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LAGUERRE PSEUDOSPECTRAL METHOD FOR NONLINEAR PARTIAL DIFFERENTIAL EQUATIONS*[1)]

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Abstract

The Laguerre Gauss-Radau interpolation is investigated. Some approximation results are obtained. As an example, the Laguerre pseudospectral scheme is constructed for the BBM equation. The stability and the convergence of proposed scheme are proved. The numerical results show the high accuracy of this approch.

Key words: Laguerre pseudospectral method, Nonlinear differential equations.

1. Introduction

In scientific computations, we often need to solve differential equations in unbounded domains numerically, e.g., see Gottlieb and Orszag [1], Canuto, Hussaini, Quarteroni and Zang [2], Bernardi and Maday [3], and Guo [4]. Usually we set up some artificial boundaries, impose certain artificial boundary conditions and then resolve them. Whereas these treatments cause additional errors. One of reasonable ways for solving such problems is to use spectral method or pseudospectral method related to orthogonal systems of polynomials in unbounded domains. In particular, the Laguerre spectral method and Laguerre pseudospectral method are applicable to problems on the half line. Maday, Bernaud-Thomas and Vandeven [5] established some results on the Laguerre approximation. We also refer to Funaro [6]. On the other hand, Mavriplis [7], Coulaud, Funaro and Kivian [8], and Iranzo and Falqués [9] proposed various algorithms based on the Laguerre approxiamtion. Recently Guo and Shen [10] derived some new approximation results on the Laguerre approximation, constructed some Laguerre spectral schemes for nonlinear problems, proved the stability and the convergence of proposed schemes, and obtained accurate numerical results. But in actual computations, the Laguerre pseudospectral method is more preferable, since it does not need quadratures on the half line, and so saves a lot of work and avoids the corresponding numerical errors. In addition, it is easier to deal with nonlinear terms, e.g., see Coulaud, Funaro and Kivian [8], and Iranzo and Falqués [9]. It is noted that Mastroianni and Monegato [11], also established another kind of approximation results on the generalized Laguerre interpolation and used them for some integral equations successfully.

The aim of this paper is to develop the Laguerre pseudospectral method and its applications to nonlinear partial differential equations. In the next section, we introduce certain spaces, establish some weighted imbedding inequalities, inverse inequalities and approximation results on the Laguerre -Gauss-Radau interpolation. These results play important roles in numerical analysis of the Laguerre pseudospectral method for nonlinear partial differential equations. In section 3, we take the Benjamin-Bona-Mahony (BBM) equation as an example to show how to construct reasonable Laguerre pseudospectral schems for nonlinear problems. The stability and

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the convergence of proposed scheme are proved. The numerical results show the high accuracy of this approach. The main idea and the techniques used in this paper are also applicable to other nonlinear problems.

2. The Laguerre-Gauss-Radau Interpolation

Let $\Lambda = \{x \mid 0 < x < \infty\}$, $\bar{\Lambda} = \Lambda \cup \{0\}$ and $\omega(x) = e^{-x}$. For $1 \le p \le \infty$,

$$L^p_\omega(\Lambda) = \{\ v \mid v \ \text{ is measurable and } \ \|v\|_{L^p_\omega} < \infty\ \}$$

where

$$\|v\|_{L^p_\omega} = \begin{cases} \left(\int_\Lambda |v(x)|^p \omega(x) dx\right)^{\frac{1}{p}}, & 1 \le p < \infty, \\ \operatorname*{ess\,sup}_{x\in\Lambda} |v(x)|, & p = \infty. \end{cases}$$

In particular, $L^2_\omega(\Lambda)$ is a Hilbert space equipped with the following inner product and norm

$$(u,v)_\omega = \int_\Lambda u(x)v(x)\omega(x)dx. \quad \|v\|_\omega = (v,v)^{\frac{1}{2}}_\omega.$$

For simplicity, let $\partial_x v(x) = \frac{\partial v}{\partial x}(x)$, etc.. For any non-negative integer m,

$$H^m_\omega(\Lambda) = \{\ v \mid \partial^k_x v \in L^2_\omega(\Lambda), 0 \le k \le m\}$$

equipped with the following inner product, semi-norm and norm

$$(u,v)_{m,\omega} = \sum_{k=0}^{m} (\partial^k_x u, \partial^k_x v)_\omega,$$

$$|v|_{m,\omega} = \|\partial^m_x v\|_\omega, \qquad \|v\|_{m,\omega} = (v,v)^{\frac{1}{2}}_{m,\omega}.$$

For any real $r > 0$, the space $H^r_\omega(\Lambda)$, its semi-norm $|v|_{r,\omega}$ and norm $\|v\|_{r,\omega}$ are defined by space interpolation as in Adams [12]. Furthermore

$$H^1_{0,\omega}(\Lambda) = \{\ v \mid v \in H^1_\omega(\Lambda) \text{ and } v(0) = 0\}.$$

In addition, $\|v\|_{L^\infty}$ stands for $\|v\|_{L^\infty(\Lambda)}$. We have the following imbedding inequalities.

Lemma 2.1. *For any $v \in H^1_{0,\omega}(\Lambda)$,*

$$\|e^{-\frac{x}{2}} v\|_{L^\infty} \le \sqrt{2}\|v\|^{\frac{1}{2}}_\omega |v|^{\frac{1}{2}}_{1,\omega}, \tag{2.1}$$

$$\|v\|_\omega \le 2|v|_{1,\omega}. \tag{2.2}$$

Moreover for any $v \in H^1_\omega(\Lambda)$,

$$\|e^{-\frac{x}{2}} v\|_{L^\infty} \le \sqrt{2}\|v\|_{1,\omega}. \tag{2.3}$$

Proof. (2.1) and (2.2) are proved in Guo and Shen [10]. Next, for any $x \in \Lambda$,

$$\begin{aligned} e^{-x}v^2(x) &= -\int_x^\infty \partial_y(e^{-y}v^2(y))dy = \int_x^\infty e^{-y}v^2(y)dy - 2\int_x^\infty e^{-y}v(y)\partial_y v(y)dy \\ &\le \|v\|^2_\omega + 2\|v\|_\omega |v|_{1,\omega} \le 2\|v\|^2_{1,\omega}. \end{aligned}$$

We next recall some properties of the Laguerre polynomials. The Laguerre polynomial of degree l is defined by

$$\mathcal{L}_l(x) = \frac{1}{l!} e^x \partial^l_x (x^l e^{-x}).$$

It is the l-th eigenfunction of the singular Strum-Liouville problem

$$\partial_x(xe^{-x}\partial_x v(x)) + \lambda e^{-x} v(x) = 0,$$

with the corresponding eigenvalue $\lambda = l$. The set of Laguerre polynomials is the normalized $L^2_\omega(\Lambda)$-orthogonal system, i.e.,

$$\int_\Lambda \mathcal{L}_l(x)\mathcal{L}_m(x)\omega(x)dx = \delta_{l,m}, \qquad l,m \geq 0$$

where $\delta_{l,m}$ is the Kronecker function. Moreover

$$\int_\Lambda \partial_x\mathcal{L}_l(x)\partial_x\mathcal{L}_m(x)xe^{-x}dx = l\delta_{l,m}, \quad l,m \geq 0. \tag{2.4}$$

For any $v \in L^2_\omega(\Lambda)$,

$$v(x) = \sum_{l=0}^{\infty} \hat{v}_l \mathcal{L}_l(x)$$

with the Laguerre coefficients

$$\hat{v}_l = \int_\Lambda v(x)\mathcal{L}_l(x)\omega(x)dx, \qquad l = 0, 1, 2, ...$$

Next, Let N be any positive integer and $\mathcal{P}_N$ be the set of restrictions to Λ of all algebraic polynomials of degree at most N. Furthermore $\mathcal{P}^0_N = \mathcal{P}_N \cap H^1_{0,\omega}(\Lambda)$. In the sequel, we denote by c a generic positive constant independent of any function and N. Maday, Bernaud-Thomas and Vendeven [5] proved the following inverse inequality, also see Bernardi and Maday [3].

