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A NOTE ON VECTOR CASCADE ALGORITHM*[1)]

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Abstract

The focus of this paper is on the relationship between accuracy of multivariate refinable vector and vector cascade algorithm. We show that, if the vector cascade algorithm (1.5) with isotropic dilation converges to a vector-valued function with regularity, then the initial function must satisfy the Strang-Fix conditions.

Key words: Cascade algorithm, Accuracy, Symbol, Refinable vector.

1. Introduction and Main Result

Fox a fixed integer d no less than 1, let A be a $d \times d$ matrix with integer entries and all eigenvalues of modulus > 1.

In wavelet theory, we are often concerned with functional equation of the form

$$\Phi(x) = \sum_{k \in Z^d} c_k \Phi(Ax - k), \tag{1.1}$$

where Φ is the unknown vector of functions defined on the d-dimensional Euclidean space R^d and $c = \{c_k\}_{k \in Z^d}$ is a finitely supported $r \times r$ matrix sequence on Z^d. We call the equation (1.1) refinement equation. Any vector-valued function satisfying a refinement equation is called a refinable vector. In Fourier domain, (1.1) is equivalent to

$$\hat{\Phi}(\xi) = m\left(\left(A^T\right)^{-1}\xi\right)\hat{\Phi}\left(\left(A^T\right)^{-1}\xi\right), \tag{1.2}$$

where $m(\xi) = |\det A|^{-1} \sum_{k \in Z^d} c_k e^{-ik\xi}$.

The matrix A is called a dilation matrix, the sequence c is called a refinement mask and $m(\xi)$ is called a symbol.

The equation (1.1) is the starting point for construction of wavelets (in scalar case) and multiwavelets (in multiple case) in one dimensional or higher-dimensional case (see [5], [2], [8], [14] et al). The usual choice of A is 2 or $2I_{d\times d}$ (the $d \times d$ identity matrix). Recently, the isotropic matrix dilations have already been considered in [2] and [14] to construct non-separable bidimensional wavelet bases. For example, in [2] A is chosen to be $\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$ and

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$\begin{pmatrix} 0 & 2 \\ 1 & 0 \end{pmatrix}$. Here we say A is isotropic if A is similar to a diagonal matrix $\mathrm{diag}(\sigma_1, \cdots, \sigma_d)$ with $|\sigma_1| = \cdots = |\sigma_d| = |\mathrm{det}A|^{\frac{1}{d}}$. In this case, there exists an invertible matrix Λ such that

$$\Lambda A \Lambda^{-1} = \mathrm{diag}\,(\sigma_1, \cdots, \sigma_d)\,. \tag{1.3}$$

Obviously, the dilation matrices studied in [2] are isotropic.

The existence of the solution of the equation (1.1) has been studied by many mathematicians ([6], [9], [1]). It is well-known that there is no closed-form analytic formula even for scalar univariate orthogonal refinable functions in the case with 2 dilation. The cascade algorithms are often used to study the solutions of the refinement equation (1.1). Associated with refinement equation (1.1), we define refinement operator

$$Q_c f(\cdot) := \sum_{k \in Z^d} c_k f(A \cdot - k), \tag{1.4}$$

for each component of the vector-valued function f belongs to $L_q(R^d)$, the familiar space $-$ the set of all q-absolutely integrable functions in R^d.

Let Φ_0 be an initial vector-valued function with compact support in $L_q(R^d)$. For $n = 1, 2, \cdots$, define

$$\Phi_n := Q_c \Phi_{n-1}. \tag{1.5}$$

Clearly $\Phi_n = Q_c^n \Phi_0$, $n = 0, 1, \cdots$ (here we let $Q_c^0 \Phi_0 = \Phi_0$). The algorithm (1.5) is called cascade algorithm with mask c and dilation A.

Convergence of cascade algorithms has been studied by many mathematicians ([3], [11], [15]). In this paper, we are interested in vector cascade algorithms with isotropic dilation. We shall prove that, in order to get a refinable vector of functions with higher accuracy and smoothness, the initial vector-valued function Φ_0 must satisfy the Strang-Fix conditions.

We say that the vector-valued function Φ has accuracy p if all multivariable polynomials with total degree less than p can be reproduced from linear combinations of multi-integer translates of the functions $\phi_1, \cdots, \phi_r$, or equivalently, $\Pi_{p-1} \subset S(\Phi)$, where Π_{p-1} denotes the linear space of all polynomials in d variable of total degree at most $p-1$ and $S(\Phi) = \{f : f = \sum_{l \in Z^d} \sum_{k=1}^{r} a_{kl} \phi_k(\cdot - l)\}$ with any sequence $\{a_{kl}\}$. We say $S(\Phi)$ is principal shift-invariant (PSI) if $r = 1$ and finite shift-invariant (FSI) if $r > 1$.

In wavelet theory, accuracy is a desirable property of scaling function. The accuracy p of orthonormal refinable function leads to the p vanishing moments of wavelets which is very important for image compression.

