1,1} + b_{1,1} - b_{1,2})(\lambda - 1 + x_{m-1})/\lambda, \tag{4.4}$$

$$b_{i,j} + d_{i,j} - d_{i+1,j} = b_{1,1} + d_{1,1} - d_{2,1}, \qquad i = 1, \cdots, m-1,\ j \in N, \tag{4.5}$$

$$\begin{array}{l} b_{m,j} + d_{m,j} - d_{1.j} = (b_{1,1} + d_{1,1} - d_{2,1})(1 - \lambda^{-1}) \\ -(a_{1,1} + c_{1,1} - c_{2,1})y_{j-1}\lambda^{-1}, \qquad j \in N, \end{array} \tag{4.6}$$

$$c_{i,j} + d_{i,j} - d_{i,j+1} = c_{1,1} + d_{1,1} - d_{1,2}, \qquad i \in M,\ j = 1, \cdots, n-1, \tag{4.7}$$

$$\begin{array}{l} c_{i,n} + d_{i,n} - d_{i,1} = (c_{1,1} + d_{1,1} - d_{1,2})(1 - \lambda^{-1}) \\ -(a_{1,1} + b_{1,1} - b_{1,2}x_{i-1}\lambda^{-1}, \quad i \in M. \end{array} \tag{4.8}$$

Proof. Suppose that $\{D \times \mathbf{R}; W_{i,j} : i \in M,\ j \in N\} \to f$. One can verify that Theorem 2.1a implies (4.2) — (4.8).

To show that the $3mn-3$ equations in (4.2) — (4.8) are independent, we set $a_{1,1} = b_{1,1} = c_{1,1} = d_{i,j} = 0$, $i \in M$, $j \in N$. Thus we obtain a system of homogeneous linear equations in $3mn - 3$ unknowns $a_{i,j}$, $b_{i,j}$, $c_{i,j}$, $(i,j) \neq (1,1)$, $i \in M$, $j \in N$. But from equations (4.5) and (4.6), in this case all $b_{i,j} = 0$. Similarly from equations (4.7) and (4.8) all $, c_{i,j} = 0$. From these results and equations (4.2) and (4.4) we have $a_{i,j} = 0$ for all $i \in M$, $j \in N$. Therefore the system mentioned above has only zero solutions. This completes the proof.

Theorem 4.1. $\dim F(\Delta_{m,n}, \lambda, \mathbf{K}) = mn + 3$.

Proof. Let H be the solution space of the system of homogeneous linear equations (4.2) — (4.8) in $4mn$ unknowns $a_{i,j}$, $b_{i,j}$, $c_{i,j}$, $d_{i,j}$, $i \in M$, $j \in N$. By Lemma 4.1 we know that H is a $mn+3$-dimensional subspace of $\mathbf{R}^{4mn}$. From Proposition 2.2 we can see that there exists a natural isomorphism $\Pi : H \to F(\Delta_{m,n}, \lambda, \mathbf{K})$,

$$\Pi \begin{pmatrix} a_{1,1} & \cdots & a_{1,n} & b_{1,1} & \cdots & b_{1,n} & c_{1,1} & \cdots & c_{1,n} & d_{1,1} & \cdots & d_{1,n} \\ \vdots & & \vdots & \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots & & \vdots \\ a_{m,1} & \cdots & a_{m,n} & b_{m,1} & \cdots & b_{m,n} & c_{m,1} & \cdots & c_{m,n} & d_{m,1} & \cdots & d_{m,n} \end{pmatrix} = f,$$

where f is generated by the IFS $\{D \times \mathbf{R}; W_{i,j} : i \in M,\ j \in N\}$ having the form as in Lemma 4.1.

The following is a corollary of Lemma 4.1.

Lemma 4.2. *Let* $S = \{z_{i,j} \in \mathbf{R} :\ i = 0, 1, \cdots, m;\ j = 0, 1, \cdots, n\}$ *be a set of* $(m+1)(n+1)$ *real numbers. Then there exists a unique* $f \in F(\Delta_{m,n}, \lambda, \mathbf{K})$ *such that*

$$f(x_i, y_j) = z_{i,j}, \quad i = 0, 1, \cdots, m;\ j = 0, 1, \cdots, n \tag{4.9}$$

if and only if S *satisfies the following* $m + n - 2$ *equations:*

$$(1 - x_i)z_{0,0} - (1 - x_i)z_{0,n} - z_{i,0} + z_{i,n} + x_i z_{m,0} - x_i z_{m,n} = 0, \quad i = 1, \cdots, m-1, \tag{4.10}$$

$$(1 - y_j)z_{0,0} - (1 - y_j)z_{m,0} - z_{0,j} + z_{m,j} + y_j z_{0,n} - y_j z_{m,n} = 0, \quad j = 1, \cdots, n-1. \tag{4.11}$$

