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ORDER RESULTS OF GENERAL LINEAR METHODS FOR MULTIPLY STIFF SINGULAR PERTURBATION PROBLEMS*1)

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Abstract

In this paper we analyze the error behavior of general linear methods applied to some classes of one-parameter multiply stiff singularly perturbed problems. We obtain the global error estimate of algebraically and diagonally stable general linear methods. The main result of this paper can be viewed as an extension of that obtained by Xiao [13] for the case of Runge-Kutta methods.

Key words: Singular perturbation problem; Stiffness; General linear method; Global error estimate.

1. Introduction

Let $\langle , \rangle$ be the standard inner product on real Euclidean space and $\|\cdot\|$ the corresponding norm. The matrix norm is subordinate to the vector norm $\|\cdot\|$. Consider the following singular perturbation problems(SPPs)

$$\begin{cases} x'(t) = f(x(t), y(t)), & t \in [0, T], \\ \epsilon y'(t) = g(x(t), y(t)), & 0 < \epsilon \ll 1, \end{cases} \tag{1.1}$$

with initial values $(x(0), y(0)) \in \check{G}$ admitting a smooth solution $(x(t), y(t))$ (i.e. all derivatives of $x(t)$ and $y(t)$ up to a sufficiently high order are bounded independently of the stiffness of the problem), where $\check{G}$ is an appropriate region on $R^M \times R^N$, and the mappings $f : \check{G} \to R^M$ and $g : \check{G} \to R^N$ are sufficiently smooth and satisfy the following conditions

$$\langle f(x_1, y) - f(x_2, y), x_1 - x_2 \rangle \le \omega \|x_1 - x_2\|^2, \tag{1.2a}$$

$$\langle g(x, y_1) - g(x, y_2), y_1 - y_2 \rangle \le -\|y_1 - y_2\|^2, \tag{1.2b}$$

$$\|f(x, y_1) - f(x, y_2)\| \le L_1 \|y_1 - y_2\|, \tag{1.2c}$$

$$\|g(x_1, y) - g(x_2, y)\| \le L_2 \|x_1 - x_2\|, \tag{1.2d}$$

with moderately-sized constants ω, L_1 and L_2.

We note that the one-sized Lipschitz condition (1.2a) is weaker than the conventional Lipschitz condition

$$\|f(x_1, y) - f(x_2, y)\| \le L \|x_1 - x_2\|, \tag{1.3}$$

since (1.3) implies (1.2a) with $\omega = L$ for moderately-sized L. If the problem (1.1) satisfies (1.3) with moderately-sized L, then it is a singly stiff singular perturbation problem(SSPP) because its stiffness is only caused by the small parameter ϵ. For the problem (1.1) with $L \gg 1$, it is

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a multiply stiff singular perturbation problem(MSPP) whose stiffness is caused by the small parameter ϵ and some other factors.

Although stiff SPPs is considered as a special class of stiff initial value problems of ordinary differential equations, B-theory (cf. [3, 6, 9]) can't cover the former because of its very special structure. Recently, some developments of quantitative convergence analysis for numerical methods applied to SSPPs have been made (cf. [4, 5, 6, 11]). The main technique approach is a transformation of the system (1.1) into a series of semi-explicit differential-algebraic equtions by $\epsilon-$asymptotic expansions, in the meantime, the numerical solutions are also expanded analogously. In 1991, Lubich [10] obtained some quantitative convergence results of $A(\alpha)-$stable linear multistep methods for SSPPs by the other approach(i.e. direct approach). In 1999, Xiao [13] discussed convergence of one-leg methods and Runge-Kutta methods for MSPPs by direct approach. This paper is concerned with the error analysis of general linear methods for MSPPs by direct approach. We obtain the global error estimate of algebraically stable and diagonally stable general linear methods. Our main result (Theorem 3.3) can be considered as an extension of that obtained by Xiao [13].