A single compactly supported function $f \in L_1(R^d)$ is said to satisfy the Strang-Fix conditions of order p if

$$\hat{f}(0) = 1 \quad \text{and} \quad D^{\mu} \hat{f}(2\pi k) = 0, \quad \text{for} \quad 0 \le \mu, |\mu| < p, k \in Z^d/\{0\}. \tag{1.6}$$

A vector-valued function $\Phi = (\phi_1, \cdots, (\phi_r)^T$ is said to satisfy the Strang-Fix conditions of order p if there exists a scalar function $\psi \in S(\Phi)$ satisfying the usual Strang-Fix conditions (1.6).

It is stated in [12] that any vector-valued function (refinable or not refinable) has accuracy p if and only if the vector of functions satisfies the Strang-Fix conditions of order p under the assumption of linear independence. For refinable functions, it is crucial to explore the conditions of the mask c and the symbol $m(\xi)$ when refinable vector Φ has accuracy p.

In the case of a single, one-dimensional refinable function, $A = 2$, the requirement for Φ to have accuracy p is the following set of sum rules:

$$\sum_{k=0}^{N} c_k = 2 \qquad \text{and} \quad \sum_{k=0}^{N} (-1)^k k^j c_k = 0, \quad \text{for} \;\; j = 0, \cdots, p-1.$$

The above sum rules can be stated in an equivalent form based on symbol function $m(\xi)$, namely

$$m\left(0\right)=1 \qquad \text{and} \quad m^{(j)}\left(\pi\right)=0 \quad \text{for} \quad j=0,\cdots,p-1.$$

These sum rules imply that the symbol function factorizes as $m(\xi)=\left(\frac{1+e^{-i\xi}}{2}\right)^p R(\xi)$, here $R(\xi)$ is a trigonometric polynomials.

The same results for FSI in one dimensional case i.e. $(d=1, r\geq 1)$ with $A=2$ were obtained independently by Heil, Strang and Strela [10] and by Plonka [16]. In 1998, Cabrelli, Heil and Molter obtained the accuracy conditions in time domain in higher-dimensional, multi-function case with an arbitary dilation matrix (see [1],Theorem 3.11 and Theorem 3.6). Based on the work in [1], the accuracy conditions in frequency domain is raised in [4] in the case of dydaic dilation. Furthermore, in [4], two kinds of equivalent characterization like the scalar univariate case was established via symbol and masks in FSI case with arbitrary matrix dilation. To faciliate our discussion, we need to restate a result in [4].

Proposition 1. *Assume that $\Phi : \mathbf{R^d} \to \mathbf{C^{r\times 1}}$ satisfies the refinement equation* (1.1), *Φ is integrable and compactly supported, and the integer translates of Φ are linearly independent. Then Φ has accuracy p if and only if the following equivalent conditions hold.*

(a) *(Conditions in frequency domain) there exists a collection of row vectors $\{\nu_\alpha \in \mathbf{C^{1\times r}} : 0\leq \alpha, |\alpha|<p\}$ such that $\nu_0\neq 0$ and the following equations hold $\alpha\in\mathbf{Z^d}$ and $0\leq\alpha,|\alpha|<p$:*

$$\sum_{0\leq\beta\leq\alpha}\begin{pmatrix}\alpha\\ \beta\end{pmatrix}(2i)^{|\beta|-|\alpha|}\,\tilde{\nu}_\beta D^{\alpha-\beta}m\left(0\right)=2^{-|\alpha|}\nu_\alpha, \tag{1.7}$$

$$\sum_{0\leq\beta\leq\alpha}\begin{pmatrix}\alpha\\ \beta\end{pmatrix}(2i)^{|\beta|-|\alpha|}\,\tilde{\nu}_\beta D^{\alpha-\beta}m\left(2\pi\left(A^T\right)^{-1}\omega\right)=0, \quad \omega\in\Omega(A^T)/\{0\}. \tag{1.8}$$

(b) *(Conditions in time domain) there exists a collection of row vectors $\{\nu_\alpha\in\mathbf{C^{1\times r}} : 0\leq \alpha, |\alpha|<p\}$ such that $\nu_0\neq 0$ and the following equations hold*

$$\sum_{0\leq\beta\leq\alpha}\begin{pmatrix}\alpha\\ \beta\end{pmatrix}(-2)^{-|\beta|}\,\tilde{\nu}_{\alpha-\beta}\sum_{k\in Z^d}\left(Ak+\mu\right)^\beta c_{Ak+\mu}=2^{-|\alpha|}\nu_\alpha, \tag{1.9}$$

for $\alpha\in\mathbf{Z^d}$ and $0\leq\alpha,|\alpha|<p$ and $\forall\mu\in\Omega(A)$.

We need to explain the notations appearing in the above proposition.