Definition 4.1. *Given* $\Delta_{m,n}$, *we denote*

$$\begin{array}{rcl} I(\Delta_{m,n}) & = & \{(i,j) :\ i = 1, \cdots, m-1,\ j = 1, \cdots, n-1\}, \\ B(\Delta_{m,n}) & = & \{(i,j) :\ i = 0, m,\ j = 0, 1, \cdots, n\} \\ & & \bigcup \{(i,j) :\ i = 0, 1, \cdots, m,\ j = 0, n\}. \end{array}$$

We call the elements of $I(\Delta_{m,n})$ the inner indices and those of $B(\Delta_{m,n})$ the boundary indices.

Theorem 4.2. *Let $J \subset \{(i,j):\ i=0,1,\cdots,m,\ j=0,1,\cdots,n\}$. Let $J_I = J\bigcap I(\Delta_{m,n})$, $J_B = J\bigcap B(\Delta_{m,n})$. Let $\{z_{i,j}:\ (i,j)\in J\}$ be a set of given real numbers.*

(a) *There exists some $f\in F(\Delta_{m,n},\,\lambda,\,\mathbf{K})$ such that*

$$f(x_i,y_j)=z_{i,j},\quad (i,j)\ \in\ J \tag{4.12}$$

if and only if the following system of linear equations in unknowns $z_{i,j}$, $(i,j)\ \in\ B(\Delta_{m,n})\backslash J_B$ is solvable:

$$(1-x_i)z_{0,0}-(1-x_i)z_{0,n}-z_{i,0}+z_{i,n}+x_iz_{m,0}-x_iz_{m,n}=0,\quad i=1,\cdots,m-1, \tag{4.13}$$

$$(1-y_j)z_{0,0}-(1-y_j)z_{m,0}-z_{0,j}+z_{m,j}+y_jz_{0,n}-y_jz_{m,n}=0,\quad j=1,\cdots,n-1. \tag{4.14}$$

where $z_{i,j},\ (i,j)\ \in\ J_B$ are known.

(b) *There exists a unique $f\in F(\Delta_{m,n},\,\lambda,\,\mathbf{K})$ satisfying (4.45) if and only if $J_I=I(\Delta_{m,n})$ and the linear equations (4.13) and (4.14) in unknowns $z_{i,j},\ (i,j)\ \in\ B(\Delta_{m,n})\backslash J_B$ are uniquely solvable.*

Proof. (a) This follows directly from Lemma 4.2.

(b) From Lemma 4.2 it is easy to see that the sufficient part of the conclusion is obviously valid. In order to establish the necessary part, we first suppose that $\tilde{z}_{i,j},\ \bar{z}_{i,j},\ (i,j)\ \in\ B(\Delta_{m,n})\backslash J_B$ are different solutions of equations (4.13) and (4.14). Then by Lemma 4.2 we have $f,\ g\in F(\Delta_{m,n},\,\lambda,\,\mathbf{K})$ with

$$\begin{aligned}
&f(x_i,y_j)=g(x_i,y_j)=z_{i,j},\quad (i,j)\in J,\\
&f(x_i,y_j)=\tilde{z}_{i,j},\quad g(x_i,y_j)=\bar{z}_{i,j},\quad (i,j)\in B(\Delta_{m,n})\backslash J_B,\\
&f(x_i,y_j)=g(x_i,y_j)=0,\quad (i,j)\in I(\Delta_{m,n})\backslash J.
\end{aligned}$$

Since $\tilde{z}_{i,j}$ and $\bar{z}_{i,j}$, $(i,j)\in B(\Delta_{m,n})\backslash J_B$ are different, $f\neq g$. this contradicts the assumption of uniqueness of f.

