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ON THE FINITE VOLUME ELEMENT VERSION OF RITZ-VOLTERRA PROJECTION AND APPLICATIONS TO RELATED EQUATIONS*

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Abstract

In this paper, we present a general error analysis framework for the finite volume element (FVE) approximation to the Ritz-Volterra projection, the Sobolev equations and parabolic integro-differential equations. The main idea in our paper is to consider the FVE methods as perturbations of standard finite element methods which enables us to derive the optimal L_2 and H^1 norm error estimates, and the L_∞ and W^1_∞ norm error estimates by means of the time dependent Green functions. Our disc ussions also include elliptic and parabolic problems as the special cases.

Key words: Finite volume element, Ritz-Volterra projection, Integro-differential equations, Error analysis.

1. Introduction

Consider the integro-differential equation of Volterra type

$$\begin{array}{rll} V(t)u \equiv A(t)u + \int_0^t B(t,\tau)u(\tau)\,d\tau & = f(t), & in\ J\times\Omega \\ u(t,x) & = 0, & on\ J\times\partial\Omega \end{array} \tag{1}$$

Where Ω is a bounded convex polygon in R^2 with a boundary $\partial\Omega$, $J=(0,T]$, $T>0$, $A(t)$ is a symmetric and positive definite linear partial differential operator of second order, and $B(t,\tau)$ an arbitrary second order linear partial differential operator, both with coefficients depending smoothly on x, t, and τ for the latter. When $f(t)\in L_\infty(J;L_p(\Omega))$, problem (1) admits a unique solution $u(t)\in L_\infty(J;W_p^2(\Omega)\bigcap H_0^1(\Omega))$ and satisfies[1]

$$\|u(t)\|_{2,p} \le C(\|f(t)\|_{0,p} + \int_0^t \|f(t)\|_{0,p}d\tau),\ 1<p<p_0,\ t\in J \tag{2}$$

Where $p_0 = 2+\alpha$, $\alpha>0$ is a positive constant depending on the maximal inner angle of Ω, and when $\partial\Omega$ is smooth enough, $p_0=\infty$.

The so called finite element Ritz-Volterra projection[2] just is the finite element approximation $u_h(t)$ of the exact solution $u(t)$ of problem (1). Obviously, Ritz-Volterra projection is a natural generalization of the finite element Ritz projection, when $B(t,\tau)\equiv 0$, both are identical. In recent years, Ritz-Volterra projection has attracted considerable attentions since that it provides a unified and powerful means in studying the Galerkin finite element methods for many evolution equations such as parabolic and hyperbolic integro-differential equations, Sobolev equations and visco-elasticity equations, etc. [2−7] In this paper, we will investigate the finite volume element(FVE) version of the finite element Ritz-Volterra projection, that is the

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FVE approximation to the exact solution of problem (1). We will present a general error analysis framework for the FVE methods of the integro-differential equations and related equations, and establish some optimal error estimates under L_2, H^1, L_∞ and W^1_∞ norms.

FVE methods for the elliptic boundary value problems have a long history just like finite element methods. In early literatures [8,9], a so called integral finite difference methods were systematically investigated, most of the results were given in one-dimensional cases. FVE methods have also been termed as box scheme, generalized finite difference schemes or integral type difference schemes [10]. Generally speaking, FVE methods are numerical techniques lie somewhere between finite difference and finite element methods. They have a flexibility similar to that of finite element methods for handing complicated solution domain geometry and boundary conditions, and have a comparable simplicity for implementation like finite difference methods when the triangulation has simple structures. More importantly, numerical solutions generated by FVE methods usually have certain conservation features which are very desirable in many applications. However, the analysis for FVE methods is far behind that for the finite element and finite difference methods. The readers are referred to articles [10-22] for some recent developments.

Many early publications can be found on the FVE methods using linear finite elements and the related optimal H^1 norm error estimates, and some superconvergence in the discrete H^1 norms. Later the authors of [10] obtained L_2 norm error estimate of the following form:

$$\|u - u_h\| \le Ch^2 \|u\|_{3,p}, \ p > 1 \tag{3}$$

Note that the order in this estimate is optimal, but its regularity requirement on the exact solution is not. In article [16,17], a framework based on functional analysis was presented to analyze the FVE methods, but they did not provide the optimal L_2 error estimate. The authors of [18] obtained some new error estimates by extending the techniques of [10]. In these articles, optimal H^1 and W^1_∞ error estimates and superconvergence in H^1 and W^1_∞ norms are obtained.

Recently, the authors of [23] present the L_2 error estimate of the following form:

$$\|u - u_h\| \le C(h^2\|u\|_2 + h^{1+\beta}\|f\|_\beta), \ 0 \le \beta \le 1 \tag{4}$$

It seems to be a better result compared with that in (3), because, except for the solution domain with a boundary smooth enough, the H^1 regularity on the source term does not automatically imply the H^3 regularity of the exact solution. Moreover, they also indicate, by a counter example, that the regularity requirement on the source term can not be reduced in order to obtain the optimal order error estimate.

These results just mentioned are mainly for the elliptic and parabolic problems. To our knowledge, up to now, there are no or few publications concerning the FVE methods for the integro-differential equations as above. In this paper, we will investigate the FVE methods using linear finite element for problem (1), and the Sobolev equations and parabolic integro-differential equations. The main idea in our paper is to consider FVE methods as perturbations of Galerkin finite element methods. This approach simplifies tremendously our analysis and allows us to employ the standard error analysis techniques developed for finite element methods to derive the optimal L_2 and H^1 norm error estimates. Moreover, by means of the time dependent Green function methods introduced in articles [1,6], we also obtain the optimal L_∞ and W^1_∞ norm error estimates. In our discussion below, for simplicity, we assume the operators $A(t)$ and $B(t,\tau)$ as follows

$$A(t) = -\nabla \cdot A\,\nabla \ \ ; \quad B(t,\tau) = \nabla \cdot B\,\nabla$$

Where $A = (a_{ij}(t,x))$ is a 2×2 symmetric and positive definite matrix uniformly in $J \times \Omega$, and $B = (b_{ij}(t,\tau,x))$ an arbitrary 2×2 matrix in $J \times J \times \Omega$. The results of this paper can be extended easily to cover more general models without any additional difficulties.

This paper is organized as follows. In Section 2, we formulate the FVE methods in linear finite element spaces defined on a triangulation. Section 3 is devoted to the L_2 and H^1 norm error estimates. In Section 4, the optimal L_∞ and W^1_∞ norm error estimates are established by using the time dependent Green function methods. Finally, in Section 5, the FVE approximations to Sobolev equations and parabolic integro-differential equations are discussed.