Lamma 2.2. *For any $\phi \in \mathcal{P}_N$ and $r \geq 0$,*

$$\|\phi\|_{r,\omega} \leq cN^r\|\phi\|_\omega. \tag{2.5}$$

We now turn to various orthogonal projections. The $L^2_\omega(\Lambda)$-orthogonal projection $P_N : L^2_\omega(\Lambda) \to \mathcal{P}_N$ is a mapping such that for any $v \in L^2_\omega(\Lambda)$,

$$(P_N v - v, \phi)_\omega = 0, \qquad \forall\phi \in \mathcal{P}_N.$$

The $H^1_{0,\omega}(\Lambda)$-orthogonal projection $P^{1,0}_N : H^1_{0,\omega}(\Lambda) \to \mathcal{P}^0_N$ is a mapping such that for any $v \in H^1_{0,\omega}(\Lambda)$,

$$(P^{1,0}_N v - v, \phi)_{1,\omega} = 0, \qquad \forall\phi \in \mathcal{P}^0_N.$$

For technical reasons, Bernardi and Maday [3] introduced the space

$$H^r_{\omega,\beta}(\Lambda) = \{\ v \mid v, x^{\frac{\beta}{2}}v \in H^r_\omega(\Lambda)\}$$

where β is any non-negative integer. Its norm $\|v\|_{r,\omega,\beta} = \|v(1+x)^{\frac{\beta}{2}}\|_{r,\omega}$. Bernardi and Maday [3] proved result stated below.

Lemma 2.3. *For any $v \in H^r_{\omega,\beta}(\Lambda), r \geq 0$ and $0 \leq \mu \leq r$,*

$$\|P_N v - v\|_{\mu,\omega} \leq cN^{\mu-\frac{r}{2}}\|v\|_{r,\omega,\beta} \tag{2.6}$$

where β is the largest integer for which $\beta < r+1$.

Recently Guo and Shen [10] proved the following result.

Lemma 2.4. *For any $v \in H^1_{0,\omega}(\Lambda) \cap H^r_{\omega,\beta}(\Lambda)$ and $r \geq 1$,*

$$\|P^{1,0}_N v - v\|_{1,\omega} \leq cN^{\frac{1}{2}-\frac{r}{2}}\|v\|_{r,\omega,\beta} \tag{2.7}$$

where β is the largest integer for which $\beta < r$.

In order to obtain better approximation results on the Laguerre pseudospectral interpolation, we also need other spaces. The first was introduced by Mastroianni and Monegato [11]. For any real $r \geq 0$,

$$H^r_{\omega,*}(\Lambda) = \{\ v \in L^2_\omega(\Lambda) \mid \|v\|_{r,\omega,*} < \infty\}$$

where

$$\|v\|_{r,\omega,*} = (\sum_{l=0}^{\infty}(l+1)^r \hat{v}_l^2)^{\frac{1}{2}}. \tag{2.8}$$

Clearly $\|v\|_{0,\omega,*} = \|v\|_\omega$. Let r be any non-negative integer, $\omega_k(x) = x^k\omega(x)$ and

$$\|v\|_{r,\omega,\sim} = (\sum_{k=0}^{r} \|\partial_x^k v\|_{\omega_k}^2)^{\frac{1}{2}}. \tag{2.9}$$

.

Lemma 2.5. *(see Lemma 2.3 of Mastroianni and Monegato [11]). For any non-negative integer r, the norm $\|v\|_{r,\omega,*}$ is equivalent to the norm $\|v\|_{r,\omega,\sim}$.*

Remark 2.1. If r is a non-negative integer and so $\beta = r$. Then $H^r_{\omega,\beta}(\Lambda) \subset H^r_{\omega,*}(\Lambda)$. In fact, let $w(x) = (1+x)^{\frac{r}{2}}v(x)$. Then $\|v\|_{r,\omega,\beta} = \|w\|_{r,\omega}$. By Lemma 2.5,

$$\begin{aligned} \|v\|^2_{r,\omega,*} &\leq c\sum_{k=0}^{r}\|\partial_x^k v\|^2_{\omega_k} = c\sum_{k=0}^{r}\|x^{\frac{k}{2}}\partial_x^k\left((1+x)^{-\frac{r}{2}}w\right)\|^2_\omega \\ &\leq c\sum_{k=0}^{r}\sum_{j=0}^{k}\|x^{\frac{k}{2}}(1+x)^{-\frac{r}{2}-k+j}\partial_x^j w\|^2_\omega \\ &\leq c\sum_{k=0}^{r}\sum_{j=0}^{k}\|\partial_x^j w\|^2_\omega \leq c\|w\|^2_{r,\omega}. \end{aligned}$$

Therefore $\|v\|_{r,\omega,*} \leq c\|v\|_{r,\omega,\beta}$.

We derive some imbedding inequalities and an inverse inequality related to the space $H^r_{\omega,*}(\Lambda)$.

Lemma 2.6. *For any $v \in H^1_{\omega,*}(\Lambda)$,*

$$\|x^{\frac{1}{2}}v\|_\omega \leq 2\|v\|_{1,\omega,*}, \tag{2.10}$$

$$\|x^{\frac{1}{2}}e^{-\frac{x}{2}}v\|_{L^\infty} \leq 2\|v\|_{1,\omega,*}. \tag{2.11}$$

Proof. For any $x \in \Lambda$,

$$\begin{aligned} xe^{-x}v^2(x) &= \int_0^x \partial_y\left(ye^{-y}v^2(y)\right)dy \\ &= 2\int_0^x ye^{-y}v(y)\partial_y v(y)dy + \int_0^x (1-y)e^{-y}v^2(y)dy. \end{aligned}$$

By the Schwartz inequality,

$$xe^{-x}v^2(x) + \int_0^x ye^{-y}v^2(y)dy \leq \frac{1}{2}\|y^{\frac{1}{2}}v\|^2_\omega + 2\|y^{\frac{1}{2}}\partial_y v\|^2_\omega + \|v\|^2_\omega. \tag{2.12}$$

Letting $x \to \infty$ in (2.12), we find that

$$\|y^{\frac{1}{2}}v\|^2_\omega \leq \frac{1}{2}\|y^{\frac{1}{2}}v\|^2_\omega + 2\|y^{\frac{1}{2}}\partial_y v\|^2_\omega + \|v\|^2_\omega$$

from which and Lemma 2.5, the desired result (2.10) follows immediately. Next, (2.11) comes from (2.10), (2.12) and Lemma 5.

Lemma 2.7. *For any $\phi \in \mathcal{P}_N$ and $r \geq 0$,*

$$\|\phi\|_{r,\omega,*} \leq (N+1)^{\frac{r}{2}}\|\phi\|_\omega. \tag{2.13}$$

Proof. The conclusion comes from the definition (2.8) directly.

Theorem 2.4 of Mastroianni and Monegato [11] gives the following result.

Lemma 2.8. *For any $v \in H^r_{\omega,*}(\Lambda)$, and $0 \leq \mu \leq r$,*

$$\|P_N v - v\|_{\mu,\omega,*} \leq cN^{\frac{\mu}{2}-\frac{r}{2}}\|v\|_{r,\omega,*}. \tag{2.14}$$

We now consider the interpolation. Let σ_j^N be the Laguerre-Gauss-Radau interpolation nodes, i.e., $\sigma_0^N = 0$, and σ_j^N are the zeros of $\partial_x \mathcal{L}_{N+1}(x), 1 \le j \le N$. By Theorem 6.31.2 of Szegö [13],

$$|\sigma_j^N| \le 4(N+1), \qquad 0 \le j \le N. \tag{2.15}$$

Let ω_j^N be the corresponding Christoffel number. According to (3.6.10) of Davis and Rabinowitz [14],

$$\omega_0^N = \frac{1}{N+1}, \qquad \omega_j^N = \frac{1}{(N+1)\mathcal{L}_N^2(\sigma_j^N)}, \quad 1 \le j \le N.$$

It is pointed out in Szegö [13] that for any $\phi \in \mathcal{P}_{2N}$,

$$\int_\Lambda \phi(x)\omega(x)dx = \sum_{j=0}^N \phi(\sigma_j^N)\omega_j^N. \tag{2.16}$$

Let $\Lambda_N = \{\sigma_j^N | \quad 1 \le j \le N\}$ and $\bar{\Lambda}_N = \Lambda_N \cup \{0\}$. For $v \in C(\bar{\Lambda})$, the Laguerre-Gauss-Radau interpolant $I_N v \in \mathcal{P}_N$ is determined by

$$I_N v(x) = v(x), \qquad \forall x \in \bar{\Lambda}_N.$$

We also introduce the discrete inner product and norm as follows

$$(u,v)_{\omega,N} = \sum_{j=0}^N u(\sigma_j^N)v(\sigma_j^N)\omega_j^N, \qquad \|v\|_{\omega,N} = (v,v)_{\omega,N}^{\frac12}.$$

By (2.16), for any $\phi,\ \psi \in \mathcal{P}_N$,

$$(\phi,\psi)_\omega = (\phi,\psi)_{\omega,N}, \qquad \|\phi\|_\omega = \|\phi\|_{\omega,N}. \tag{2.17}$$

Obviously

$$(I_N v - v, \phi)_{\omega,N} = 0, \qquad \forall \phi \in \mathcal{P}_N.$$

If the integral $\int_\Lambda f(x)\omega(x)dx$ exists, then we say that $f(x)$ is ω-integrable on Λ. In order to estimate the difference between $I_N v(x)$ and $v(x)$, we need the following important lemma which will be proved in Appendix of this paper.