Secondly, if $J_I\neq I(\Delta_{m,n})$, then we can choose an inner index $(i_0,j_0)\ \in\ I(\Delta_{m,n})\backslash J_I$. By Lemma 4.2 there exist $f,g\in F(\Delta_{m,n},\lambda,\mathbf{K})$ such that

$$\begin{aligned}
&f(x_i,y_j)=g(x_i,y_j)=z_{i,j},\quad (i,j)\in J,\\
&f(x_i,y_j)=g(x_i,y_j)=\tilde{z}_{i,j},\quad (i,j)\in B(\Delta_{m,n})\backslash J_B,\\
&f(x_i,y_j)=0,\quad g(x_i,y_j)=1,\quad (i,j)\in I(\Delta_{m,n})\backslash J,
\end{aligned}$$

where $\tilde{z}_{i,j},\ (i,j)\ \in\ B(\Delta_{m,n})\backslash J$ are the solutions of the system of linear equations (4.13) and (4.14) in unknowns $z_{i,j},\ (i,j)\in B(\Delta_{m,n})\backslash J_B$. Obviously $f\neq g$. This contradicts the assumption of uniqueness again. Thus $J_I=I(\Delta_{m,n})$.

Corollary 1. *Let*

$$J_0=\{(i,j):\ i=0,1,\cdots,m-1,\ j=0,1,\cdots,n-1\}\bigcup\{(0,n),(m,0),(m,n)\}.$$

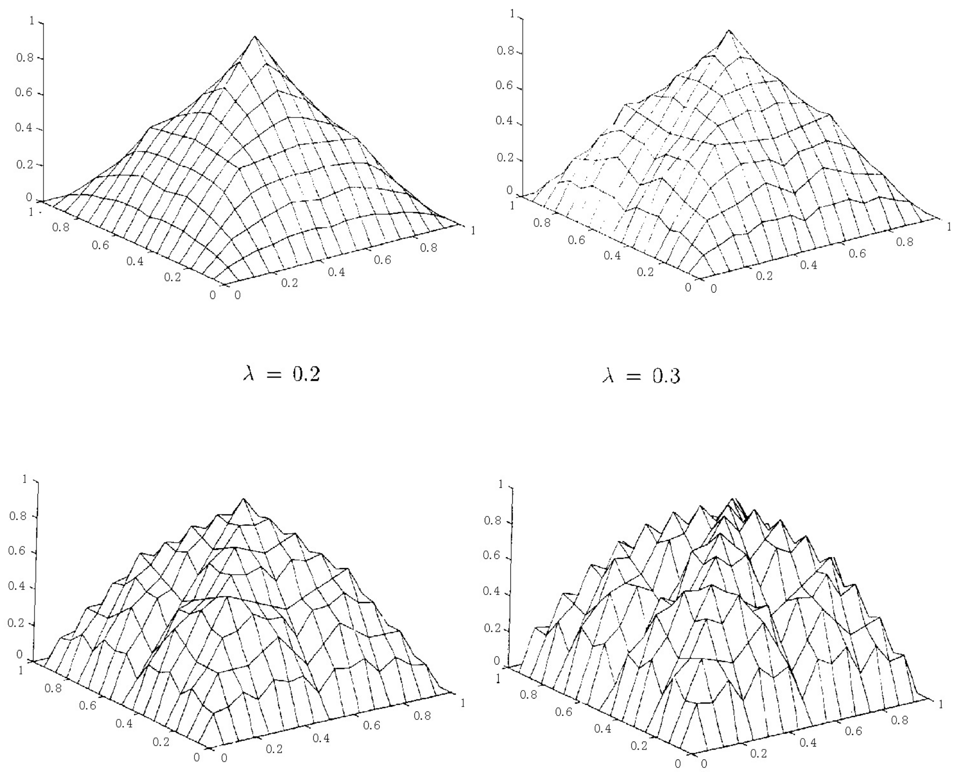
Then for any given set of real numbers $\{z_{i,j}:(i,j)\in J_0\}$ there is a unique $f\in F(\Delta_{m,n},\,\lambda,\,\mathbf{K})$ satisfying $f(x_i,y_j)=z_{i,j},\quad (i,j)\in J_0$.

Corollary 2. *Let J_0 be defined as above. For each $(p,q)\in J_0$, define $g_{p,q}\in F(\Delta_{m,n},\lambda,\mathbf{K})$ by*

$$g_{p,q}(x_i,y_j)=\begin{cases}1 & \textit{if } (i,j)=(p,q),\\ 0 & \textit{if } (i,j)\in J_0\backslash\{(p,q)\}.\end{cases}$$

Then $\{g_{i,j}:(i,j)\in J\}$ is a basis of the linear space $F(\Delta_{m,n},\lambda,\mathbf{K})$.

Example 4.1. Figure 4.1 sketches out the graphs of $g_{1,1}$, where $m=n=2$, under different parameters λ.



$\lambda = 0.2$ $\lambda = 0.3$

$\lambda = 0.4$ $\lambda = 0.5$

Figure 4.1

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