2. The Finite Volume Methods

We will use the standard notations of Sobolev spaces $W^s_p(\Omega)$ with norms $\|\cdot\|_{s,p}$, $1 \le p \le \infty$. When $p = 2$, denote $W^s_2(\Omega) = H^s(\Omega)$, $\|\cdot\|_{s,2} = \|\cdot\|_s$. $H^1_0(\Omega)$ is the completion of $C^\infty_0(\Omega)$ under norm $\|\cdot\|_1$. And denote by $(\cdot,\cdot)$ and $\|\cdot\|$ the standard inner and norm in $L_2(\Omega) = H^0(\Omega)$. Let X be a Banach space. We use the notation $L_p(J;X)$, $1 \le p \le \infty$ to represent the usual X-value integral function spaces, ie.

$$L_p(J;X) = \{\, u(t):\ J \to X;\ \Big(\int_0^T \|u(\tau)\|^p_X\, d\tau\,\Big)^{\frac{1}{p}} < \infty\,\},\ 1 \le p \le \infty$$

with the usual modification for $p = \infty$. For simplicity, we will also use the following notation

$$|||u(t)|||_{s,p} = \|u(t)\|_{s,p} + \int_0^t \|u(\tau)\|_{s,p}\, d\tau,\ 1 \le p \le \infty,\ t \in J$$

and $|||u(t)||| = |||u(t)|||_{0,2}$, $|||u(t)|||_s = |||u(t)|||_{s,2}$.

Now, for the convex polygon domain Ω, we consider a regular triangulation T_h consisting of closed triangle elements K such that $\overline{\Omega} = \bigcup_{K\in T_h} K$, where h represents the maximal diameter of all elements $K \in T_h$. We will use N_h to denote the set of all nodes of T_h:

$$N_h = \{\, p\,; p \text{ is a vertex of element } K \in T_h\}$$

and let $N^0_h = N_h \bigcap \Omega$. We now introduce a dual partition T^*_h based on T_h whose elements are called control volumes. In the finite volume methods, there are various ways to introduce the control volumes[10], here we will use the popular barycenter configuration. In each element $K \in T_h$ consisting of vertices p_i, p_j and p_k, let $q = \frac{1}{3}(p_i + p_j + p_k)$ be the barycenter of K and $m_{ij} = \frac{1}{2}(p_i + p_j)$ be the middle point of line $\overline{p_i p_j}$, then connect q to the points m_{ij} by straight lines $l_{ij,K}$. Now we let V_i be the polygon around p_i whose edges are $l_{ij,K}$, here K is any element whose a vertex is p_i. We call V_i a control volume centered at point p_i (Figure 1 gives a sketch of a control volume centered at p_i). Obviously we have $\bigcup_{p_i\in N_h} V_i = \overline{\Omega}$, and the dual partition T^*_h is then defined as the union of these control volumes. The partition T^*_h is called regular if $c_0 h^2_{V_i} \le meas(V_i)$, $\forall\, V_i \in T^*_h$, where h_{V_i} is the diameter of V_i. It is easily to see that when T_h is regular, T^*_h is also regular. Let $S_h \subset H^1_0(\Omega)$ be the standard linear finite element space defined on the triangulation T_h, and S^*_h be its dual volume element space defined on the dual partition T^*_h by

$$S^*_h = \{v \in L_2(\Omega);\ v|_V \text{ is constant for all } V \in T^*_h \text{ and } v|_{\partial\Omega} = 0\,\}$$

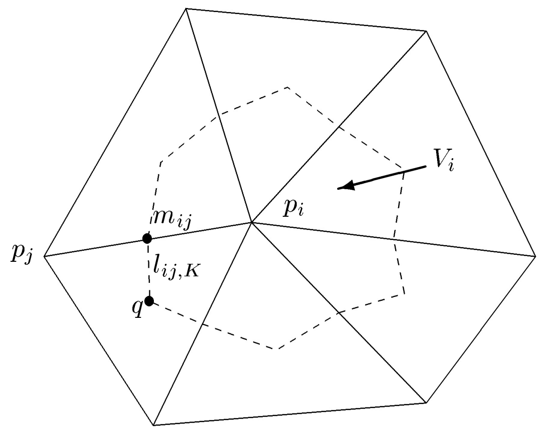
Obviously

$$S_h = span\,\{\,\varphi_i(x);\, p_i \in N^0_h\,\}\,; \qquad S^*_h = span\,\{\,\chi_i(x);\, p_i \in N^0_h\,\}$$

Where φ_i are the standard piecewise linear basis functions and χ_i are the characteristic functions of the volume V_i. Let $I_h : C(\Omega) \to S_h$ and $I^*_h : C(\Omega) \to S^*_h$ be the usual interpolation operators, ie.,

$$I_h u = \sum_{p_i\in N^0_h} u(x_i)\varphi_i(x)\,; \qquad I^*_h u = \sum_{p_i\in N^0_h} u(x_i)\chi_i(x)$$

Obviously, for any $v^*_h \in S^*_h$, there exists unique one $v_h \in S_h$ such that $I^*_h v_h = v^*_h$.



Figur 1. Control volume centered at p_i

Introduce the bilinear forms associated with the operators $A(t)$ and $B(t,\tau)$:

$$A(t;u,v) = \begin{cases} -\displaystyle\sum_{V_i \in T_h^*} v_i \int_{\partial V_i} A\nabla u \cdot n\, ds & , (u,v) \in (H^2(\Omega)\bigcup S_h) \times S_h^* \\ \displaystyle\int_\Omega A\nabla u \cdot \nabla v\, ds & , (u,v) \in H_0^1(\Omega) \times H_0^1(\Omega) \end{cases} \tag{5}$$

and $B(t,\tau;u,v)$ similarly by replacing the 2×2 matrix $A = (a_{ij}(t,x))$ by $B = (b_{ij}(t,\tau,x))$ in the definition above. We have used the same notation for the bilinear form $A(t;u,v)$ (or $B(t,\tau;u,v)$) defined in two different ways on the pair of spaces $(H^2(\Omega)\bigcup S_h) \times S_h^*$ and $H_0^1(\Omega) \times H_0^1(\Omega)$, correspondingly. We hope that this will not lead to serious confusion while it simplifies tremendously the notations and operations in this paper. There is one more reason to use this representation. Namely, we have (see (9) and Lemma 2.1 below) when matrix A is constant over each element $K \in T_h$

$$A(t;u_h,v_h) = A(t;u_h,I_h^* v_h),\ \forall\, u_h, v_h \in\ S_h$$

Introduce the bilinear form associated with the operator $V(t)$ as follows

$$V(t;u(t),v(t)) = A(t;u(t),v(t)) + \int_0^t B(t,\tau;u(\tau),v(t))\, d\tau \tag{6}$$