Lemma 2.9. *Suppose that $f(x)$ is ω-integrable on Λ, and for large $x > 0$,*

$$|f(x)| \le \frac{ce^x}{x^{1+\rho}}, \qquad 0 < \rho < 1. \tag{2.18}$$

Then

$$\sum_{j=0}^N f(\sigma_j^N)\omega_j^N \to \int_\Lambda f(x)\omega(x)dx, \qquad \text{as } N \to \infty.$$

The following Lemma is related to the stability of the Laguerre-Gauss-Radau interpolation.

Lemma 2.10. *For any $v \in H_{0,\omega}^1(\Lambda) \cap H_{\omega,*}^1(\Lambda)$ and $0 < \epsilon < \frac12$,*

$$\|I_N v\|_\omega \le c\|v\|_\omega^{\frac12}|v|_{1,\omega}^{\frac12} + c(\epsilon)N^\epsilon \|v\|_{1,\omega,*} \tag{2.19}$$

where $c(\epsilon)$ is a positive constant depending only on ϵ.

Proof. By (2.17),

$$\|I_N v\|_\omega^2 = \|I_N v\|_{\omega,N}^2 = \sum_{j=0}^N v^2(\sigma_j^N)\omega_j^N = A_N + B_N$$

where

$$A_N = \sum_{\sigma_j^N < 1} v^2(\sigma_j^N)\omega_j^N$$

$$B_N = \sum_{\sigma_j^N \geq 1} (\sigma_j^N)^{2\epsilon} \left(\omega^{-1}(\sigma_j^N)(\sigma_j^N)^{-1-2\epsilon}\omega_j^N\right) \left(\sigma_j^N v^2(\sigma_j^N)\omega(\sigma_j^N)\right).$$

By virtue of (2.1), and the fact that $\sum_{j=0}^{N} \omega_j^N = 1$,

$$A_N \leq \|v\|^2_{L^\infty(0,1)} \sum_{\sigma_j^N < 1} \omega_j^N \leq c\|e^{-\frac{x}{2}} v\|^2_{L^\infty} \leq c\|v\|_\omega |v|_{1,\omega}.$$

Thanks to (2.11), (2.15) and Lemma 2.9,

$$\begin{aligned} B_N &\leq cN^{2\epsilon}\|x^{\frac{1}{2}} e^{-\frac{x}{2}} v\|^2_{L^\infty} \sum_{\sigma_j^N \geq 1} \omega^{-1}(\sigma_j^N)(\sigma_j^N)^{-1-2\epsilon}\omega_j^N \\ &\leq cN^{2\epsilon}\|v\|^2_{1,\omega,*} \int_1^\infty \frac{1}{x^{1+2\epsilon}} dx \leq c(\epsilon) N^{2\epsilon}\|v\|^2_{1,\omega,*}. \end{aligned}$$

The previous estimates lead to the desired result.

We are now in position of deriving the main results of this section. Let $\beta \geq 0$ and $H^r_{\omega,\beta,*}(\Lambda) = H^r_{\omega,\beta}(\Lambda) \cap H^r_{\omega,*}(\Lambda)$, equipped with the following norm

$$\|v\|_{r,\omega,\beta,*} = (\|v\|^2_{r,\omega,\beta} + \|v\|^2_{r,\omega,*})^{\frac{1}{2}}.$$

Theorem 2.1. *For any $v \in H^1_{0,\omega}(\Lambda) \cap H^r_{\omega,\beta,*}(\Lambda)$, $r \geq 1$, $0 \leq \mu \leq r$* and $0 < \epsilon < \frac{1}{2}$,

$$\|I_N v - v\|_{\mu,\omega} \leq c(\epsilon) N^{\mu+\frac{1}{2}+\epsilon-\frac{r}{2}}\|v\|_{r,\omega,\beta,*} \tag{2.20}$$

where β is the largest integer for which $\beta < r+1$.

Proof. By virtue of (2.5),(2.6),(2.14) and (2.19),

$$\begin{aligned} \|P_N v - I_N v\|_{\mu,\omega} &\leq cN^\mu \|P_N v - I_N v\|_\omega = cN^\mu \|I_N(P_N v - v)\|_\omega \\ &\leq cN^\mu \|P_N v - v\|^{\frac{1}{2}}_\omega |P_N v - v|^{\frac{1}{2}}_{1,\omega} + c(\epsilon)N^{\mu+\epsilon}\|P_N v - v\|_{1,\omega,*} \\ &\leq c(\epsilon)N^{\mu+\frac{1}{2}+\epsilon-\frac{r}{2}}\|v\|_{r,\omega,\beta,*}. \end{aligned}$$

Furthermore by using (2.6) again, we find that

$$\begin{aligned} \|I_N v - v\|_{\mu,\omega} &\leq \|P_N v - v\|_{\mu,\omega} + \|P_N v - I_N v\|_{\mu,\omega} \\ &\leq cN^{\mu-\frac{r}{2}}\|v\|_{r,\omega,\beta} + c(\epsilon)N^{\mu+\frac{1}{2}+\epsilon-\frac{r}{2}}\|v\|_{r,\omega,\beta,*} \\ &\leq c(\epsilon)N^{\mu+\frac{1}{2}+\epsilon-\frac{r}{2}}\|v\|_{r,\omega,\beta,*}. \end{aligned}$$

Theorem 2.2. *For any $v \in H^1_{0,\omega}(\Lambda) \cap H^r_{\omega,\beta,*}(\Lambda)$, $r \geq 1, 0 \leq \mu \leq r$ and $0 < \epsilon < \frac{1}{2}$,*

$$\|I_N v - v\|_{\mu,\omega,*} \leq c(\epsilon) N^{\frac{\mu}{2}+\frac{1}{2}+\epsilon-\frac{r}{2}}\|v\|_{r,\omega,\beta,*} \tag{2.21}$$

where β is the largest integer for which $\beta < r+1$.