2. General linear methods for SPPs

A $r-$step and $s-$stage general linear method(cf.[3, 9]) applied to (1.1) reads

$$X_i^{(n)} = h\sum_{j=1}^{s} C_{ij}^{11} f(X_j^{(n)}, Y_j^{(n)}) + \sum_{j=1}^{r} C_{ij}^{12} x_j^{(n-1)}, \quad i = 1, \ldots, s, \tag{2.1a}$$

$$\epsilon Y_i^{(n)} = h\sum_{j=1}^{s} C_{ij}^{11} g(X_j^{(n)}, Y_j^{(n)}) + \epsilon\sum_{j=1}^{r} C_{ij}^{12} y_j^{(n-1)}, \quad i = 1, \ldots, s, \tag{2.1b}$$

$$x_i^{(n)} = h\sum_{j=1}^{s} C_{ij}^{21} f(X_j^{(n)}, Y_j^{(n)}) + \sum_{j=1}^{r} C_{ij}^{22} x_j^{(n-1)}, \quad i = 1, \ldots, r, \tag{2.1c}$$

$$\epsilon y_i^{(n)} = h\sum_{j=1}^{s} C_{ij}^{21} g(X_j^{(n)}, Y_j^{(n)}) + \epsilon\sum_{j=1}^{r} C_{ij}^{22} y_j^{(n-1)}, \quad i = 1, \ldots, r, \tag{2.1d}$$

$$\xi_n^{(x)} = \sum_{j=1}^{r} \beta_j x_j^{(n)}, \qquad \xi_n^{(y)} = \sum_{j=1}^{r} \beta_j y_j^{(n)}, \tag{2.1e}$$

here $h > 0$ is the fixed stepsize, the coefficients C_{ij}^{IJ} and β_j are real constants, $X_i^{(n)}, x_i^{(n)}$ and $\xi_n^{(x)}$ are approximations to $x(t_n+\mu_i h), H_i^{(x)}(t_n+\nu_i h)$ and $x(t_n+\eta h)$, respectively, $Y_i^{(n)}, y_i^{(n)}$ and $\xi_n^{(y)}$ are approximations to $y(t_n+\mu_i h), H_i^{(y)}(t_n+\nu_i h)$ and $y(t_n+\eta h)$, respectively. $H_i^{(x)}(t_n+\nu_i h)$ and $H_i^{(y)}(t_n+\nu_i h)$ denote a piece of information about the true solution $x(t)$ and $y(t)$ respectively, i.e.

$$H_i^{(x)}(t) = a_i x(t) + hb_i x'(t), \qquad H_i^{(y)}(t) = a_i y(t) + hb_i y'(t), \qquad i = 1, \ldots, r,$$

a_i, b_i, μ_i, ν_i and η are real constants.

For any matrix H, let $\bar{H} = H \otimes I_M, \tilde{H} = H \otimes I_N$, where $\otimes$ denotes the Kronecker product of two matrices, I_l denotes an $l \times l$ unit matrix. Let $C_{IJ} = [C_{ij}^{IJ}]$ and $\beta = [\beta_1, \ldots, \beta_r]$, the process (2.1) can be written in more compact form

$$X^{(n)} = h\bar{C}_{11} F(X^{(n)}, Y^{(n)}) + \bar{C}_{12} x^{(n-1)}, \tag{2.2a}$$

$$\epsilon Y^{(n)} = h\tilde{C}_{11} G(X^{(n)}, Y^{(n)}) + \epsilon\tilde{C}_{12} y^{(n-1)}, \tag{2.2b}$$

$$x^{(n)} = h\bar{C}_{21} F(X^{(n)}, Y^{(n)}) + \bar{C}_{22} x^{(n-1)}, \tag{2.2c}$$

$$\epsilon y^{(n)} = h\tilde{C}_{21}G(X^{(n)}, Y^{(n)}) + \epsilon\tilde{C}_{22}y^{(n-1)}, \tag{2.2d}$$

$$\xi_n^{(x)} = \bar{\beta}x^{(n)}, \quad \xi_n^{(y)} = \tilde{\beta}y^{(n)}, \tag{2.2e}$$