Now we define the finite volume element approximation of problem (1) by finding $u_h(t):\ J \to S_h$ such that

$$V(t;u_h,v_h^*) = (f,v_h^*),\ \forall\, v_h^* \in S_h^*$$

$$Or \qquad V(t;u_h,I_h^* v_h) = (f,I_h^* v_h),\ \forall\, v_h \in S_h \tag{7}$$

Then, the exact solution u and the FVE solution u_h will satisfy

$$V(t;u-u_h,I_h^* v_h) = 0,\ \forall\, v_h \in S_h \tag{8}$$

Some important properties of the interpolation operator I_h^* should be presented. First, since $I_h^* u$ is the piecewise constant approximation of function u on the regular dual mesh T_h^*, then we have

$$\|u - I_h^* u\|_{0,p} \le Ch\|u\|_{1,p},\ 1 \le p \le \infty$$

Furthermore, by straight calculating and note that $v_h \in S_h$ is piecewise linear, we can find that

$$\int_K (v_h - I_h^* v_h)\, dx = 0\,; \qquad \int_l (v_h - I_h^* v_h)\, ds = 0,\ \forall\ line\ l \subset \partial K,\ K \in T_h \tag{9}$$

$$\|I_h^* v_h\|_{0,K}^2 = \int_K |I_h^* v_h|^2 dx = \frac{|K|}{3}(v_i^2 + v_j^2 + v_k^2) \tag{10}$$

$$\|v_h\|_{0,K}^2 = \int_K |v_h|^2 dx = \frac{|K|}{12}(v_i^2 + v_j^2 + v_k^2 + (v_i + v_j + v_k)^2) \tag{11}$$

From (10) and (11), we have

$$\frac{1}{2}\|I_h^* v_h\| \le \|v_h\| \le \|I_h^* v_h\|, \ \forall\, v_h \in \ S_h \tag{12}$$

Lemma 2.1. Let $w \in H^2(\Omega)\bigcup S_h$ and $v_h \in S_h$. Then

$$A(t;w,v_h) = A(t;w,I_h^* v_h) + E_A(t;w,v_h)$$
$$B(t,\tau;w,v_h) = B(t,\tau;w,I_h^* v_h) + E_B(t,\tau;w,v_h)$$

where

$$E_A(t;w,v_h) = \sum_{K\in T_h}\int_{\partial K} A\nabla w\cdot n(v_h - I_h^* v_h)\,ds - \sum_{K\in T_h}\int_K \nabla\cdot A\nabla w(v_h - I_h^* v_h)\,dx$$
$$E_B(t,\tau;w,v_h) = \sum_{K\in T_h}\int_{\partial K} B\nabla w\cdot n(v_h - I_h^* v_h)\,ds - \sum_{K\in T_h}\int_K \nabla\cdot B\nabla w(v_h - I_h^* v_h)\,dx$$

Proof. It follows Green's formula that

$$\sum_{K\in T_h}(\nabla\cdot A\nabla w, v_h)_K = \sum_{K\in T_h}\int_{\partial K} A\nabla w\cdot n\, v_h\,ds - A(t;w,v_h)$$

$$\begin{aligned}\sum_{K\in T_h}(\nabla\cdot A\nabla w, I_h^* v_h)_K &= \sum_{K\in T_h}\sum_{V_j\in T_h^*}(\nabla\cdot A\nabla w, I_h^* v_h)_{K\cap V_j}\\ &= \sum_{K\in T_h}\int_{\partial K} A\nabla w\cdot n\, I_h^* v_h\,ds + \sum_{V_j\in T_h^*}\int_{\partial V_j} A\nabla w\cdot n\, I_h^* v_h\,ds\\ &= \sum_{K\in T_h}\int_{\partial K} A\nabla w\cdot n\, I_h^* v_h\,ds - A(t;w,I_h^* v_h)\end{aligned}$$

Then, the first equality is proved by taking the difference of these two identities. The second equality can be derived similarly. The proof is completed.

From Lemma 2.1 and (6), we obtain

$$V(t;w,v_h) = V(t;w,I_h^* v_h) + E(t;w,v_h), \ \forall\, w \in H^2(\Omega)\cup S_h,\ v_h\in S_h \tag{13}$$
$$E(t;w,v_h) = E_A(t;w,v_h) + \int_0^t E_B(t,\tau;w(\tau),v_h)\,d\tau$$

Lemma 2.2. Let $A \in W^1_\infty(\Omega)$. Then, for any $u_h,\, v_h \in S_h$, we have

$$|\,E_A(t;u_h,v_h)\,| + |\,E_B(t,\tau;u_h,v_h)\,| \le Ch\|u_h\|_1\|v_h\|_1$$

Proof. Note that

$$\begin{aligned}E_A(t;u_h,v_h) &= \sum_{K\in T_h}\int_{\partial K} A\nabla u_h\cdot n(v_h - I_h^* v_h)\,ds - \sum_{K\in T_h}\int_K \nabla\cdot A\nabla u_h(v_h - I_h^* v_h)\,dx\\ &= J_1(u_h,v_h) + J_2(u_h,v_h)\end{aligned} \tag{14}$$

Let M be the middle point of line l, for $l\subset\partial K, K\in T_h$. We introduce the piecewise constant approximation of A on the boundary of elements:

$$A_l = A(M),\ \ \forall\, l\subset\partial K,\, K\in T_h$$

Since $A_l\nabla u_h\cdot n$ is constant on l, then we have from (9), the trace theorem and approximation properties that

$$J_1(u_h,v_h) = \sum_{K\in T_h}\int_{\partial K}(A - A_l)\nabla u_h\cdot n(v_h - I_h^* v_h)\,ds$$
$$\le C\sum_K h_K\|A\|_{1,\infty,K}h_K^{-\frac{1}{2}}\|u_h\|_{1,K}(h_K^{\frac{1}{2}}\|v_h\|_{1,K} + h_K^{-\frac{1}{2}}\|v_h - I_h^* v_h\|_{0,K}) \le Ch\|u_h\|_1\|v_h\|_1$$

For $J_2(u_h, v_h)$, we have

$$J_2(u_h, v_h) \le C\|A\|_{1,\infty} \sum_K \|u_h\|_{1,K}\|v_h - I_h^* v_h\|_{0,K} \le Ch\|u_h\|_1\|v_h\|_1$$

Combining these two estimates, we obtain the estimate of $E_A(t; u_h, v_h)$. Here and afterwards, we will always omit the related estimate concerning $E_B(t, \tau; u_h, v_h)$, because it can be derived by a completely similar argument to that for $E_A(t; u_h, v_h)$. The proof is completed.