Proof. By virtue of (2.13), (2.14) and (2.19),

$$\begin{aligned} \|P_N v - I_N v\|_{\mu,\omega,*} &\leq cN^{\frac{\mu}{2}}\|I_N(P_N v - v)\|_\omega \\ &\leq cN^{\frac{\mu}{2}}\|P_N v - v\|^{\frac{1}{2}}_\omega |P_N v - v|^{\frac{1}{2}}_{1,\omega} + c(\epsilon)N^{\frac{\mu}{2}+\epsilon}\|P_N v - v\|_{1,\omega,*} \\ &\leq c(\epsilon)N^{\frac{\mu}{2}+\frac{1}{2}+\epsilon-\frac{r}{2}}\|v\|_{r,\omega,\beta,*}. \end{aligned}$$

Furthermore by using (2.14) again, we find that

$$\begin{aligned} \|I_N v - v\|_{\mu,\omega,*} &\leq \|P_N v - v\|_{\mu,\omega,*} + \|P_N v - I_N v\|_{\mu,\omega,*} \\ &\leq c(\epsilon)N^{\frac{\mu}{2}-\frac{r}{2}}\|v\|_{r,\omega,*} + c(\epsilon)N^{\frac{\mu}{2}+\frac{1}{2}+\epsilon-\frac{r}{2}}\|v\|_{r,\omega,\beta,*} \\ &\leq c(\epsilon)N^{\frac{\mu}{2}+\frac{1}{2}+\epsilon-\frac{r}{2}}\|v\|_{r,\omega,\beta,*}. \end{aligned}$$

Remark 2.2. Let $\omega_\alpha(x) = e^{-(1-\alpha)x}$, $\alpha \geq 0$. The space $H^r_{\omega_\alpha}(\Lambda)$ is definded in the same way as for $H^r_\omega(\Lambda)$. Maday, Bernaud-Thomas and Vendeven [5] gave another approximation result, i.e.,

$$\|I_N v - v\|_{\mu,\omega} \leq c(\epsilon)N^{\mu+\frac{1}{2}-\frac{r}{2}}\|v\|_{r,\omega_\alpha}, \quad \alpha > 0. \tag{2.22}$$

This result is different from (2.20) and (2.21). Since $\epsilon > 0$, the power of N in (2.20) is slightly biger than the power of N in (2.22). But since $\alpha > 0$, the norm $\|v\|_{r,\omega,\beta,*}$ in (2.20) is smaller than the norm at the right hand of (2.22). On the other hand, for $\mu > 0$, the power of N in (2.21) is also smaller than that of (2.22). Generally speaking, (2.20) and (2.21) improve the result (2.22). Indeed the main motivation of deriving (2.20) and (2.21) is to use it for the convenience of numerical analysis of Laguerre pseudospectral method for nonlinear partial differential equations (see Section 3 of this paper).

Theorem 2.3. *For any $v \in H^1_{0,\omega}(\Lambda) \cap H^r_{\omega,\beta,*}(\Lambda)$, $r \geq 1$ and $0 < \epsilon < \frac{1}{2}$,*

$$\|(1+x)^{\frac{1}{2}}e^{-\frac{x}{2}}(I_N v - v)\|_{L^\infty} \leq c(\epsilon)N^{1+\epsilon-\frac{r}{2}}\|v\|_{r,\omega,\beta,*} \tag{2.23}$$

where β is the largest integer for which $\beta < r+1$.

Proof. We know from (2.11) and (2.21) that

$$\begin{aligned}\|x^{\frac{1}{2}}e^{-\frac{x}{2}}(I_N v - v)\|_{L^\infty} &\leq 2\|I_N v - v\|_{1,\omega,*}\\ &\leq c(\epsilon)N^{1+\epsilon-\frac{r}{2}}\|v\|_{r,\omega,\beta,*}.\end{aligned} \tag{2.24}$$

On the other hand, (2.1) and (2.20) imply that

$$\begin{aligned}\|e^{-\frac{x}{2}}(I_N v - v)\|_{L^\infty} &\leq \sqrt{2}\|I_N v - v\|^{\frac{1}{2}}_{\omega}\|I_N v - v\|^{\frac{1}{2}}_{1,\omega}\\ &\leq c(\epsilon)N^{1+\epsilon-\frac{r}{2}}\|v\|_{r,\omega,\beta,*}.\end{aligned} \tag{2.25}$$

The combination of (2.24) with (2.25) leads to the desired result.

Remark 2.3. Maday, Bernaud-Thomas and Vendeven [5] proved that

$$\|e^{-\frac{x}{2}}(I_N v - v)\|_{L^\infty} \leq c(\alpha)N^{\frac{5}{4}-\frac{r}{2}}\|v\|_{r,\omega_\alpha}, \qquad \alpha > 0. \tag{2.26}$$

Clearly (2.25) improves (2.26).

Theorem 2.4. *For any $v \in H^1_{0,\omega}(\Lambda) \cap H^r_{\omega,\beta,*}(\Lambda)$, $r \geq 1$ and $0 < \epsilon < \frac{1}{2}$,*

$$|(v,\phi)_\omega - (v,\phi)_{\omega,N}| \leq c(\epsilon)N^{\frac{1}{2}+\epsilon-\frac{r}{2}}\|v\|_{r,\omega,\beta,*}\|\phi\|_\omega \tag{2.27}$$

where β is the largest integer for which $\beta < r+1$.

Proof. By (2.17) and (2.21),

$$\begin{aligned}|(v,\phi)_\omega - (v,\phi)_{\omega,N}| &= |(v,\phi)_\omega - (I_N v,\phi)_{\omega,N}|\\ &= |(v,\phi)_\omega - (I_N v,\phi)_\omega| \leq \|I_N v - v\|_\omega\|\phi\|_\omega\\ &\leq c(\epsilon)N^{\frac{1}{2}+\epsilon-\frac{r}{2}}\|v\|_{r,\omega,\beta,*}\|\phi\|_\omega.\end{aligned}$$

3. Laguerre Pseudospectral Method

This section is devoted to the Laguerre pseudospectral method. We take the BBM equation as an example. It is of the form

$$\begin{cases}\partial_t V + \dfrac{1}{2}\partial_y V^2 - \delta\partial_t\partial_y^2 V = F, & 0 < y < \infty, 0 < t \leq T,\\ V(0,t) = g(t), & 0 < t \leq T,\\ \lim\limits_{y\to\infty} V(y,t) = \lim\limits_{y\to\infty}\partial_y V(y,t) = 0, & 0 < t \leq T,\\ V(y,0) = V_0(y), & 0 \leq y < \infty\end{cases} \tag{3.1}$$

where $\delta > 0$, $F(y,t)$, $g(t)$ and $V_0(y)$ are given functions, $g(0) = V_0(0)$, and $V_0(y)$, $\partial_y V_0(y) \to 0$ as $y \to \infty$. Without loss of generality, assume $g(t) \equiv 0$. As explained in Guo and Shen [10], it is not suitable to approximate (3.1) directly. So we make the following variable transformation

$$U(x,t) = e^{\frac{x}{2}} V(\frac{\sqrt{\delta}}{2}x,t), \quad U_0(x) = e^{\frac{x}{2}} V_0(\frac{\sqrt{\delta}}{2}x), \quad f(x,t) = \frac{1}{4} e^{\frac{x}{2}} F(\frac{\sqrt{\delta}}{2}x,t). \tag{3.2}$$

Then (3.1) becomes

$$\begin{cases} \dfrac{1}{4\sqrt{\delta}} e^{\frac{x}{2}} \partial_x (e^{-x} U^2) + \partial_t \partial_x U - \partial_t \partial_x^2 U = f, & x \in \Lambda, 0 < t \le T, \\ U(0,t) = 0, & 0 < t \le T, \\ \lim\limits_{x\to\infty} e^{-\frac{x}{2}} U(x,t) = \lim\limits_{x\to\infty} e^{-\frac{x}{2}} \partial_x U(x,t) = 0, & 0 < t \le T, \\ U(x,0) = U_0(x), & x \in \bar{\Lambda}. \end{cases} \tag{3.3}$$

The Laguerre pseudospectral scheme for (3.3) is to find $u_N(x,t) \in \mathcal{P}_N^0$ for all $0 \le t \le T$, such that

$$\begin{cases} \dfrac{1}{4\sqrt{\delta}} e^{\frac{x}{2}} \partial_x (e^{-x} u_N^2) + \partial_t \partial_x u_N - \partial_t \partial_x^2 u_N = f, & x \in \Lambda_N, 0 < t \le T, \\ u_N(0) = u_{N,0} = I_N U_0, & x \in \bar{\Lambda}. \end{cases} \tag{3.4}$$

Let

$$a(u,v) = (\partial_x u, \partial_x v)_\omega, \qquad \forall u,v \in H_\omega^1(\Lambda),$$

$$B_N(u,w,v) = \left(e^{\frac{x}{2}} \partial_x (e^{-x} uw), v\right)_{\omega,N}, \quad \forall u,w \in C^1(\Lambda), v \in C(\Lambda).$$

By taking the discrete inner product of (3.4) with $\phi \in \mathcal{P}_N^0$, using (2.17) and integrating by parts, we find that (3.4) is equivalent to the following form

$$\begin{cases} a(\partial_t u_N, \phi) + \frac{1}{4\sqrt{\delta}} B_N(u_N, u_N, \phi) = (f, \phi)_{\omega,N}, & \forall \phi \in P_N^0, 0 < t \le T, \\ u_N(0) = u_{N,0}. \end{cases} \tag{3.5}$$