Througout the paper, We use the standard multi-index notation $x^\alpha=x_1^{\alpha_1}\cdots x_d^{\alpha_d}$, where $x=(x_1,\cdots,x_d)^T\in\mathbf{R}^d$ and $\alpha=(\alpha_1,\cdots,\alpha_d)^T$ with each α_i a nonnegative integer. The length of α is defined to be $|\alpha|=\alpha_1+\cdots+\alpha_d$. The number of multi-indices α of degree s is $d_s=\begin{pmatrix}s+d-1\\ d-1\end{pmatrix}$.

We write $\beta\leq\alpha$ if $\beta_i\leq\alpha_i$ for $i=1,\cdots,d$. Denote

$$\begin{pmatrix}\alpha\\ \beta\end{pmatrix}=\begin{cases}\begin{pmatrix}\alpha_1\\ \beta_1\end{pmatrix}\cdots\begin{pmatrix}\alpha_d\\ \beta_d\end{pmatrix}, & \text{if} \quad \beta_i\leq\alpha_i \quad \text{for every i,}\\ 0, & \text{if} \quad \beta_i>\alpha_i \quad \text{for some i.}\end{cases}$$

For $j=1,\cdots,d$, D_j denotes the partial differential operator with respect to the jth coordinate. Let $D=(D_1,\cdots,D_d)^T$. For multi-integer $\mu=(\mu_1,\cdots,\mu_d)^T$, D^μ is the differential operator $D_1^{\mu_1}\cdots D_d^{\mu_d}$.

Let $C^{r\times 1}$ $(C^{1\times r})$ be the space of $r\times 1$ column ($1\times r$ row) vectors with complex entries.

For a $d\times d$ dilation matrix A, let $\Omega(A)$ be a complete set of representatives of the distinct cosets of Z^d/AZ^d, and let $\Omega(A^T)$ be a complete set of representatives of the distinct cosets of Z^d/A^TZ^d, where A^T denotes the transpose of A. Evidently, $\#\Omega(A)=\#\Omega(A^T)=|\text{det}A|$. Without loss of any generality, we may assume that $0\in\Omega(A)$ and $0\in\Omega(A^T)$.

For each integer $s \geq 0$, define the vector-valued function $X_{[s]} : \mathbf{R}^d \to \mathbf{C}^{d_s \times 1}$ by

$$X_{[s]}(x) = (x^\alpha)_{|\alpha|=s} \qquad x \in \mathbf{R}^d, \tag{1.10}$$

where $(x^\alpha)_{|\alpha|=s}$ is a $d_s \times 1$ vector with each entry being monomial with exact degree s and coefficient 1. Of course, we alway write the entries of the vector by a fixed order.

The matrix $A_{[s]}$ arises when considering the dilation of the vector of monomials $X_{[s]}$. It satisfies the fundamental equations

$$X_{[s]}(Ax) = A_{[s]} X_{[s]}(x). \tag{1.11}$$

In fact, $A_{[s]}$ is given by

$$A_{[s]} = \left(a^s_{\alpha,\beta}\right)_{|\alpha|=s, |\beta|=s},$$

where $a^s_{\alpha,\beta}$ are the coefficients of the polynomial $(Ax)^\alpha = \sum_{|\beta|=s} a^s_{\alpha,\beta} x^\beta$, for $|\alpha| = s$.

Let $\{\nu_\alpha \in C^{1\times r} : 0 \leq \alpha, |\alpha| \leq p-1\}$ be a collection of $1 \times r$ row vectors. We define the $d_s \times 1$ column vectors $\nu_{[s]}$ with block entries that are the $1 \times r$ row vectors ν_α by

$$\nu_{[s]} := (\nu_\alpha)_{|\alpha|=s}, \qquad 0 \leq s < p. \tag{1.12}$$

Thus, $\nu_{[s]}$ is a $d_s \times r$ matrix.

We define some vectors $\tilde{\nu}[s]$ for $0 \leq s < p$ from $A_{[s]}$ and a collection of row vectors $\{\nu_\alpha \in C^{1\times r} : 0 \leq \alpha, |\alpha| < p\}$. Let

$$\tilde{\nu}_{[s]} := 2^{-s} A_{[s]} \nu_{[s]}, \quad \text{for} \quad 0 \leq s < p. \tag{1.13}$$

Then $\tilde{\nu}_{[s]}$ $(0 \leq s < p)$ are $d_s \times 1$ column vectors with block entries that are $1 \times r$ vectors. So we can define $1 \times r$ vector $\tilde{\nu}_\alpha$ by

$$(\tilde{\nu}_\alpha)_{|\alpha|=s} := \tilde{\nu}_{[s]}, \tag{1.14}$$

for $0 \leq \alpha, |\alpha| < p$.

Finally, we define the following differential operator for dilation matrix $A = (a_{k,l})_{1\leq k,l\leq d}$

$$(AD)^\mu := (a_{1,1}D_1 + \cdots + a_{1,d}D_d)^{\mu_1} \cdots (a_{d,1}D_1 + \cdots + a_{d,d}D_d)^{\mu_d}. \tag{1.15}$$

Correspondingly, we can define the vector with each entry being a differential operator:

$$X_{[s]}(D) := (D^\alpha)_{|\alpha|=s}. \tag{1.16}$$

Then, we have

$$X_{[s]}(AD) = A_{[s]} X_{[s]}(D). \tag{1.17}$$

In this paper we shall establish a connection between accuracy condition (1.7) and cascade algorithms. The following theorem is the main result of this paper.