A immediate result from Lemma 2.1 and Lemma 2.2 is that there exist positive constants C and h_0 such that for $0 < h \le h_0$, h_0 small, and u_h, $v_h \in S_h$

$$C^{-1}\|u_h\|_1^2 \le A(t; u_h, I_h^* u_h)\,; \quad |A(t; u_h, I_h^* v_h)| \le C\|u_h\|_1\|v_h\|_1$$

Theorem 2.1. There exists $h_0 > 0$ such that, for $0 < h \le h_0$, the solution $u_h(t)$ of problem (7) uniquely exists and satisfies

$$\|u_h(t)\|_1 \le C|||f(t)|||\,,\ t \in J$$

Proof. It follows Lemma 2.2 that

$$\begin{aligned} &C_0\|u_h(t)\|_1^2 + \int_0^t B(t, \tau; u_h(\tau), u_h(t))\, d\tau \le V(t; u_h, u_h) \\ &= V(t; u_h, I_h^* u_h) + E(t; u_h, u_h) \le (f, I_h^* u_h) + Ch|||u_h|||_1\|u_h\|_1 \end{aligned}$$

then, by (12) we have

$$C_0\|u_h(t)\|_1 \le 2\|f(t)\| + Ch\|u_h(t)\|_1 + C\int_0^t \|u_h(\tau)\|_1\, d\tau$$

For $0 < h \le h_0$, h_0 small, the proof is completed by using Gronwall lemma.

3. Optimal H^1 and L_2 Error Estimates

The H^1 estimate can be obtained easily by Lemma 2.2.

Theorem 3.1. Let $u(t)$ and $u_h(t)$ be the solutions of problems (1) and (7), respectively. Then

$$\|u(t) - u_h(t)\|_1 \le Ch|||f(t)|||\,,\ t \in J$$

Proof. Denote $v_h = I_h u - u_h$. By using equations satisfied by $u(t)$ and $u_h(t)$, respectively, (13) and Lemma 2.2, we obtain

$$\begin{aligned} &C_0\|u(t) - u_h(t)\|_1^2 + \int_0^t B(t, \tau; (u - u_h)(\tau), (u - u_h)(t))\, d\tau \\ &\le V(t; u - u_h, u - u_h) = V(t; u - u_h, u - I_h u) + V(t; u - u_h, v_h) \\ &= V(t; u - u_h, u - I_h u) + (f, v_h - I_h^* v_h) - E(t; u_h, v_h) \\ &\le |||u - u_h|||_1\|u - I_h u\|_1 + h\|f\|\|v_h\|_1 + Ch|||u_h|||_1\|v_h\|_1 \end{aligned}$$

Note that $\|u_h\|_1 \le C|||f|||$ and

$$\|v_h\|_1 \le \|u - I_h u\|_1 + \|u - u_h\|_1 \le Ch|||f||| + \|u - u_h\|_1$$

then, we have

$$\|u(t) - u_h(t)\|_1 \le Ch|||f(t)||| + C\int_0^t \|u(\tau) - u_h(\tau)\|_1\, d\tau$$

The proof is completed by Gronwall lemma.

For L_2 estimate, we need the following lemma.

Lemma 3.1. Let $A \in W_\infty^2(\Omega)$, $u \in H^2(\Omega)$. Then, for any u_h, $v_h \in S_h$, we have

$$|E_A(t; u_h, v_h)| + |E_B(t, \tau; u_h, v_h)| \le Ch(h\|u\|_2 + \|u - u_h\|_1)\|v_h\|_1 + Ch^2\|u_h\|_1\|v_h\|_1$$

Proof. Note that $E_A(t; u_h, v_h) = J_1(u_h, v_h) + J_2(u_h, v_h)$ (see (14)). Since, except the middle point M of l, $l \subset \partial K$, $v_h - I_h^* v_h$ is continuos over l, then, when $u \in H^2(\Omega)$, we have $J_1(u, v_h) = 0$

(In fact, we may assume that $u \in H^2(\Omega) \cap C^1(\Omega)$, otherwise, consider $u_\varepsilon \in H^2(\Omega) \cap C^1(\Omega)$ so that $\|u_\varepsilon - u\|_{1,\partial\Omega} \le C\|u_\varepsilon - u\|_{2,\Omega} \to 0, \varepsilon \to 0$). Therefore

$$J_1(u_h, v_h) = J_1(u_h - u, v_h) = \sum_{K \in T_h} \int_{\partial K} (A - A_l)\nabla(u_h - u) \cdot n(v_h - I_h^* v_h)\, ds \tag{15}$$

Thus, from the trace theorem we see that

$$\begin{aligned} J_1(u_h, v_h) \quad &\le \quad C \sum_K h_K \|A\|_{1,\infty,K} (h_K^{\frac{1}{2}} \|u\|_{2,K} + h_K^{-\frac{1}{2}} \|u - u_h\|_{1,K}) \times \\ &\times \quad (h_K^{\frac{1}{2}} \|v_h\|_{1,K} + h_K^{-\frac{1}{2}} \|v_h - I_h^* v_h\|_{0,K}) \le Ch(h\|u\|_2 + \|u - u_h\|_1)\|v_h\|_1 \end{aligned}$$

For $J_2(u_h, v_h)$, let w_K be the average value of function w on element K. From (9) we obtain

$$\begin{aligned} |J_2(u_h, v_h)| &= \Big| \sum_{K \in T_h} \int_K (\nabla \cdot A\nabla u_h - (\nabla \cdot A\nabla u_h)_K)(v_h - I_h^* v_h)\, dx \Big| \\ &\le Ch\|A\|_{2,\infty} \sum_K \|u_h\|_{1,K} \|v_h - I_h^* v_h\|_{0,K} \le Ch^2 \|u_h\|_1 \|v_h\|_1 \end{aligned}$$

Combining the estimates of J_1 and J_2, the proof is completed.

Theorem 3.2. Let $u(t)$ and $u_h(t)$ be the solutions of problems (1) and (7), respectively, and $f(t) \in L_\infty(J; H^\beta(\Omega))$, $0 \le \beta \le 1$. Then

$$\|u(t) - u_h(t)\| \le Ch^2 |||f(t)||| + Ch^{1+\beta} |||f(t)|||_\beta$$

Proof. Let $w \in H^2(\Omega) \cap H_0^1(\Omega)$ be the solution of

$$-\nabla \cdot A\nabla w = u - u_h,\ x \in \Omega \ \textit{and}\ w = 0,\ \textit{on}\ \partial\Omega \tag{16}$$

then we have $\|w\|_2 \le C\|u - u_h\|$. Now, for any $w_h \in S_h$, we see that

$$\begin{aligned} & C_0 \|u(t) - u_h(t)\|^2 \le V(t; u - u_h, w) - \int_0^t B(t, \tau; u(\tau) - u_h(\tau), w(t))\, d\tau \\ & = V(t; u - u_h, w - w_h) + V(t; u - u_h, w_h) + \int_0^t (u(\tau) - u_h(\tau), B^*(t, \tau) w(t))\, d\tau \\ & = V(t; u - u_h, w - w_h) + (f, w_h - I_h^* w_h) - E(t; u_h, w_h) + \\ & + \int_0^t (u(\tau) - u_h(\tau), B^*(t, \tau) w(t))\, d\tau \end{aligned}$$

where $B^*(t, \tau)$ is the adjoint operator of $B(t, \tau)$. From (9) we have

$$|(f, w_h - I_h^* w_h)| = |(f - f_K, w_h - I_h^* w_h)| \le Ch^{1+\beta} \|f\|_\beta \|w_h\|_1$$

Then, it follows by taking $w_h = I_h w$ and using Theorem 3.1 and Lemma 3.1 that

$$\|u(t) - u_h(t)\|^2 \le Ch^2 |||f(t)|||\, \|w\|_2 + Ch^{1+\beta} \|f\|_\beta \|w\|_2 + C \int_0^t \|u(\tau) - u_h(\tau)\|\, d\tau\, \|w\|_2$$

Thus, the proof is completed by Gronwall lemma.