We now analyze the stability of (3.5). Assume that f and $u_{N,0}$ have the errors $\tilde{f}$ and $\tilde{u}_{N,0}$, respectively. They induce the error of u_N, denoted by $\tilde{u}_N$. By (3.5), the error $\tilde{u}_N$ satisfies the equation

$$\begin{cases} a(\partial_t \tilde{u}_N, \phi) + \frac{1}{4\sqrt{\delta}} B_N(\tilde{u}_N, \tilde{u}_N, \phi) + \frac{1}{2\sqrt{\delta}} B_N(\tilde{u}_N, u_N, \phi) = (\tilde{f}, \phi)_{\omega,N}, & \forall \phi \in \mathcal{P}_N^0, 0 < t \le T, \\ \tilde{u}_N(0) = \tilde{u}_{N,0}. \end{cases} \tag{3.6}$$

By taking $\phi = 2\tilde{u}_N$ in (3.6), it follows from (2.2) and (2.16) that

$$\frac{d}{dt} |\tilde{u}_N(t)|_{1,\omega}^2 + \frac{1}{2\sqrt{\delta}} B_N(\tilde{u}_N, \tilde{u}_N, \tilde{u}_N) + \frac{1}{\sqrt{\delta}} B_N(\tilde{u}_N, u_N, \tilde{u}_N) \le \|\tilde{u}_N\|_\omega^2 + \|\tilde{f}\|_{\omega,N}^2 \le 4|\tilde{u}_N|_{1,\omega}^2 + \|\tilde{f}\|_{\omega,N}^2. \tag{3.7}$$

Lemma 3.1. *For any $u, v, w \in H_{0,\omega}^1(\Lambda) \cap C^1(\bar{\Lambda})$,*

$$\begin{aligned} |B_N(u,w,v)| \quad & \le \sqrt{2} \|v\|_{\omega,N} (\|w\|_\omega^{\frac{1}{2}} |w|_{1,\omega}^{\frac{1}{2}} |u|_{1,\omega,N} + \|u\|_\omega^{\frac{1}{2}} |u|_{1,\omega}^{\frac{1}{2}} |w|_{1,\omega,N}) \\ & + \sqrt{2} \|v\|_\omega^{\frac{1}{2}} |v|_{1,\omega}^{\frac{1}{2}} \|u\|_{\omega,N} \|w\|_{\omega,N}. \end{aligned} \tag{3.8}$$

In particular, for any $u, v, w \in \mathcal{P}_N^0$,

$$|B_N(u,w,v)| \le 16 |u|_{1,\omega} |v|_{1,\omega} |w|_{1,\omega}. \tag{3.9}$$

Proof. We have

$$B_N(u,w,v) = \sum_{i=1}^{3} B_N^{(i)}(u,w,v)$$

where

$$B_N^{(1)}(u,w,v) = (e^{-\frac{x}{2}}\partial_x u, w, v)_{\omega,N},$$

$$B_N^{(2)}(u,w,v) = B_N^{(1)}(w,u,v)$$

$$B_N^{(3)}(u,w,v) = -(e^{-\frac{x}{2}}uw, v)_{\omega,N}.$$

By (2.1),

$$|B_N^{(1)}(u,w,v)| \le \sqrt{2}\|w\|_\omega^{\frac{1}{2}}|w|_{1,\omega}^{\frac{1}{2}}|u|_{1,\omega,N}\|v\|_{\omega,N},$$

$$|B_N^{(2)}(u,w,v)| \le \sqrt{2}\|u\|_\omega^{\frac{1}{2}}|u|_{1,\omega}^{\frac{1}{2}}|w|_{1,\omega,N}\|v\|_{\omega,N},$$

$$|B_N^{(3)}(u,w,v)| \le \sqrt{2}\|v\|_\omega^{\frac{1}{2}}|v|_{1,\omega}^{\frac{1}{2}}\|u\|_{\omega,N}\|w\|_{\omega,N}.$$

The combination of the above statements leads to (3.8). If $u,v,w \in \mathcal{P}_N^0$, then (3.9) follows from (2.2) , (2.17) and (3.8) immediately.

According to (3.9), we claim that

$$|B_N(\tilde{u}_N,\tilde{u}_N,\tilde{u}_N)| \le 16|\tilde{u}_N|_{1,\omega}^3, \tag{3.10}$$

$$|B_N(\tilde{u}_N,u_N,\tilde{u}_N)| \le 16|u_N|_{1,\omega}|\tilde{u}_N|_{1,\omega}^2. \tag{3.11}$$

By inserting (3.10) and (3.11) into (3.7), we get that

$$\frac{d}{dt}|\tilde{u}_N|_{1,\omega}^2 \le 4(\frac{4}{\sqrt{\delta}}\|u_N\|_{1,\omega}+1)|\tilde{u}_N|_{1,\omega}^2 + \frac{8}{\sqrt{\delta}}|\tilde{u}_N|_{1,\omega}^3 + \|\tilde{f}\|_{\omega,N}^2. \tag{3.12}$$

Let

$$c^* \equiv c^*(u_N,\delta,T) = 4(\frac{4}{\sqrt{\delta}}\|u_N\|_{L^\infty(0,T;H_\omega^1(\Lambda))}+1),$$

$$\rho(\tilde{u}_{N,0},\tilde{f},t) = |\tilde{u}_{N,0}|_{1,\omega}^2 + \int_0^t \|\tilde{f}(\eta)\|_{\omega,N}^2 d\eta.$$

Then integrating (3.12) for time t yields that

$$|\tilde{u}_N(t)|_{1,\omega}^2 \le \int_0^t \left(c^*|\tilde{u}_N(\eta)|_{1,\omega}^2 + \frac{8}{\sqrt{\delta}}|\tilde{u}_N(\eta)|_{1,\omega}^3\right)d\eta + \rho(\tilde{u}_{N,0},\tilde{f},t). \tag{3.13}$$

Lemma 3.2. (see Guo [15]). *Assume that*

(i) $b_1, b_2, d \ge 0$,

(ii) $Z(t)$ *is a non-negative function of* t,

(iii) *for certain* $t_1 > 0$, $d \le e^{-(b_1+b_2)t_1}$,

(iv) *for all* $t \le t_1$,

$$Z(t) \le \int_0^t (b_1 + b_2 Z(\eta))d\eta + d.$$

Then for all $t \le t_1$,

$$Z(t) \le de^{(b_1+b_2)t}.$$

Applying Lemma 3.2 to (3.13), we obtain the following result.

Theorem 3.1. *If for certain* $t_1 > 0$,

$$\rho(\tilde{u}_{N,0},\tilde{f},t_1) \le e^{-(c^*+\frac{8}{\sqrt{\delta}})t_1},$$

then for all $t \le t_1$,

$$|\tilde{u}_N(t)|_{1,\omega}^2 \le \rho(\tilde{u}_{N,0},\tilde{f},t)e^{(c^*+\frac{8}{\sqrt{\delta}})t}.$$

Remark 3.1. Theorem 3.1 indicates that if the average data error $\rho(\tilde{u}_{N,0},\tilde{f},t)$ does not exceed certain critical value, then the error of numerical solution $\tilde{u}_N(t)$ is controlled. Obviously, (3.4) is not stable in the sense of Courant, Friedrichs and Lewy [16]. But it is stable in the sense of Guo [17].