Theorem 1.2. *Suppose that A is a $d \times d$ isotropic dilation matrix and $m(\xi)$ is the matrix symbol function of refinement equation (1.1). Assume that there exists a collection of vectors $\{\nu_\alpha \in C^{1\times r} : 0 \leq \alpha, |\alpha| \leq p-1, \nu_0 \neq 0\}$ such that (1.7) holds. Suppose that $\Phi \in W_1^{p-1}(R^d)$ is a compactly supported solution of the refinement equation (1.1) with symbol $m(\xi)$ and Φ satisfies $\nu_0 \hat{\Phi}(0) \neq 0$. Let Q_c be the linear operator defined in (1.4). If Φ_0 is a compactly supported vector of functions in $W_1^{p-1}(R^d)$ such that*

$$\lim_{n\to\infty} ||Q_c^n \Phi_0 - \Phi||_{W_1^{p-1}(R^d)} = 0, \tag{1.18}$$

then Φ_0 satisfies the Strang-Fix conditions of order p.

Note 1. Throughout the paper $L_1(R^d)$ denotes the space of absolutely integrable functions. The Sobolev space $W_1^{p-1}(R^d)$ is defined by the set of functions

$$W_1^{p-1}(R^d) := \{f \in L_1(R^d); \sum_{0\le\alpha,|\alpha|\le p-1} ||D^\alpha f||_{L_1(R^d)} < \infty\}.$$

A vector-valued function $\Phi = (\phi_1, \cdots, \phi_r)^T \in L_1(R^d)$ means that $\sum_{k=1}^{r} ||\phi_k||_{L_1} < \infty$. For simplicity, we also denote by $||\Phi||_{L_1}$ the norm of vector-valued function Φ in $L_1(R^d)$. Similarily, the norm of a vector-valued function $\Phi \in W_1^{p-1}(R^d)$ is defined by

$$||\Phi||_{W_1^{p-1}(R^d)} := \sum_{k=1}^{r} \sum_{0\le\alpha,|\alpha|\le p} ||D^\alpha \phi_k||_{L_1}.$$

2. Lemmas

Lemma 2.1. *Suppose that A is an isotropic matrix, i.e. there exists an invertible matrix Λ such that $\Lambda A \Lambda^{-1} = diag(\sigma_1, \cdots, \sigma_d)$ with $|\sigma_1| = \cdots |\sigma_d| = |detA|^{\frac{1}{d}}$. Let $\sigma = (\sigma_1, \cdots, \sigma_d)^T$ and p be any non-negative integer. Then for p-differentiable function f, the following equation*

$$(\Lambda D)^\mu f\left(\left(A^T\right)^n \cdot\right)(x) = \sigma^{n\mu} (\Lambda D)^\mu f\left(\left(A^T\right)^n x\right) \tag{2.1}$$

holds for any non-negative integer n and $0 \le \mu, |\mu| \le p$.

Proof. For any $d \times d$ matrix with integer entries, the following identity is true for $0 \le \mu, |\mu| \le p$

$$D^\mu f\left(M^T \cdot\right)(x) = (MD)^\mu f\left(M^T x\right).$$

Noting that the operator $(\Lambda D)^\mu$ is the linear combination of all D^β with $|\beta| = |\mu|$, i.e. $(\Lambda D)^\mu = \sum_{|\beta|=|\mu|} b_\beta D^\beta$, where b_β are determined by Λ and μ, we have

$$(\Lambda D)^\mu f\left(\left(A^T\right)^n \cdot\right)(x) = \sum_{|\beta|=|\mu|} b_\beta D^\beta f\left(\left(A^T\right)^n \cdot\right)(x)$$

$$= \sum_{|\beta|=|\mu|} b_\beta \left(A^n D\right)^\beta f\left(\left(A^T\right)^n x\right) = \left(\Lambda A^n D\right)^\mu f\left(\left(A^T\right)^n x\right).$$

Since A is isotropic, we have

$$\Lambda A^n = \mathrm{diag}\left(\sigma_1^n, \cdots, \sigma_d^n\right)\Lambda.$$

Furthermore, we get

$$\left(\Lambda A^n D\right)^\mu = \left(\mathrm{diag}\left(\sigma_1^n, \cdots, \sigma_d^n\right)\Lambda D\right)^\mu = \sigma^{n\mu} (\Lambda D)^\mu.$$

This completes the proof of lemma 2.1.