The result in Theorem 3.2 reveals how the regularity of the source can affect the error of FVE solution in L_2 norm. When f is in H^1, the order of convergence is optimal with respect to the approximation property of linear finite element space. And the authors [23] have indicated that the regularity assumption on f can not be reduced. There, a counter example is presented for the elliptic problem ($B(t, \tau) \equiv 0$ case) to show that when f is only in $L_2(\Omega)$, there exists some $0 < \alpha < \frac{1}{2}$ such that $\|u - u_h\| \ge Ch^{2-\alpha}$.

The following results will be used in Section 5.

Theorem 3.3. Let $u(t)$ and $u_h(t)$ be the solutions of problems (1) and (7), respectively. Then

$$\|(u - u_h)_t(t)\|_1 \le Ch(|||f||| + \|u_t\|_2),\ t \in J$$

$$\|(u - u_h)_t(t)\| \le Ch^2(|||f|||_1 + \|u_t\|_2),\ t \in J$$

Proof. Denote $e(t) = u(t) - u_h(t)$. Differentiate (8) in time to obtain

$$A(t; e_t, I_h^* v_h) + D(t; e, I_h^* v_h) = 0, \; \forall \, v_h \in S_h \tag{17}$$

$$D(t; e, v) = A_t(t; e, v) + B(t, t; e, v) + \int_0^t B_t(t, \tau; e(\tau), v) \, d\tau$$

Where the subscript t represents the differentiate in time with respect to the coefficient functions of the operators. Writing $e = u - I_h u + I_h u - u_h = \eta + \theta$, $\theta \in S_h$, from (17) we have

$$\begin{aligned} &C_0 \|e_t\|_1^2 \le A(t; e_t, e_t) = A(t; e_t, \eta_t) + A(t; e_t, \theta_t) = A(t; e_t, \eta_t) + A(t; e_t, I_h^* \theta_t) \\ &+ E_A(t; e_t, \theta_t) = A(t; e_t, \eta_t) - D(t; e, \theta_t) + E_D(t; e, \theta_t) + E_A(t; e_t, \theta_t) \\ &\le C \|e_t\|_1 \|\eta_t\|_1 + C \||e\||_1 \|\theta_t\|_1 + E_D(t; e, \theta_t) + E_A(t; e_t, \theta_t) \end{aligned}$$

where

$$E_D(t; e, \theta_t) = D(t; e, \theta_t) - D(t; e, I_h^* \theta_t)$$

Following a similar argument used in Lemma 2.2, we see that

$$\begin{aligned} |E_A(t; e_t, \theta_t)| &\le Ch(\|e_t\|_1 + \|u_t\|_1) \|\theta_t\|_1 + Ch \|e_t\|_1 \|\theta_t\|_1 \\ |E_D(t; e, \theta_t)| &\le Ch(\||e\||_1 + \||u\||_1) \|\theta_t\|_1 + Ch \||e\||_1 \|\theta_t\|_1 \end{aligned}$$

Thus, note that $\|\theta_t\|_1 \le \|e_t\|_1 + \|\eta_t\|_1$, the H^1 estimate is derived. For the L_2 estimate, we still use the auxiliary problem (16) with the right side function e_t to obtain

$$\begin{aligned} &\|e_t\|^2 = A(t; e_t, w - w_h) + A(t; e_t, w_h) = A(t; e_t, w - w_h) + A(t; e_t, I_h^* w_h) + \\ &+ E_A(t; e_t, w_h) = A(t; e_t, w - w_h) - D(t; e, w_h) + E_D(t; e, w_h) + E_A(t; e_t, w_h) \\ &= A(t; e_t, w - w_h) + D(t; e, w - w_h) - D(t; e, w) + E_D(t; e, w_h) + E_A(t; e_t, w_h) \\ &\le C \|e_t\|_1 \|w - w_h\|_1 + C \||e\||_1 \|w - w_h\|_1 + C \||e\|| \, \|w\|_2 + E_D(t; e, w_h) + E_A(t; e_t, w_h) \end{aligned}$$

Following a similar argument used in Lemma 3.1, we see that

$$\begin{aligned} |E_A(t; e_t, w_h)| &\le Ch(h \|u_t\|_2 + \|e_t\|_1) \|w_h\|_1 + Ch \|e_t\|_1 \|w_h\|_1 \\ |E_D(t; e, w_h)| &\le Ch(h \||u\||_2 + \||e\||_1) \|w_h\|_1 + Ch \||e\||_1 \|w_h\|_1 \end{aligned}$$

Taking $w_h = I_h w$, the proof is completed by means of Theorem 3.1 and Theorem 3.2.

4. W_∞^1 and L_∞ Norm Error Estimates

In this section, we will investigate the W_∞^1 and L_∞ norm error estimates for the FVE methods of problem (1). It is well known that the Green function methods can be used in deriving the W_∞^1 and L_∞ error estimates for finite element approximations to elliptic problems [20,21]. However, this kind of Green functions can not be applied to problem (1) for the same purpose, excepting that operator $A(t)$ and $B(t, \tau)$ are in special forms (see [25]). Therefore, we shall introduce a new type of Green functions associated with the Volterra integro-differential equations, some detail discussion on this kind of Green functions can be found in articles [1,6]. Below we will assume the triangulation T_h is quasi-regular so that the inverse properties hold on the finite element space S_h.

Introduce the adjoint operator of the integro-differential operator $V(t)$ and corresponding bilinear form

$$\begin{aligned} V^*(t) u(t) &= A(t) u(t) + \textstyle\int_t^T B^*(\tau, t) u(\tau) \, d\tau, \; t \in J \\ V^*(t; u(t), v(t)) &= A(t; u(t), v(t)) + \textstyle\int_t^T B^*(\tau, t; u(\tau), v(t)) \, d\tau \end{aligned}$$

By using the *Dirichlet* formula of integration

$$\int_0^T \int_0^t f(t, \tau) \, d\tau \, dt = \int_0^T \int_t^T f(\tau, t) \, d\tau \, dt \tag{18}$$

we have

$$\int_0^T V(t;u(t),v(t))\,dt = \int_0^T V^*(t;v(t),u(t))\,dt,\ \forall\, u,v\ \in\ L_1(J;H_0^1(\Omega)) \tag{19}$$