We next deal with the convergence of scheme (3.4) . Let U be the solution of (3.3) and $U_N = P_N^{1,0}U$. By (3.3) and integration by parts, we have that

$$\begin{cases} a(\partial_t U_N,\phi)+\frac{1}{4\sqrt{\delta}}B_N(U_N,U_N,\phi)+G(t,\phi)=(f,\phi)_{\omega,N}, & \forall\phi\in\mathcal{P}_N^0, 0<t\le T,\\ U_N(0)=P_N^{1,0}U_0 \end{cases} \tag{3.14}$$

where $G(t,\phi)=G_1(t,\phi)+G_2(t,\phi)$ with

$$G_1(t,\phi)=\frac{1}{4\sqrt{\delta}}\left(e^{\frac{x}{2}}\partial_x(e^{-x}U^2),\phi\right)_\omega-\frac{1}{4\sqrt{\delta}}\left(e^{\frac{x}{2}}\partial_x(e^{-x}U_N^2),\phi\right)_{\omega,N},$$

$$G_2(t,\phi)=(f,\phi)_{\omega,N}-(f,\phi)_\omega.$$

Further let $\tilde{U}_N=u_N-U_N$. Subtracting (3.14) from (3.4), we obtain that

$$\begin{cases} a(\partial_t\tilde{U}_N,\phi)+\frac{1}{4\sqrt{\delta}}B_N(\tilde{U}_N,\tilde{U}_N,\phi)+\frac{1}{2\sqrt{\delta}}B_N(\tilde{U}_N,U_N,\phi)=G(t,\phi), & \forall\phi\in\mathcal{P}_N^0, 0<t\le T,\\ \tilde{U}_N(0)=I_NU_0-P_N^{1,0}U_0. \end{cases} \tag{3.15}$$

Comparing (3.15) with (3.6), we can derive a result similar to that of Theorem 3.1. But $u_N,\tilde{u}_N,\tilde{u}_{N,0},\tilde{f}$ and $c^*(u_N,\delta,T)$ are now replaced by $U_N,\tilde{U}_N,\tilde{U}_N(0)$, $G(t,\tilde{u}_N)$ and $c^*(U_N,\delta,T)$, respectively. Thus it remains to estimate $|G(t,\tilde{U}_N)|,|\tilde{U}_N(0)|_{1,\omega}$ and $\|U_N\|_{L^\infty(0,T;H^1_\omega(\Lambda))}$. Firstly we have

$$4\sqrt{\delta}G_1(t,\tilde{U}_N)=\sum_{i=1}^5 A_i(t,\tilde{U}_N)$$

where

$$A_1(t,\tilde{U}_N)=(e^{-\frac{x}{2}}U^2,\tilde{U}_N)_{\omega,N}-(e^{-\frac{x}{2}}U^2,\tilde{U}_N)_\omega,$$

$$A_2(t,\tilde{U}_N)=2(e^{-\frac{x}{2}}U\partial_xU,\tilde{U}_N)_\omega-2(e^{-\frac{x}{2}}U\partial_xU,\tilde{U}_N)_{\omega,N},$$

$$A_3(t,\tilde{U}_N)=(e^{-\frac{x}{2}}(U_N^2-U^2),\tilde{U}_N)_{\omega,N},$$

$$A_4(t,\tilde{U}_N)=2(e^{-\frac{x}{2}}\partial_xU(U-U_N),\tilde{U}_N)_{\omega,N},$$

$$A_5(t,\tilde{U}_N)=2(e^{-\frac{x}{2}}U_N(\partial_xU-\partial_xU_N),\tilde{U}_N)_{\omega,N}.$$

Let $r\ge1$ and β be the largest integer for which $\beta<r+1$. By (2.27),

$$|A_1(t,\tilde{U}_N)|\le c(\epsilon)N^{\frac12+\epsilon-\frac{r}{2}}\|e^{-\frac{x}{2}}U^2\|_{r,\omega,\beta,*}\|\tilde{U}_N\|_\omega. \tag{3.16}$$

It can be checked that

$$\partial_x^k(e^{-\frac{x}{2}}U^2)=e^{-\frac{x}{2}}\sum_{j=0}^k\sum_{i=0}^j C_k^jC_j^i(-\frac12)^{k-j}\partial_x^iU\partial_x^{j-i}U.$$

By (2.3) ,

$$\|e^{-\frac{x}{2}}U^2\|_{r,\omega,\beta,*}\le c\|U\|_{[\frac{r}{2}]+1,\omega}\|U\|_{r,\omega,\beta,*}\le c(\epsilon)\|U\|^2_{r,\omega,\beta,*}.$$

So we get from (2.2) and (3.16) that

$$|A_1(t,\tilde{U}_N)|\le c(\epsilon)N^{\frac12+\epsilon-\frac{r}{2}}\|U\|^2_{r,\omega,\beta,*}\|\tilde{U}_N\|_\omega\le c(\epsilon)N^{1+2\epsilon-r}\|U\|^4_{r,\omega,\beta,*}+|\tilde{U}_N|^2_{1,\omega}.$$

Similarly

$$|A_2(t,\tilde{U}_N)|\le c(\epsilon)N^{\frac12+2\epsilon-r}\|U\|^4_{r+1,\omega,\beta,*}+|\tilde{U}_N|^2_{1,\omega}.$$

Let β_1 be the largest integer such that $\beta_1 < r$. According to (2.1), (2.2) and (2.7),

$$\|e^{-\frac{x}{2}}U_N\|_{L^\infty} \le c\|U_N\|_{1,\omega} \le c(\|U\|_{1,\omega} + N^{\frac{1}{2}-\frac{r}{2}}\|U\|_{r,\omega,\beta_1}) \le c\|U\|_{r,\omega,\beta_1}. \tag{3.17}$$

By virtue of (2.7), (2.17) and (2.20),

$$\begin{aligned}\|U-U_N\|_{\omega,N} = \|I_NU-U_N\|_{\omega,N} &= \|I_NU-U_N\|_\omega \le \|I_NU-U\|_\omega + \|U-U_N\|_\omega \\ &\le c(\epsilon)N^{\frac{1}{2}+\epsilon-\frac{1}{2}}\|U\|_{r,\omega,\beta,*}.\end{aligned} \tag{3.18}$$

Therefore by (2.1), (2.2), (3.17) and (3.18),

$$\begin{aligned}|A_3(t,\tilde{U}_N)| &\le c\|e^{-\frac{x}{2}}(U+U_N)\|_{L^\infty}\|U-U_N\|_{\omega,N}\|\tilde{U}_N\|_\omega \le c(\epsilon)N^{\frac{1}{2}+\epsilon-\frac{r}{2}}\|U\|^2_{r,\omega,\beta,*}\|\tilde{U}_N\|_\omega \\ &\le c(\epsilon)N^{1+2\epsilon-r}\|U\|^4_{r,\omega,\beta,*} + |\tilde{U}_N|^2_{1,\omega}.\end{aligned} \tag{3.19}$$

Furthermore (2.2), (2.3) and (3.18) imply that

$$\begin{aligned}|A_4(t,\tilde{U}_N)| &\le c\|e^{-\frac{x}{2}}\partial_xU\|_{L^\infty}\|U-U_N\|_{\omega,N}\|\tilde{U}_N\|_\omega \le c(\epsilon)N^{\frac{1}{2}+\epsilon-\frac{r}{2}}\|U\|^2_{r+1,\omega,\beta,*}\|\tilde{U}_N\|_\omega \\ &\le c(\epsilon)N^{1+2\epsilon-r}\|U\|^4_{r+1,\omega,\beta,*} + |\tilde{U}_N|^2_{1,\omega}.\end{aligned} \tag{3.20}$$

We have from (2.7), (2.17) and (2.20) that

$$\begin{aligned}\|\partial_xU-\partial_xU_N\|_{\omega,N} &= \|I_N\partial_xU-\partial_xU_N\|_\omega \le \|I_N\partial_xU-\partial_xU\|_\omega + \|\partial_xU-\partial_xU_N\|_\omega \\ &\le c(\epsilon)N^{\frac{1}{2}+\epsilon-\frac{r}{2}}\|U\|_{r+1,\omega,\beta,*}.\end{aligned}$$

Thus by (2.2) and (3.17),

$$\begin{aligned}|A_5(t,\tilde{U}_N)| &\le c\|e^{-\frac{x}{2}}U_N\|_{L^\infty}\|\partial_xU-\partial_xU_N\|_{\omega,N}\|\tilde{U}_N\|_\omega \le c(\epsilon)N^{\frac{1}{2}+\epsilon-\frac{r}{2}}\|U\|^2_{r+1,\omega,\beta,*}\|\tilde{U}_N\|_\omega \\ &\le c(\epsilon)N^{1+2\epsilon-r}\|U\|^4_{r+1,\omega,\beta,*} + |\tilde{U}_N|^2_{1,\omega}.\end{aligned}$$

In addition, we know from (2.2), (2,7), (2.20) and (2.27) that

$$|G_2(t,\tilde{U}_N)| \le c(\epsilon)N^{1+2\epsilon-r}\|f\|^2_{r,\omega,\beta,*} + |\tilde{U}_N|^2_{1,\omega},$$

$$\|\tilde{U}_N(0)\|_\omega \le c(\epsilon)N^{\frac{1}{2}+\epsilon-\frac{r}{2}}\|U_0\|_{r,\omega,\beta,*}$$

and

$$\|U_N\|_{L^\infty(0,T;H^1_\omega(\Lambda))} \le c\|U\|_{L^\infty(0,T;H^1_{\omega,\beta}(\Lambda))}.$$

By the above estimates and an argument as in the proof of the previous theorem, we reach the following conclusion.