where

$$X^{(n)} = (X_1^{(n)^T}, \ldots, X_s^{(n)^T})^T \in R^{Ms}, \quad x^{(n)} = (x_1^{(n)^T}, \ldots, x_r^{(n)^T})^T \in R^{Mr},$$
$$F(X^{(n)}, Y^{(n)}) = (f(X_1^{(n)}, Y_1^{(n)})^T, \ldots, f(X_s^{(n)}, Y_s^{(n)})^T)^T \in R^{Ms},$$

and likewise for $Y^{(n)}, y^{(n)}$ and $G(X^{(n)}, Y^{(n)})$. For simplicity, we write

$$w_0 = (a_1, \ldots, a_r)^T, \qquad \mu = (\mu_1, \ldots, \mu_s)^T, \qquad \nu = (\nu_1, \ldots, \nu_r)^T,$$
$$e_l = (1, \ldots, 1)^T \in R^l, \quad w_k(\nu) = (a_1\nu_1^k + kb_1\nu_1^{k-1}, \ldots, a_r\nu_r^k + kb_r\nu_r^{k-1})^T.$$

Introduce the pre-consistent conditions(cf.[1, 9])

$$C_{12}w_0 = e_s, \quad C_{22}w_0 = w_0, \quad \beta w_0 = 1, \tag{2.3}$$

and the simplifying conditions

$$\left\{\begin{array}{ll} \hat{A}(q): & \beta w_j(\nu) = \eta^j, \qquad 1 \le j \le q, \\ B(q): & w_j(\nu) = jC_{21}\mu^{j-1} + C_{22}w_j(\nu - e_r), \qquad 1 \le j \le q, \\ C(q): & \mu^j = jC_{11}\mu^{j-1} + C_{12}w_j(\nu - e_r), \qquad 1 \le j \le q, \end{array}\right. \tag{2.4}$$

where $\mu^i = (\mu_1^i, \ldots, \mu_s^i)^T$.

Definition 2.1[2,8]**.** A general linear method (2.1) is said to have stage order p if it satisfies the condition (2.3) and the simplifying conditions $\hat{A}(p-1), B(p)$ and $C(p)$.

Definition 2.2[1]**.** A general linear method (2.1) is called algebraically stable, if there exist a symmetric and positive definite $r \times r$ matrix G and a non-negative diagonal $s \times s$ matrix D, such that the matrix

$$H = \begin{pmatrix} G - C_{22}^T G C_{22} & C_{12}^T D - C_{22}^T G C_{21} \\ DC_{12} - C_{21}^T G C_{22} & DC_{11} + C_{11}^T D - C_{21}^T G C_{21} \end{pmatrix}$$

is non-negative definite.

Definition 2.3[9]**.** A general linear method (2.1) is said to be diagonally stable, if there exists a diagonal $s \times s$ matrix $E > 0$ such that the matrix

$$EC_{11} + C_{11}^T E$$

is positive definite.

3. Main result and its proof

In order to prove our result, we need the following Lemmas, and suppose that σ in the Lemmas is a given real constant.

Lemma 3.1. *Suppose the method (2.1) is diagonally stable. Then there exist positive constants h_1, d_1 and d_2 which depend only on the method such that for any given $h > 0, z \in E_\sigma$ with $h\sigma \le h_1$, $\bar{I}_s - h\bar{C}_{11}z$ is regular and*

$$\|(\bar{I}_s - h\bar{C}_{11}z)^{-1}\| \le d_1, \quad \|h\bar{C}_{21}z(\bar{I}_s - h\bar{C}_{11}z)^{-1}\| \le d_2,$$

where $E_\sigma = \{z : z = blockdiag(z_1, \ldots, z_s), z_i \in R^{M\times M}, \langle\zeta, z_i\zeta\rangle \le \sigma\|\zeta\|^2, i = 1, \ldots, s, \forall\zeta \in R^M\}$.