For any given point $z=(z_1,z_2)\in\Omega$, there exists uniquely $\delta_h^z(x)\in S_h$ to be the discrete δ-function at point $x=z$ which satisfies

$$(\delta_h^z, v_h) = v_h(z),\ \forall v_h\in S_h$$

Denote by L any appointed direction, and define the direction derivative

$$\partial_z\delta_h^z = \lim_{\triangle z\to 0,\,\triangle z//L}(\delta_h^{z+\triangle z}-\delta_h^z)/|\triangle z|$$

It is easy to see that $\partial_z\delta_h^z(x)\in S_h$ and

$$(\partial_z\delta_h^z, v_h) = \partial_z v_h(z),\ \forall v_h\in S_h$$

Now, define the *Green* type functions in function and derivative forms by $G^z(t)$, $\partial_z G^z(t)\in L_\infty(J;H_0^1(\Omega)\bigcap H^2(\Omega))$ such that

$$V^*(t)G^z(t) = \delta_h^z(x)\sigma(t),\ t\ \in\ J$$
$$V^*(t)\partial_z G^z(t) = \partial_z\delta_h^z(x)\sigma(t),\ t\ \in\ J$$

where $\sigma(t)\ge 0$ is an arbitrary function in $C[0,T]$. In practical application, we usual take $\sigma(t)$ as the mollifier function, for instance

$$\sigma(t)=\sigma_\mu(t,s)=\begin{cases}\dfrac{C_1}{\mu}\ exp(-\dfrac{1}{1-|(t-s)/\mu|^2}) & , \quad |t-s|<\mu\\ 0 & , \quad |t-s|\ge\mu\end{cases}$$

here $s\in[0,T]$ is any given point and constant C_1 is taken to insure $\int_0^T\sigma(t)\,dt=1$. Thus, we have

$$f(s)=\lim_{\mu\to 0}\int_0^T f(t)\,\sigma_\mu(t,s)\,dt,\quad for\ \ f(t)\in L_\infty(0,T) \tag{20}$$

By the definitions of $G^z(t)$ and $\partial_z G^z(t)$, we see that

$$V^*(t;G^z(t),v) = P_h v(z)\sigma(t),\quad \forall\, v\ \in\ H_0^1(\Omega),\ t\ \in\ J \tag{21}$$

$$V^*(t;\partial_z G^z(t),v) = \partial_z P_h v(z)\sigma(t),\quad \forall\, v\ \in\ H_0^1(\Omega),\ t\ \in\ J \tag{22}$$

where $P_h:\ L_2(\Omega)\to S_h$ is the L_2 projection operator which possesses the stability properties[24]

$$\|P_h v\|_{s,p}\le C\|v\|_{s,p},\ s=0,1,\ 2\le p\le\infty \tag{23}$$

Now, define the finite element approximation of $G^z(t)$ and $\partial_z G^z(t)$ by $G_h^z(t)$, $\partial_z G_h^z(t)$ $\in\ L_\infty(J;S_h)$ such that

$$V^*(t;G^z(t)-G_h^z(t),v_h)=0,\ \forall v_h\ \in S_h,\ t\in J \tag{24}$$

$$V^*(t;\partial_z G^z(t)-\partial_z G_h^z(t),v_h)=0,\ \forall v_h\ \in S_h,\ t\in J \tag{25}$$

According to articles [1, 5, 6], we know that $G^z(t)$ and $\partial_z G^z(t)$ exist uniquely and satisfy

$$\|G^z(t)-G_h^z(t)\|_{1,1}\le Ch\,ln\frac{1}{h}(\sigma(t)+\int_t^T\sigma(\tau)\,d\tau) \tag{26}$$

$$\|\partial_z G^z(t)-\partial_z G_h^z(t)\|_{1.1}+h|ln\,h|^{-\frac{1}{2}}\|\partial_z G_h^z(t)\|_1\le C(\sigma(t)+\int_t^T\sigma(\tau)\,d\tau) \tag{27}$$

$$\|G_h^z(t)\|_1+\|\partial_z G_h^z(t)\|_{1,1}\le Cln\frac{1}{h}(\sigma(t)+\int_t^T\sigma(\tau)\,d\tau) \tag{28}$$

Theorem 4.1. Let $u(t)$ and $u_h(t)$ be the solutions of problems (1) and (7), respectively. Then

$$\|u(t)-u_h(t)\|_{1,\infty}\le Ch|ln\,h|^{1/2}(|||u(t)|||_{2,\infty}+|||f(t)|||_1),\ t\in J$$

Proof. It follows from (22),(25) and (19) that

$$\begin{aligned} \int_0^T \partial_z (P_h u - u_h)(t,z)\sigma(t)\,dt &= \int_0^T V^*(t;\partial_z G^z - \partial_z G_h^z + \partial_z G_h^z, u-u_h)\,dt \\ &= \int_0^T V^*(t;\partial_z G^z - \partial_z G_h^z, u - I_h u)\,dt + \int_0^T V^*(t;\partial_z G_h^z, u-u_h)\,dt \\ &= \int_0^T V(t;u-I_h u, \partial_z G^z - \partial_z G_h^z)\,dt + \int_0^T V(t;u-u_h,\partial_z G_h^z)\,dt \\ &\le C\int_0^T |||u-I_h u|||_{1,\infty}\|\partial_z G^z - \partial_z G_h^z\|_{1,1}dt + \int_0^T (f, \partial_z G_h^z - I_h^* \partial_z G_h^z)\,dt \\ &\quad - \int_0^T E(t;u_h,\partial_z G_h^z)\,dt \end{aligned} \tag{29}$$

First, we have

$$|(f,\partial_z G_h^z - I_h^*\partial_z G_h^z)| = |(f - f_K, \partial_z G_h^z - I_h^*\partial_z G_h^z)| \le Ch^2\|f\|_1\|\partial_z G_h^z\|_1$$

Then, by using Lemma 3.1 and Theorem 3.1, we obtain

$$|E(t;u_h,\partial_z G_h^z)| \le Ch^2 |||f(t)|||\,\|\partial_z G_h^z\|_1$$

Substituting these estimates into (29) and in view of (27) we see that

$$\begin{aligned} |\int_0^T \partial_z(P_h u - u_h)(t,z)\sigma(t)\,dt| &\le Ch|ln\,h|^{\frac12}\int_0^T (|||u(t)|||_{2,\infty} + |||f(t)|||_1)(\sigma(t) + \int_t^T \sigma(\tau)\,d\tau)\,dt \\ &\le Ch|ln\,h|^{\frac12}\int_0^T (|||u(t)|||_{2,\infty} + |||f(t)|||_1)\sigma(t)dt \end{aligned}$$

where formula (18) has been used. Taking $\sigma(t) = \sigma_\mu(t,s)$ and letting $\mu \to 0$, it follows from (20) that

$$|\partial_z(P_h u - u_h)(t,z)| \le Ch|ln\,h|^{\frac12}(|||u(t)|||_{2,\infty} + |||f(t)|||_1),\ \forall\, t\in J,\ z\in\Omega$$

Thus, by the approximation properties of $P_h u$ (see (23)), the proof is completed.