Lemma 2.2. *Let $\{\nu_\alpha \in C^{1\times r} : 0 \le \alpha, |\alpha| \le p-1\}$ be a collection of $1 \times r$ vectors. Then, above collection of vectors can uniquely determine a $1 \times r$ vector function $\tau(\xi) = (\tau_1(\xi), \cdots, \tau_r(\xi))$ of the form $\tau(\xi) = \sum_{0\le j,|j|<p} h(j) e^{ij\xi}$ such that $\tau(\xi)$ satisfies the following linear equations*

$$D^\alpha \tau(0) = i^{|\alpha|} \nu_\alpha, \quad \textit{for} \quad 0 \le \alpha, |\alpha| \le p-1, \tag{2.2}$$

where $h(j)$ is $1 \times r$ vectors.

Proof. The equations (2.2) can be written as

$$\sum_{0 \le j, |j| < p} j^{\alpha} h(j) = \nu_{\alpha} \quad \text{for} \quad 0 \le \alpha, |\alpha| \le p-1.$$

Since $\{e^{ij\xi} : 0 \le j, |j| < p-1\}$ is linearly independent, the matrix $(j^{\alpha})_{\alpha,j}$ is nonsingular for $0 \le \alpha, 0 \le j$ and $|j| \le p-1, |\alpha| \le p-1$. Then the above equations have a unique solution and $\tau(\xi)$ can be uniquely determined by linear system (2.2). This completes the proof of Lemma 2.2.

Lemma 2.3. *Let $\{\nu_{\alpha} \in C^{1\times r} : 0 \le \alpha, |\alpha| \le p-1\}$ be a collection of $1 \times r$ vectors and $\tau(\xi)$ be a trigonometric polynomials vector defined in Lemma 2.2. Then*

$$\tilde{\nu}_{\alpha} = (2i)^{-|\alpha|} (AD)^{\alpha} \tau(0), \quad \textit{for} \quad 0 \le \alpha, |\alpha| \le p-1, \tag{2.3}$$

where $\tilde{\nu}_{\alpha}$ is defined in (1.14).

Proof. By (2.2), we have

$$\nu_{[s]} = i^{-s} X_{[s]}(D)\tau(0) \quad \text{for} \quad 0 \le s \le p-1.$$

Multiplying $A_{[s]}$ on the both sides of the above identity, we obtain

$$A_{[s]}\nu_{[s]} = i^{-s} A_{[s]} \left(X_{[s]}(D)\tau\right)(0) \quad \text{for} \quad 0 \le s \le p-1.$$

From the definition of $\tilde{\nu}_{[s]}$ and (1.13), we conclude

$$\tilde{\nu}_{[s]} = (2i)^{-s} A_{[s]} \left(X_{[s]}(D)\tau(0)\right) = (2i)^{-s} \left(X_{[s]}(AD)\tau\right)(0),$$

or equivalently

$$(\tilde{\nu}_{\alpha})_{|\alpha|=s} = \left((2i)^{-|\alpha|} (AD)^{\alpha} \tau(0)\right)_{|\alpha|=s}.$$

This completes the proof of Lemma 2.3.

Lemma 2.4. *Let $m(\xi)$ be the matrix symbol function of the refinement equation (1.1). Suppose that there exists a collection of $1 \times r$ vector $\{\nu_{\alpha} \in C^{1\times r} : 0 \le \alpha, |\alpha| \le p-1\}$ such that the symbol $m(\xi)$ satisfies (1.7). Let $\tau(\xi)$ be a vector trigonometric polynomials defined in Lemma 2.2. Then for $0 \le \alpha, |\alpha| \le p-1$, any positive integer n and $\forall k \in Z^d$, we have*

$$D^{\alpha}\left(\tau\left(\left(A^T\right)^n \cdot\right) \prod_{j=1}^{n} m\left(\left(A^T\right)^{j-1} \cdot\right)\right)(2\pi k) = D^{\alpha}\tau(0). \tag{2.4}$$

Proof. For simplicity, define a sequence of vector-valued function $\{t_n(\xi)\}_{n=1}^{\infty}$ by

$$t_n(\xi) := \tau\left(\left(A^T\right)^n \xi\right) \cdot \prod_{j=1}^{n} m\left(\left(A^T\right)^{j-1} \xi\right). \tag{2.5}$$

Applying Leibnitz's formula, noticing $\tau(\cdot)$ and $m(\cdot)$ are of period 2π, we have

$$\begin{aligned} & D^{\alpha} t_1(2\pi k) \\ &= \sum_{0 \le \beta \le \alpha} \binom{\alpha}{\beta} D^{\beta}\tau\left(A^T \cdot\right)(2\pi k)\, D^{\alpha-\beta} m(2\pi k) \\ &= \sum_{0 \le \beta \le \alpha} \binom{\alpha}{\beta} (AD)^{\beta} \tau(0)\, D^{\alpha-\beta} m(0), \end{aligned}$$

using (2.3), (1.7) and (2.2), it follows

$$D^{\alpha}t_1\left(2\pi k\right)=\sum_{0\leq\beta\leq\alpha}\begin{pmatrix}\alpha\\ \beta\end{pmatrix}(2i)^{|\beta|}\,\tilde{\nu}_{\beta}D^{\alpha-\beta}m\left(0\right)=D^{\alpha}\tau\left(0\right).$$

This establishes (2.4) for n=1.