Theorem 3.2. *Let* $0<\epsilon<\frac{1}{2}$, $r\ge 1$ *and* β *be the largest integer for which* $\beta < r+1$. *If* $U \in L^4\left(0,T;H^{r+2+2\epsilon}_{\omega,\beta,*}(\Lambda)\right)\cap L^\infty\left(0,T;H^1_{\omega,\beta}(\Lambda)\right)$, $U_0 \in H^{r+1+2\epsilon}_{\omega,\beta}(\Lambda)$ *and* $f\in L^2(0,T;H^{r+1+2\epsilon}_{\omega,\beta}(\Lambda))$, *then for all* $0\le t\le T$,

$$\|U-u_N\|_{L^\infty(0,T;H^1_\omega(\Lambda))} \le d^*N^{-\frac{r}{2}}$$

where d^* *is a positive constant depending only on* ϵ *and the norms of* U, U_0 *and* f *in the mentioned spaces.*

Remark 3.2. Let $v_N(y,t) = e^{-\frac{y}{\sqrt{\delta}}}u_N(\frac{2}{\sqrt{\delta}}y,t)$. By (3.2),

$$\|V-v_N\|_{L^2(\Lambda)} = \|U-u_N\|_\omega, \qquad \|V-v_N\|_{H^1(\Lambda)} \le \frac{c}{\sqrt{\delta}}\|U-u_N\|_{1,\omega}.$$

Thus we also have from Theorem 3.2 that

$$\|V-v_N\|_{L^\infty(0,T;H^1(\Lambda))} \le d^*N^{-\frac{r}{2}}.$$

Remark 3.3. Since we introduce the space $H^r_{\omega,\beta,*}(\Lambda)$, we can derive the estimates for $|A_i(t,\tilde{U}_N)|$, $1\le i\le 5$. Thus it not only improves the approximation result (2.22), but also simplifies the error estimation for the nonlinear terms. Indeed, we can use (2.22) to obtain the error estimate for the Laguerre pseudospectral scheme of the corresponding linearized problem as follows

$$\frac{\partial U}{\partial t} + b\frac{\partial U}{\partial x} - \frac{\partial^3 U}{\partial t \partial^2 x} = f.$$

In this case, we may get that for $\alpha > 0$,

$$\|U - u_N\|_{L^\infty(0,T;H^1_\omega(\Lambda))} \le d^* N^{-\frac{r}{2}}$$

where d^* is a positive constant depending only on the norms $\|U\|_{L^2(0,T;H^{r+2}_{\omega_\alpha}(\Lambda))}, \|U_0\|_{r+1,\omega_\alpha}$ and $\|f\|_{L^2(0,T;H^{r+1}_{\omega_\alpha}(\Lambda))}$.

In the end of this section, we present some numerical results. Take the following test function

$$U(x,t) = \frac{\sin kxt}{(1+x)^h}$$

with $k = 0.2, h = 5.0$. We use scheme (3.4) to solve (3.3). In actual computation, the standard fourth order Runge-Kutta method with the mesh size τ is used for discretization in time t. Let $E_N(t) = \|U(t) - u_N(t)\|_{\omega,N}$ be the errors of numerical solution u_N. The errors at $t = 1$ are listed in Table 1, which shows the high accuracy and the convergence of the scheme (3.4). Moreover the errors $E_N(t)$ with $N = 64,\ \tau = 0.001$ and various values of t are listed in Table 2, which show the stability of calculation.

Table 1. The errors $E_N(1)$.

τ	$N=16$	$N=32$	$N=64$	$N=80$
0.01	4.378E-04	1.854E-05	9.484E-07	3.072E-07
0.001	4.379E-04	1.856E-05	7.474E-07	7.431E-08
0.0001	4.379E-04	1.854E-05	7.289E-07	5.624E-08

Table 2. The errors $E_N(t)$.

t	$E^{(1)}(t)$
1.0	7.474E-07
2.0	1.522E-06
3.0	2.308E-06
4.0	3.087E-06
5.0	3.838E-06

4. Appendix

In order to prove Lemma 2.9, we need some preparations. Let Λ be the same as in Section 2 and $\hat{\omega}(x)$ be certain weight function in the usual sense. As pointed out in Upensky [18], there exist the Gauss interpolation nodes $\hat{\sigma}_j^N$ and the corresponding Christoffel numbers $\hat{\omega}_j^N, 1 \le j \le N$, such that

$$c_m \equiv \int_\Lambda x^m \hat{\omega}(x)dx = \sum_{j=1}^N \hat{\omega}_j^N (\hat{\sigma}_j^N)^m, \qquad 0 \le m \le 2N-1, \tag{A.1}$$

provided that all c_m exist. Next for any $f \in C(\Lambda)$, let

$$S_N(f, \hat{\omega}, \Lambda) = \sum_{j=1}^N \hat{\omega}_j^N f(\hat{\sigma}_j^N). \tag{A.2}$$

Further, if the integral

$$I(f,\hat{\omega},\Lambda) \equiv \int_{\Lambda} f(x)\hat{\omega}(x)dx$$

exists, then we say that $f(x)$ is $\hat{\omega}$-integrable.

Lemma A.1. *(see Section 3 of Uspensky [18]). If $f(x)$ is $\hat{\omega}$-integrable on a finite interval [0,A] and $f(x)=0$ for $x>A$. Then $S_N(f,\hat{\omega},\Lambda)\to I(f,\hat{\omega},\Lambda)$ as $N\to\infty$.*

Lemma A.2. *Suppose that $f(x)$ and $F(x)$ are $\hat{\omega}$-integrable, and for all $x\in\Lambda$, $|f(x)|\le F(x)$. If*

$$S_N(F,\hat{\omega},\Lambda)\to I(F,\hat{\omega},\Lambda), \qquad \text{as } N\to\infty, \tag{A.3}$$

then

$$S_N(f,\hat{\omega},\Lambda)\to I(f,\hat{\omega},\Lambda), \qquad \text{as } N\to\infty.$$

Proof. By the $\hat{\omega}$-integrability of $F(x)$, for any $\epsilon>0$, there exists a positive number A_ϵ such that

$$\int_{A_\epsilon}^{\infty} F(x)\hat{\omega}(x)dx<\epsilon.$$

Let

$$\tilde{f}(x)=\begin{cases} f(x), & 0\le x\le A_\epsilon,\\ 0, & x>A_\epsilon\end{cases}$$

and

$$\tilde{F}(x)=\begin{cases} F(x), & 0\le x\le A_\epsilon,\\ 0, & x>A_\epsilon.\end{cases}$$

According to Lemma A.1, when $N\to\infty$,

$$S_N(\tilde{f},\hat{\omega},\Lambda)\to I(\tilde{f},\hat{\omega},\Lambda), \quad S_N(\tilde{F},\hat{\omega},\Lambda)\to I(\tilde{F},\hat{\omega},\Lambda).$$

Thus there exists a positive integer N_1 depending only on f,F and ϵ, such that for all $N\ge N_1$,

$$D^{(1)}\equiv\left|\sum_{\hat{\sigma}_j^N\le A_\epsilon}\hat{\omega}_j^N f(\hat{\sigma}_j^N)-\int_0^{A_\epsilon} f(x)\hat{\omega}(x)dx\right|<\epsilon,$$

$$D_N^{(2)}\equiv\left|\sum_{\hat{\sigma}_j^N\le A_\epsilon}\hat{\omega}_j^N F(\hat{\sigma}_j^N)-\int_0^{A_\epsilon} F(x)\hat{\omega}(x)dx\right|<\epsilon.$$