Theorem 4.2. Let $u(t)$ and $u_h(t)$ be the solutions of problems (1) and (7), respectively. Then

$$\|u(t) - u_h(t)\|_{0,\infty} \le Ch^2 ln\frac{1}{h}(|||u(t)|||_{2,\infty} + |||f(t)|||_1),\ t\in J$$

Proof. It follows from (21),(24) and (19) that

$$\begin{aligned} \int_0^T (P_h u - u_h)(t,z)\sigma(t)\,dt &= \int_0^T V^*(t;G^z - G_h^z + G_h^z, u-u_h)\,dt \\ &= \int_0^T V^*(t;G^z - G_h^z, u - I_h u)\,dt + \int_0^T V^*(t;G_h^z, u-u_h)\,dt \\ &= \int_0^T V(t;u-I_h u, G^z - G_h^z)\,dt + \int_0^T V(t;u-u_h, G_h^z)\,dt \\ &\le C\int_0^T |||u-I_h u|||_{1,\infty}\|G^z - G_h^z\|_{1,1}dt + \int_0^T (f, G_h^z - I_h^* G_h^z)\,dt - \int_0^T E(t;u_h,G_h^z)\,dt \end{aligned}$$

The rest argument is similar to that of Theorem 4.1. The proof is completed.

5. FVE Methods for Related Equations

The Ritz-Volterra projection on finite element spaces can be applied to the error analysis for finite element approximations to many evolution equations. In this section, we will discuss the FVE methods for Sobolev equations and parabolic integro-differential equations by means of the FVE version of Ritz-Volterra projection analyzed in Section 2-4. It will be seen that the Ritz-Volterra projection plays an important role in our demonstration.

Let $u(t) \in H^2(\Omega) \cap H_0^1(\Omega)$, the FVE Ritz-Volterra projection of function u is defined as (see (8)) $u_h(t): J \to S_h$ such that

$$V(t; u(t) - u_h(t), I_h^* v_h) = 0, \ \forall v_h \in S_h, \ t \in J \tag{30}$$

5.1 Sobolev Equations

Consider the initial boundary value problems of *Sobolev* equations

$$\begin{array}{c} A(t)u_t + B(t)u = g(t), \ \ in \ J \times \Omega \\ u(0,x) = u_0(x), \ in \ \Omega \quad \ u(t,x) = 0 \, , on \ J \times \partial\Omega \end{array} \tag{31}$$

Where g and u_0 are known functions, both $A(t)$ and $B(t)$ are second order partial differential operators with the forms given in Section 1, and $A(t)$ is uniformly positive definite.

The semidiscrete FVE approximation of problem (31) is defined by finding $U(t): J \to S_h$ such that

$$\begin{array}{rcll} A(t; U_t, I_h^* v_h) + B(t; U, I_h^* v_h) & = & (g, I_h^* v_h), & v_h \in S_h, t \in J \\ A(0; u_0 - U(0), I_h^* v_h) & = & 0, & v_h \in S_h \end{array} \tag{32}$$

It is easy to show that problem (32) admits a unique solution. Below we do the error analysis. From equations (31) and (32), we have

$$A(t; (u-U)_t(t), I_h^* v_h) + B(t; u(t) - U(t), I_h^* v_h) = 0, \ v_h \in S_h, \ t \in J$$

Integrate with respect to variable t and using (32) to obtain

$$A(t; u(t) - U(t), I_h^* v_h) + \int_0^t \overline{B}(\tau; u(\tau) - U(\tau), I_h^* v_h) \, d\tau = 0, \ v_h \in S_h, \ t \in J \tag{33}$$

Where $\overline{B}(t; u, v) = B(t; u, v) - A_t(t; u, v)$ is the bilinear form associated with operator $B(t) - A_t(t)$. Now comparing (33) with the definition (30), we find immediately that $U(t)$ turns to be the FVE *Ritz-Volterra* projection of the exact solution $u(t)$ with $B(t,\tau) = B(t) - A_t(t)$, ie., $U(t) = u_h(t)$. Thus, we directly obtain the following theorem.

Theorem 5.1. Let $u(t)$ and $U(t)$ be the solutions of problems (31) and (32), respectively. Then, all conclusions of theorems in Section 2-4 keep holding for $u(t) - U(t)$ with $f(t) = g(t) + A(0)u_0$.

5.2 Parabolic Integro-Differential Equations

Consider the initial boundary value problems of parabolic integro-differential equations:

$$\begin{array}{c} u_t + A(t)u + \int_0^t B(t,\tau)u(\tau) \, d\tau = g(t), \ \ in \ J \times \Omega \\ u(0,x) = u_0(x), \ in \ \Omega \quad \ u(t,x) = 0 \, , on \ J \times \partial\Omega \end{array} \tag{34}$$

Its semidiscrete FVE approximation is defined by finding $U(t): J \to S_h$ such that

$$\begin{array}{l} (U_t, I_h^* v_h) + V(t; U, I_h^* v_h) = (g, I_h^* v_h), \ \forall v_h \in S_h \\ U(0) \in S_h \end{array} \tag{35}$$

Lemma 5.1[10]**.** We have

1) $(u_h, I_h^* v_h) = (I_h^* u_h, v_h), \ \forall u_h, v_h \in S_h$

2) $\|u_h\|_{0,h} = \sqrt{(u_h, I_h^* u_h)}$ is a equivalent norm to $\|u_h\|$ on S_h

From Lemma 5.1 we see that the mass matrix M derived from $(u_h, I_h^* v_h) = (M\vec{U}, \vec{V})$ is symmetry and positive definite. Therefore, the ordinary integro-differential equation system (35) on S_h is well posed.

In order to curry out the error analysis, we write

$$u(t) - U(t) = u(t) - u_h(t) + u_h(t) - U(t) = \eta(t) + \theta(t)$$

where $u_h(t)$ is the FVE Rize-Volterra projetion of the exact solution $u(t)$. Thus, we only need to estimate $\theta(t) \in S_h$. From equations (30), (34) and (35) we find that $\theta(t)$ satisfies

$$(\theta_t, I_h^* v_h) + V(t; \theta, I_h^* v_h) = -(\eta_t, I_h^* v_h), \ v_h \in S_h \tag{36}$$

Taking $v_h = \theta$, by Lemma 5.1, Lemma 2.2 and (12) we have

$$\frac{1}{2}\frac{d}{dt}\|\theta(t)\|_{0,h}^2 + C_0\|\theta(t)\|_1^2 + \int_0^t B(t,\tau;\theta(\tau),\theta(t))\,d\tau$$
$$\leq E(t;\theta,\theta) - (\eta_t, I_h^*\theta) \leq Ch|||\theta|||_1\|\theta\|_1 + 2\|\eta_t\|\,\|\theta\|$$