Suppose that (2.4) is valid for $n-1$, i.e. $D^{\alpha}t_{n-1}(2\pi k)=D^{\alpha}\tau(0)$. Noting that

$$t_n\left(\xi\right)=t_{n-1}\left(A^T\xi\right)m\left(\xi\right) \tag{2.6}$$

and using Leibnitz's formula again, we obtain

$$D^{\alpha}t_n\left(2\pi k\right)=\sum_{0\leq\beta\leq\alpha}\begin{pmatrix}\alpha\\ \beta\end{pmatrix}D^{\beta}t_{n-1}\left(A^T\cdot\right)\left(2\pi k\right)D^{\alpha-\beta}m\left(2\pi k\right). \tag{2.7}$$

By the induction hypothesis $D^{\alpha}t_{n-1}(2\pi k)=D^{\alpha}\tau(0)$ for $0\leq\alpha,|\alpha|\leq p-1$, we claim that

$$D^{\beta}t_{n-1}\left(A^T\cdot\right)\left(2\pi k\right)=(2i)^{|\beta|}\,\tilde{\nu}_{\beta}. \tag{2.8}$$

In fact, this can be derived as follows

$$\begin{aligned}&\left(D^{\beta}t_{n-1}\left(A^T\cdot\right)\left(2\pi k\right)\right)_{|\beta|=s}=\left(\left(AD\right)^{\beta}t_{n-1}\left(0\right)\right)_{|\beta|=s}\\ &=X_{[s]}\left(AD\right)t_{n-1}\left(0\right)=A_{[s]}X_{[s]}\left(D\right)t_{n-1}\left(0\right)\\ &=A_{[s]}\left(D^{\beta}t_{n-1}\left(0\right)\right)_{|\beta|=s}=A_{[s]}\left(D^{\beta}\tau\left(0\right)\right)_{|\beta|=s}\\ &=A_{[s]}i^s\nu_{[s]}=(2i)^s\,\tilde{\nu}_{[s]}\\ &=\left((2i)^{|\beta|}\,\tilde{\nu}_{\beta}\right)_{|\beta|=s}.\end{aligned}$$

By (2.7), (2.8) and (1.7), we obtain

$$D^{\alpha}t_n\left(2\pi k\right)=\sum_{0\leq\beta\leq\alpha}\begin{pmatrix}\alpha\\ \beta\end{pmatrix}(2i)^{|\beta|}\,\tilde{\nu}_{\beta}D^{\alpha-\beta}m\left(2\pi k\right)=i^{|\alpha|}\nu_{\alpha}=D^{\alpha}\tau\left(0\right).$$

This completes the proof of Lemma 2.4.

3. Proof of Theorem 1.2

Proof. In order to prove the initial vector of functions $\Phi_0=(\phi_{01},\cdots,\phi_{0r})^T$ satisfies the Strang-Fix conditions of order p, it suffices to find a function ψ_0 which is finitely linear combination of the integer translates of $\phi_{0j}, j=1,\cdots,r$ such that

$$\hat{\psi}_0=1\quad\text{and}\quad D^{\alpha}\hat{\psi}_0\left(2\pi k\right)=0 \tag{3.1}$$

hold for $0\leq\alpha,|\alpha|\leq p-1$ and $k\in Z^d/\{0\}$.

Let $\tau(\xi)=\sum\limits_{0\leq j,|j|\leq p-1}h(j)e^{ij\xi}$ be the $1\times r$ vector-valued trignometric polynomial defined in Lemma 2.2. Define $\psi_0(x)$ by

$$\hat{\psi}_0\left(\xi\right)=\tau\left(\xi\right)\hat{\Phi}_0\left(\xi\right). \tag{3.2}$$

In what following, we shall prove that $\psi_0(x)$ satisfies (3.1).

By (1.5), we get

$$\hat{\Phi}_n\left(\xi\right)=m\left(\left(A^T\right)^{-1}\xi\right)\hat{\Phi}_{n-1}\left(\left(A^T\right)^{-1}\xi\right),\quad \xi\in R^d. \tag{3.3}$$

A repeated application of (3.3) yields that, for all positive integer n

$$\hat{\Phi}_n\left(\left(A^T\right)^n \xi\right) = \prod_{j=1}^{n} m\left(\left(A^T\right)^{j-1}\xi\right)\cdot \hat{\Phi}_0(\xi), \quad \xi \in R^d. \tag{3.4}$$

Define a sequence of scalar functions $\{\psi_n\}_{n=1}^{\infty}$ by

$$\hat{\psi}_n(\xi) = \tau(\xi)\hat{\Phi}_n(\xi). \tag{3.5}$$

From (3.4) and (3.5), we have

$$D^\alpha \hat{\psi}_n\left(\left(A^T\right)^n \cdot\right)(2\pi k)$$

$$= \sum_{0\le\beta\le\alpha}\left(\begin{array}{c}\alpha\\ \beta\end{array}\right) D^{\alpha-\beta}\left(\tau\left(\left(A^T\right)^n\cdot\right)\cdot\prod_{j=1}^{n} m\left(\left(A^T\right)^{j-1}\cdot\right)\right)(2\pi k)\, D^\beta \hat{\Phi}_0(2\pi k). \tag{3.6}$$