The *proof* is omitted, whose main techniques are similar to those of ([12], Lemma 2.1).

Lemma 3.2. *Suppose the method (2.1) is algebraically stable and diagonally stable. Then there exist positive constants h_2 and d_3 which depend only on the method such that for any given $h > 0, z \in E_\sigma$ with $h\sigma \le h_2$, $\bar{I}_s - h\bar{C}_{11}z$ is regular and*

$$\|\bar{C}_{22} + h\bar{C}_{21}z(\bar{I}_s - h\bar{C}_{11}z)^{-1}\bar{C}_{12}\|_{\bar{G}} \le 1 + d_3h\sigma\delta(\sigma),$$

where $\delta(\sigma)=1$ for $\sigma>0$ and $\delta(\sigma)=0$ for $\sigma\le 0$, the matrix norm $\|\cdot\|_{\bar G}$ is subordinate to the vector norm $\|\cdot\|_{\bar G}$, the vector norm $\|\cdot\|_{\bar G}$ defined by

$$\|u\|_{\bar G}^2=\langle u,\bar G u\rangle,\qquad u\in R^{Mr}.$$

Proof. By Lemma 3.1 there exists $h_2>0$ such that for $h\sigma\le h_2, \bar I_s-h\bar C_{11}z$ is regular. Let $L(z)=\bar C_{22}+h\bar C_{21}z(\bar I_s-h\bar C_{11}z)^{-1}\bar C_{12}$. In view of algebraic stability, we have

$$\begin{aligned}\|L(z)u\|_{\bar G}^2 &= -[u^T,v^T]\bar H[u^T,v^T]^T+u^T\bar G u+2(\bar C_{12}u+\bar C_{11}v)^T\bar D v\\ &\le u^T\bar G u+2(\bar C_{12}u+\bar C_{11}v)^T\bar D v,\end{aligned}\tag{3.1}$$

where $v=hz(\bar I_s-h\bar C_{11}z)^{-1}\bar C_{12}u$. Let $w=(\bar I_s-h\bar C_{11}z)^{-1}\bar C_{12}u$, we can obtain

$$(\bar C_{12}u+\bar C_{11}v)^T\bar D v=hw^T\bar D z w\le\begin{cases} h\sigma\max\limits_{1\le i\le s} d_i\|w\|^2, & \sigma>0,\\ 0, & \sigma\le 0.\end{cases}\tag{3.2}$$

The conclusion follows from (3.1), (3.2), Lemma 3.1 and the equivalence of norms.

Theorem 3.3. *Suppose that the general linear method (2.1) is of stage order $p\ge 1$, algebraically stable and diagonally stable, and satisfies $\rho(C_{22}-C_{21}C_{11}^{-1}C_{12})<1$ and that the eigenvalues of C_{11} have positive real part. Then when this method is applied to the problem (1.1), the global error estimates hold for $\epsilon\le D_0h^2, h\le h_0, nh\le T$*

$$\|x(t_n+\eta h)-\xi_n^{(x)}\|\le D_1(\|\Delta x^{(0)}\|+\epsilon\|\Delta y^{(0)}\|+h^p),$$
$$\|y(t_n+\eta h)-\xi_n^{(y)}\|\le D_1(\|\Delta x^{(0)}\|+(\tilde\epsilon+\alpha^n)\|\Delta y^{(0)}\|+h^p),$$

where $\rho(C_{22}-C_{21}C_{11}^{-1}C_{12})$ is the spectral radius of the matrix $C_{22}-C_{21}C_{11}^{-1}C_{12}$, α is a constant and $0\le\alpha<1$, and $\tilde\epsilon=\epsilon(1+\frac{1}{h})$.