Hence, for $0 < h \leq h_0$, h_0 small, we have

$$\frac{d}{dt}\|\theta(t)\|_{0,h}^2 + \|\theta(t)\|_1^2 \leq C(\|\eta_t\|^2 + \int_0^t \|\theta(\tau)\|_1^2 d\tau)$$

Integrate and apply Gronwall lemma to obtain

$$\|\theta(t)\|^2 \leq C(\,\|\theta(0)\|^2 + \int_0^t \|\eta_t(\tau)\|^2\,d\tau)$$

Then, the triangular inequality, Theorem 3.2 and Theorem 3.3 (with $f = g - u_t$) now imply

Theorem 5.2. Let $u(t)$ and $U(t)$ be the solutions of problems (34) and (35), respectively. Then, the following L_2 norm error estimate holds.

$$\begin{aligned}\|u(t) - U(t)\| \quad &\leq \quad Ch^2\{\,\|u_0 - U(0)\| + \|g(0) - u_t(0)\|_1 + \|g(t) - u_t(t)\|_1 \\ &+ \quad (\int_0^t (\,\|g(\tau)\|_1^2 + \|u_t(\tau)\|_2^2)\,d\tau)^{\frac{1}{2}}\ \},\ t \in J\end{aligned}$$

For special choice of initial value $U(0)$, we can derive some better error estimates. Now, we choose $U(0)$ such that $\theta(0) = u_h(0) - U(0) = 0$, that is

$$A(0; u_0 - U(0), I_h^* v_h) = 0,\ \forall v_h \in S_h \tag{37}$$

Taking $v_h = \theta_t$ in (36), we obtain

$$\|\theta_t\|_{0,h}^2 + \frac{1}{2}\frac{d}{dt}V(t;\theta,I_h^*\theta) - \frac{1}{2}V_t(t;\theta,I_h^*\theta) + \frac{1}{2}V(t;\theta,I_h^*\theta_t) - \frac{1}{2}V(t;\theta_t,I_h^*\theta) = -(\eta_t, I_h^*\theta_t)$$

This can be rewrited as

$$\begin{aligned}&\|\theta_t\|_{0,h}^2 + \frac{1}{2}\frac{d}{dt}V(t;\theta,I_h^*\theta) - \frac{1}{2}V_t(t;\theta,\theta) + \frac{1}{2}V(t;\theta,\theta_t) - \frac{1}{2}V(t;\theta_t,\theta)\\ &= -(\eta_t, I_h^*\theta_t) - E_t(t;\theta,\theta) + E(t;\theta,\theta_t) - E(t;\theta_t,\theta)\end{aligned} \tag{38}$$

Note that

$$\begin{aligned}V(t;\theta,\theta_t) \quad &- \quad V(t;\theta_t,\theta) = \int_0^t B(t,\tau;\theta(\tau),\theta_t(t))\,d\tau - \int_0^t B(t,\tau;\theta_\tau(\tau),\theta(t))\,d\tau\\ &= \quad \frac{d}{dt}\int_0^t B(t,\tau;\theta(\tau),\theta(t))\,d\tau - B(t,t;\theta(t),\theta(t)) - \int_0^t B_t(t,\tau;\theta(\tau),\theta(t))\,d\tau\\ &- \quad B(t,t;\theta(t),\theta(t)) + B(0,0;\theta(0),\theta(0)) + \int_0^t B_\tau(t,\tau;\theta(\tau),\theta(t))\,d\tau\end{aligned}$$

and by Lemma 2.2 and the inverse inequality

$$|E_t(t;\theta,\theta)| + |E(t;\theta,\theta_t)| + |E(t;\theta_t,\theta)| \leq C(\,h|||\theta|||_1\|\theta\|_1 + |||\theta_t|||\;|||\theta|||_1)$$

then, integrating (38) we see that (note that $\theta(0) = 0$)

$$V(t;\theta,I_h^*\theta) \leq C\int_0^t (\,\|\eta_t\|^2 + \|\theta\|_1^2)\,d\tau$$

Hence, we have

$$\begin{aligned}C_0\|\theta(t)\|_1^2 + \int_0^t B(t,\tau;\theta(\tau),\theta(t))\,d\tau \quad &\leq \quad V(t;\theta,\theta) = V(t;\theta,I_h^*\theta) + E(t;\theta,\theta)\\ &\leq \quad C\int_0^t (\,\|\eta_t\|^2 + \|\theta\|_1^2)\,d\tau + Ch|||\theta|||_1\|\theta\|_1\end{aligned}$$

For $0 < h \leq h_0$, h_0 small, it implies by Gronwall lemma that

$$\|\theta(t)\|_1^2 \leq C \int_0^t \|\eta_t(\tau)\|^2 d\tau \tag{39}$$

Thus, we obtain the following conclusions.

Theorem 5.3. Let $u(t)$ and $U(t)$ be the solutions of problems (34) and (35), respectively, and the initial value $U(0)$ be chosen by (37). Then we have the following H^1, W_∞^1 and L_∞ norm error estimates.

$$\|u(t) - U(t)\|_1 \leq Ch|||g - u_t||| + Ch^2 (\int_0^t (\|g(\tau)\|_1^2 + \|u_t(\tau)\|_2^2)\, d\tau)^{\frac{1}{2}} \tag{40}$$

$$\|u(t) - U(t)\|_{1,\infty} \leq Ch|ln\, h|^{\frac{1}{2}} [\, \|g - u_t\|_1 + |||u|||_{2,\infty} + (\int_0^t (\|g(\tau)\|_1^2 + \|u_t(\tau)\|_2^2)\, d\tau)^{\frac{1}{2}}] \tag{41}$$

$$\|u(t) - U(t)\|_{0,\infty} \leq Ch^2\, ln\frac{1}{h} [\|g - u_t\|_1 + |||u|||_{2,\infty} + (\int_0^t (\|g(\tau)\|_1^2 + \|u_t(\tau)\|_2^2) d\tau)^{\frac{1}{2}}] \tag{42}$$

Proof. First, from (39) and Theorem 3.3 we obtain the superconvergence estimate

$$\|u_h(t) - U(t)\|_1 \leq Ch^2 (\int_0^t (\|g(\tau\|_1^2 + \|u_t(\tau)\|_2^2)\, d\tau)^{\frac{1}{2}} \tag{43}$$

Then, estimate (40) comes from Theorem 3.1 and (43), and estimate (41) comes from Theorem 4.1, (43) and the inverse inequality in S_h. For L_∞ estimate, by the weak embedding inequality in S_h, we have

$$\|u_h(t) - U(t)\|_{0,\infty} \leq C(ln\frac{1}{h})^{\frac{1}{2}} \|u_h(t) - U(t)\|_1$$

The proof is completed by using (43) and Theorem 4.2.

Our discussion in Section 5.2 is also suitable for the FVE approximation to parabolic problems which is the special case of $B(t, \tau) \equiv 0$.

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