Furthermore, applying Lemma 2.4 to (3.6), using (3.2), we conclude that

$$D^\alpha \hat{\psi}_0(2\pi k) = D^\alpha \hat{\psi}_n\left(\left(A^T\right)^n\cdot\right)(2\pi k), \quad \text{for} \quad 0\le\alpha\le p-1, \forall k\in Z^d. \tag{3.7}$$

Define a scalar function $\psi(x)$ by

$$\hat{\psi}(\xi) = \tau(\xi)\hat{\Phi}(\xi). \tag{3.8}$$

By the definitions of ψ, ψ_0 and ψ_n we know that ψ, ψ_0 and ψ_n are a finitely linear combination of the multi-integer translates of ϕ_j, ϕ_{0j} and $\phi_{n,j}$ for $j = 1,\cdots,r$ respectively. Preciesely, by (3.2),(3.5),(3.8) and Lemma 2.2, we obtain

$$\psi(x) = \sum_{0\le j,|j|\le p-1} h(j)\,\Phi(x+j).$$

$$\psi_0(x) = \sum_{0\le j,|j|\le p-1} h(j)\,\Phi_0(x+j).$$

$$\psi_n(x) = \sum_{0\le j,|j|\le p-1} h(j)\,\Phi_n(x+j).$$

Since $\Phi, \Phi_0, \Phi_n \in W_1^{p-1}(R^d)$, we can conclude that $\psi, \psi_0, \psi_n \in W_1^{p-1}(R^d)$.

Let $\alpha = 0$ in (3.7), we get

$$\hat{\psi}_0(2\pi k) = \hat{\psi}_n\left(2\pi(A^T)^n k\right) = \tau(0)\hat{\Phi}_n\left(2\pi(A^T)^n k\right).$$

Since $|\hat{\psi}_n - \hat{\psi}| \le ||\psi_n - \psi||_{L_1} \le C||\Phi_n - \Phi||_{W_1^{p-1}} \to 0$ when $n\to\infty$, so $\hat{\psi}_n \to \hat{\psi}$ for any $\xi \in R^d$ when $n\to\infty$. By Riemann-Lebesgue Lemma, we get that

$$\hat{\psi}_0(2\pi k) = \lim_{n\to\infty}\tau(0)\hat{\Phi}_n\left(2\pi(A^T)^n k\right) = \lim_{n\to\infty}\tau(0)\hat{\Phi}\left(2\pi(A^T)^n k\right) = 0 \tag{3.9}$$

for $k\in Z^d\backslash\{0\}$. For $k=0$, we have $\hat{\psi}_0(0) = \lim\limits_{n\to\infty}\tau(0)\hat{\Phi}_n(0) = \tau(0)\hat{\Phi}(0) \ne 0$. We can normalize ψ_0 such that $\hat{\psi}_0(0) = 1$. This establishes (3.1) for $\alpha = 0$ and $k \in Z^d$.

Let $\psi_\mu(x) = (-i\Lambda x)^\mu\,\psi(x)$ and $\psi_{n,\mu}(x) = (-i\Lambda x)^\mu\,\psi_n(x)$, then the Fourier transforms of $D^\mu\psi_\mu$ and $D^\mu\psi_{n,\mu}$ are

$$(i\xi)^\mu\,(\Lambda D)^\mu\,\hat{\psi}(\xi) \quad \text{and} \quad (i\xi)^\mu\,(\Lambda D)^\mu\,\hat{\psi}_n(\xi).$$

Since ψ_n, ψ are also compactly supported functions and the sequence of the vector-valued functions $\{\Phi_n\}_{n=1}^{\infty}$ converges to Φ in the Sobolev space $W_1^{p-1}(R^d)$, we have

$$\begin{aligned}
&|\,(i\xi)^{\mu}\,(\Lambda D)^{\mu}\,\hat{\psi}_n\,(\xi) - (i\xi)^{\mu}\,(\Lambda D)^{\mu}\,\hat{\psi}\,(\xi)\,| \\
&\le ||D^{\mu}\psi_{n,\mu} - D^{\mu}\psi_{\mu}||_{L_1(R^d)} \\
&\le C||D^{\mu}\psi_n - D^{\mu}\psi||_{L_1(R^d)} \\
&= ||\sum_{0\le j,|j|\le p-1} h(j)\,(D^{\alpha}\Phi_n(\cdot - j) - D^{\alpha}\Phi(\cdot - j))\,||_{L_1(R^d)} \\
&\le C\int_{R^d}\sum_{0\le j,|j|\le p-1}|h(j)\,(D^{\mu}\Phi_n(x-j) - D^{\mu}\Phi(x-j))\,|dx \\
&\le C\int_{R^d}\sum_{0\le j,|j|\le p-1}\sum_{k=1}^{r}|h_k(j)|\,|\,(D^{\mu}\phi_{n,k}(x-j) - D^{\mu}\phi_k(x-j))\,|dx \\
&\le C\int_{R^d}\sum_{k=1}^{r}||D^{\mu}\phi_{n,k} - D^{\mu}\phi_k||_{L_1(R^d)} \\
&\le C||\Phi_n - \Phi||_{W_1^{p-1}(R^d)} \to 0.
\end{aligned}$$