Proof. Let

$$X(t)=(x(t+\mu_1h)^T,\dots,x(t+\mu_sh)^T)^T\in R^{Ms},$$
$$H^{(x)}(t)=(H_1^{(x)}(t+\nu_1h)^T,\dots,H_r^{(x)}(t+\nu_rh)^T)^T\in R^{Mr},$$
$$F(X(t),Y(t))=(f(x(t+\mu_1h),y(t+\mu_1h))^T,\dots,f(x(t+\mu_sh),y(t+\mu_sh))^T)^T\in R^{Ms},$$

and likewise for $Y(t)$, $H^{(y)}(t)$ and $G(X(t),Y(t))$. Since the method (2.1) is of stage order p, we get

$$X(t_n)=h\bar C_{11}F(X(t_n),Y(t_n))+\bar C_{12}H^{(x)}(t_n-h)+\mathcal O(h^{p+1}),\tag{3.3a}$$
$$\epsilon Y(t_n)=h\tilde C_{11}G(X(t_n),Y(t_n))+\epsilon\tilde C_{12}H^{(y)}(t_n-h)+\mathcal O(\epsilon h^{p+1}),\tag{3.3b}$$
$$H^{(x)}(t_n)=h\bar C_{21}F(X(t_n),Y(t_n))+\bar C_{22}H^{(x)}(t_n-h)+\mathcal O(h^{p+1}),\tag{3.3c}$$
$$\epsilon H^{(y)}(t_n)=h\tilde C_{21}G(X(t_n),Y(t_n))+\epsilon\tilde C_{22}H^{(y)}(t_n-h)+\mathcal O(\epsilon h^{p+1}),\tag{3.3d}$$
$$x(t_n+\eta h)=\bar\beta H^{(x)}(t_n)+\mathcal O(h^p),\tag{3.3e}$$
$$y(t_n+\eta h)=\tilde\beta H^{(y)}(t_n)+\mathcal O(h^p).\tag{3.3f}$$

Let

$$\Delta X^{(n)}=X(t_n)-X^{(n)},\qquad \Delta x^{(n)}=H^{(x)}(t_n)-x^{(n)},$$
$$\Delta F^{(n)}=F(X(t_n),Y(t_n))-F(X^{(n)},Y^{(n)}),\quad \Delta x_{n+\eta}=x(t_n+\eta h)-\xi_n^{(x)},$$

and likewise for $\Delta Y^{(n)},\Delta y^{(n)},\Delta G^{(n)}$ and $\Delta y_{n+\eta}$. It follows from (2.2) and (3.3) that

$$\Delta X^{(n)}=h\bar C_{11}\Delta F^{(n)}+\bar C_{12}\Delta x^{(n-1)}+\mathcal O(h^{p+1}),\tag{3.4a}$$
$$\epsilon\Delta Y^{(n)}=h\tilde C_{11}\Delta G^{(n)}+\epsilon\tilde C_{12}\Delta y^{(n-1)}+\mathcal O(\epsilon h^{p+1}),\tag{3.4b}$$
$$\Delta x^{(n)}=h\bar C_{21}\Delta F^{(n)}+\bar C_{22}\Delta x^{(n-1)}+\mathcal O(h^{p+1}),\tag{3.4c}$$
$$\epsilon\Delta y^{(n)}=h\tilde C_{21}\Delta G^{(n)}+\epsilon\tilde C_{22}\Delta y^{(n-1)}+\mathcal O(\epsilon h^{p+1}),\tag{3.4d}$$

$$\Delta x_{n+\eta} = \bar{\beta}\Delta x^{(n)} + \mathcal{O}(h^p), \tag{3.4e}$$

$$\Delta y_{n+\eta} = \tilde{\beta}\Delta y^{(n)} + \mathcal{O}(h^p). \tag{3.4f}$$