Moreover by (A.3), there exists $N_2>0$ depending only on F and ϵ, such that for all $N\ge N_2$,

$$|S_N(F,\hat{\omega},\Lambda)-I(F,\hat{\omega},\Lambda)|<\epsilon.$$

The combination of the above statements leads to that for all $N\ge\max(N_1,N_2)$,

$$\begin{aligned}|S_N(f,\hat{\omega},\Lambda)-I(f,\hat{\omega},\Lambda)| &\le |\sum_{\hat{\sigma}_j^N>A_\epsilon}\hat{\omega}_j^N f(\hat{\sigma}_j^N)|+|\int_{A_\epsilon}^{\infty} f(x)\hat{\omega}(x)dx|+D_N^{(1)}\\ &\le \sum_{\hat{\sigma}_j^N>A_\epsilon}\hat{\omega}_j^N F(\hat{\sigma}_j^N)+\int_{A_\epsilon}^{\infty} F(x)\hat{\omega}(x)dx+D_N^{(1)}\\ &\le |S_N(F,\hat{\omega},\Lambda)-I(F,\hat{\omega},\Lambda)|+2\int_{A_\epsilon}^{\infty} F(x)\hat{\omega}(x)dx+D_N^{(1)}+D_N^{(2)}\\ &\le 5\epsilon.\end{aligned}$$

The proof is complete.

Lemma A.3. *Let*

$$F(x)=\frac{e^x+e^{-x}}{1+x^{2+\rho}},\qquad 0<\rho<1. \tag{A.4}$$

If for all $m=0,1,\cdots$, $c_m\le c(m+1)!$ *then*

$$S_N(F,\hat\omega,\Lambda)\to I(F,\hat\omega,\Lambda),\qquad \text{as } N\to\infty. \tag{A.5}$$

Proof. Let

$$\gamma_m^N=\sum_{j=1}^N\hat\omega_j^N\frac{(\hat\sigma_j^N)^{2m}}{1+(\hat\sigma_j^N)^{2+\rho}}.$$

By (A.2) and (A.4),

$$\begin{aligned} S_N(F,\hat\omega,\Lambda) &=2\sum_{j=1}^N\hat\omega_j^N\left(\sum_{m=0}^\infty\frac{1}{(2m)!}\,\frac{(\hat\sigma_j^N)^{2m}}{1+(\hat\sigma_j^N)^{2+\rho}}\right)\\ &=2\sum_{m=0}^\infty\frac{\gamma_m^N}{(2m)!}. \end{aligned} \tag{A.6}$$

On the other hand, by the formula (4) of Uspensky [18],

$$\sum_{j=1}^N\hat\omega_j^N(\hat\sigma_j^N)^m\le c_m.$$

So using (A.1), (A.5) and the Hölder inequality yields that

$$\begin{aligned} \gamma_m^N &\le\sum_{j=1}^N\hat\omega_j^N(\hat\sigma_j^N)^{2m-2-\rho}=\sum_{j=1}^N\left((\hat\omega_j^N)^\rho(\hat\sigma_j^N)^{(2m-3)\rho}\right)\left((\hat\omega_j^N)^{1-\rho}(\hat\sigma_j^N)^{(2m-2)(1-\rho)}\right)\\ &\le\left(\sum_{j=1}^N(\hat\omega_j^N)(\hat\sigma_j^N)^{(2m-3)}\right)^\rho\left(\sum_{j=1}^N\hat\omega_j^N(\hat\sigma_j^N)^{2m-2}\right)^{1-\rho}\\ &\le c_{2m-3}^\rho c_{2m-2}^{1-\rho}\le c(2m-1)^{-\rho}(2m-1)!. \end{aligned}$$

Consequently the series $\displaystyle\sum_{m=0}^\infty\frac{\gamma_m^N}{(2m)!}$ converges uniformly for all N. Moreover it is shown in Section 4 of Uspensky [18] that

$$\gamma_m^N\to\int_\Lambda\frac{x^{2m}}{1+x^{2+\rho}}\hat\omega(x)dx,\qquad\text{as } N\to\infty.$$

Therefore (A.6) implies that

$$\begin{aligned} \lim_{N\to\infty}\ S_N(F,\hat\omega,\Lambda) &=2\sum_{m=0}^\infty\frac{1}{(2m)!}\int_\Lambda\frac{x^{2m}}{1+x^{2+\rho}}\hat\omega(x)dx\\ &=I_N(F,\hat\omega,\Lambda). \end{aligned}$$

Remark A.1. If we take $\hat\omega(x)=xe^{-x}$, then by (A.1), $c_m=(m+1)!$. So condition (A.5) is fulfilled.

Proof of Lemma 2.9. As we know, an orthogonal system of polynomials is uniquely determined by the related weight function apart from some constants. Since $\partial_x\mathcal{L}_{N+1}(\sigma_j^N)=0$, we know from (2.4) and the definition of Gauss interpolation nodes that the Laguerre-Gauss-Radau interpolation nodes $\sigma_j^N(1\le j\le N)$ are exactly the same as the Gauss interpolation nodes $\hat\sigma_j^N(1\le j\le N)$ associated with the weight function $\hat\omega(x)=xe^{-x}$. Let $F(x)$ be the same as in (A.4). By Remark A.1 and Lemma A.3,

$$S_N(F,\hat\omega,\Lambda)\to I(F,\hat\omega,\Lambda),\qquad\text{as } N\to\infty. \tag{A.7}$$

Next let $\hat{I}_{N-1} : C(\Lambda) \to \mathcal{P}_{N-1}$ be the Lagrange interpolation with respect to the interpolation nodes $\sigma_j^N (1 \le j \le N)$. We first assume that $f(x)$ is differentiable at $x = 0$ and $f(0) = 0$. Then it can be verified that

$$I_N f = x \hat{I}_{N-1}(\frac{f}{x}). \qquad (A.8)$$

Since $f(x)$ is ω-integrable on Λ, and differentiable at $x = 0$, $\frac{f(x)}{x}$ is integrable on Λ. Moreover by (2.18), for sufficiently large $x > 0$,

$$\left|\frac{f(x)}{x}\right| \le c\frac{e^x}{x^{2+\rho}} \le c\frac{e^x + e^{-x}}{1 + x^{2+\rho}}.$$

Therefore (A.7) and Lemma A.2 imply that

$$S_N(\frac{f}{x}, \hat{\omega}, \Lambda) \to I(\frac{f}{x}, \hat{\omega}, \Lambda) = \int_\Lambda f(x)\omega(x)dx. \qquad (A.9)$$

On the other hand, by virtue of (A.1), (A.8) and (2.16), we know that

$$\begin{aligned} S_N(\tfrac{f}{x}, \hat{\omega}, \Lambda) &= \sum_{j=1}^{N} \hat{I}_{N-1}(\frac{f}{x})(\hat{\sigma}_j^N)\hat{\omega}_j^N = \int_\Lambda \hat{I}_{N-1}(\frac{f}{x})\hat{\omega}(x)dx \\ &= \int_\Lambda I_N f(x)\omega(x)dx = \sum_{j=0}^{N} f(\sigma_j)\omega_j^N. \end{aligned}$$

Therefore by (A.9),

$$\sum_{j=0}^{N} f(\sigma_j)\omega_j^N \to \int_\Lambda f(x)\omega(x)dx, \qquad \text{as } N \to \infty. \qquad (A.10)$$

Next, let

$$f_1(x) = \begin{cases} f(x), & 0 \le x \le 1, \\ 0, & x > 1 \end{cases}$$

and

$$f_2(x) = \begin{cases} f(x), & x > 1, \\ 0, & 0 \le x \le 1. \end{cases}$$

By Lemma A.1 and (A.10),

$$S_N(f_1, \omega, \Lambda) \to I_N(f_1, \omega, \Lambda) \ \text{ as } \ N \to \infty$$

and

$$S_N(f_2, \omega, \Lambda) \to I_N(f_2, \omega, \Lambda) \ \text{ as } \ N \to \infty.$$

The above statements lead to the desired result.

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