So $(i\xi)^{\mu}(\Lambda D)^{\mu}\hat{\psi}_n(\xi) \to (i\xi)^{\mu}(\Lambda D)^{\mu}\hat{\psi}(\xi)$ when n tends to ∞. Correspondingly

$$\lim_{n\to\infty}\left(i\left(\Lambda^{-1}\right)^{T}\xi\right)^{\mu}(\Lambda D)^{\mu}\,\hat{\psi}_n\,(\xi) = \left(i\left(\Lambda^{-1}\right)^{T}\xi\right)^{\mu}(\Lambda D)^{\mu}\,\hat{\psi}\,(\xi) \tag{3.10}$$

for any $\xi \in R^d$.

Since $D^{\alpha}\psi_{\mu} \in L_1(R^d)$ and the Fourier transform of $D^{\alpha}\psi_{\mu}$ is $\xi^{\alpha}(\Lambda D)^{\mu}\hat{\psi}(\xi)$, by Riemann-Lebesgue lemma, we have $\xi^{\alpha}(\Lambda D)^{\mu}\hat{\psi}(\xi) \to 0$ when $\xi \to \infty$ for any $0 < \alpha, \mu$ and $|\alpha, \mu| \le p-1$. Of course we also have $\lim_{\xi\to\infty}\left(i\left(\Lambda^{-1}\right)^{T}\xi\right)^{\mu}(\Lambda D)^{\mu}\,\hat{\psi}(\xi) = 0$. Combining this and (3.10), we obtain

$$\lim_{n\to\infty}\left(i\left(\Lambda^{-1}\right)^{T}\left(A^{T}\right)^{n}2\pi\beta\right)^{\mu}(\Lambda D)^{\mu}\,\hat{\psi}_n\left(2\pi\left(A^{T}\right)^{n}\beta\right) = 0 \quad \text{for} \quad \beta \in Z^d\backslash\{0\}. \tag{3.11}$$

Noting that $|\delta_1| = \cdots |\delta_d| = |\det A|^{\frac{1}{d}}$, we see that there exists a positive constant C independent of n such that

$$\begin{aligned}
&|\left(\left(\Lambda^{-1}\right)^{T}\left(A^{T}\right)^{n}\beta\right)^{\mu}| \\
&= |\left(\operatorname{diag}\left(\delta_1^{n}, \cdots, \delta_d^{n}\right)\left(\Lambda^{-1}\right)^{T}\beta\right)^{\mu}| \\
&= |\delta_1^{n\mu_1}\cdots\delta_d^{n\mu_d}\left(\left(\Lambda^{-1}\right)^{T}\beta\right)^{\mu}| \\
&\ge C|\det A|^{\frac{n|\mu|}{d}}|\left(\left(\Lambda^{-1}\right)^{T}\beta\right)^{\mu}|.
\end{aligned}$$

Combining this and (3.11) together, we get

$$\lim_{n\to\infty}\delta^{n\mu}\,(\Lambda D)^{\mu}\,\hat{\psi}_n\left(2\pi\left(A^{T}\right)^{n}\beta\right) = 0 \quad \text{for} \quad \beta \in Z^d\backslash\{0\}, \tag{3.12}$$

where $\delta = (\delta_1, \cdots, \delta_d)^T$. Using (3.7) again, we have

$$(\Lambda D)^{\mu}\,\hat{\psi}_0\,(2\pi k) = (\Lambda D)^{\mu}\,\hat{\psi}_n\left(\left(A^{T}\right)^{n}\cdot\right)(2\pi k)\,.$$

Furtheremore, by Lemma 2.1, we obtain

$$(\Lambda D)^{\mu}\, \hat{\psi}_0\,(2\pi k) = \delta^{n\mu}\,(\Lambda D)^{\mu}\, \hat{\psi}_n\left(2\pi\left(A^T\right)^n k\right). \tag{3.13}$$

by (3.12) and (3.13), we conclude that $(\Lambda D)^{\mu}\, \hat{\psi}_0\,(2\pi k) = 0$ for $0 < \mu, |\mu| \le p-1$ and $k \in Z^d\backslash\{0\}$. Since Λ is nonsingular, it is natural to conclude that $D^{\mu}\hat{\psi}_0\,(2\pi k) = 0$ for $0 < \mu, |\mu| \le p-1$ and $k \in Z^d\backslash\{0\}$. This establishes (3.1) for $0 < \alpha, |\alpha| \le p-1$ and $k \in Z^d\backslash\{0\}$. This completes the proof of Theorem 1.2.

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