By the assumptions in the theorem, C_{11} is nonsingular, we can compute $\Delta F^{(n)}$ and $\Delta G^{(n)}$ from (3.4a) and (3.4b) and insert them into (3.4c) and (3.4d). This gives

$$\Delta x^{(n)} = \bar{Q}\Delta x^{(n-1)} + \bar{C}_{21}\bar{C}_{11}^{-1}\Delta X^{(n)} + \mathcal{O}(h^{p+1}), \tag{3.5a}$$

$$\Delta y^{(n)} = \tilde{Q}\Delta y^{(n-1)} + \tilde{C}_{21}\tilde{C}_{11}^{-1}\Delta Y^{(n)} + \mathcal{O}(h^{p+1}), \tag{3.5b}$$

where

$$\bar{Q} = \bar{C}_{22} - \bar{C}_{21}\bar{C}_{11}^{-1}\bar{C}_{12}, \quad \tilde{Q} = \tilde{C}_{22} - \tilde{C}_{21}\tilde{C}_{11}^{-1}\tilde{C}_{12}.$$

On the other hand,

$$\Delta F^{(n)} = F_X\Delta X^{(n)} + F_Y\Delta Y^{(n)}, \quad \Delta G^{(n)} = G_X\Delta X^{(n)} + G_Y\Delta Y^{(n)}, \tag{3.6}$$

where

$$\begin{aligned} F_X = blockdiag \quad &(\textstyle\int_0^1 f_x(X_1^{(n)} + \theta(x(t_n+\mu_1 h) - X_1^{(n)}), y(t_n+\mu_1 h))d\theta, \ldots, \\ &\textstyle\int_0^1 f_x(X_s^{(n)} + \theta(x(t_n+\mu_s h) - X_s^{(n)}), y(t_n+\mu_s h))d\theta), \end{aligned}$$

$$\begin{aligned} F_Y = blockdiag \quad &(\textstyle\int_0^1 f_y(X_1^{(n)}, Y_1^{(n)} + \theta(y(t_n+\mu_1 h) - Y_1^{(n)}))d\theta, \ldots, \\ &\textstyle\int_0^1 f_y(X_s^{(n)}, Y_s^{(n)} + \theta(y(t_n+\mu_s h) - Y_s^{(n)}))d\theta), \end{aligned}$$

and likewise for G_X and G_Y. Using (1.2b), diagonal stability of the method and the fact that the eigenvalues of C_{11} have positive real parts, by means of the same technique used in the proof of ([12], Lemma 2.1), we have for any given $h > 0$

$$\|\frac{h}{\epsilon}(\tilde{I}_s - \frac{h}{\epsilon}\tilde{C}_{11}G_Y)^{-1}\| \le D_3. \tag{3.7}$$

It follows from (3.6) and (3.4b) that

$$\Delta Y^{(n)} = \frac{h}{\epsilon}(\tilde{I}_s - \frac{h}{\epsilon}\tilde{C}_{11}G_Y)^{-1}(\frac{\epsilon}{h}\tilde{C}_{12}\Delta y^{(n-1)} + \tilde{C}_{11}G_X\Delta X^{(n)} + \mathcal{O}(\epsilon h^p)). \tag{3.8}$$

Inserting (3.6) and (3.8) into (3.4a) we obtain

$$\begin{aligned} (\bar{I}_s - h\bar{C}_{11}F_X)\Delta X^{(n)} \quad &= h\bar{C}_{11}F_Y\tfrac{h}{\epsilon}(\tilde{I}_s - \tfrac{h}{\epsilon}\tilde{C}_{11}G_Y)^{-1}\tilde{C}_{11}G_X\Delta X^{(n)} + \bar{C}_{12}\Delta x^{(n-1)} + \\ &h\bar{C}_{11}F_Y(\tilde{I}_s - \tfrac{h}{\epsilon}\tilde{C}_{11}G_Y)^{-1}(\tilde{C}_{12}\Delta y^{(n-1)} + \mathcal{O}(h^{p+1})) + \mathcal{O}(h^{p+1}). \end{aligned} \tag{3.9}$$

It follows from (3.7), (3.8), (3.9), the condition (1.2a) and Lemma 3.1 that for $h \le h_0$

$$\|\Delta X^{(n)}\| \le D_4(\|\Delta x^{(n-1)}\| + \epsilon\|\Delta y^{(n-1)}\| + \epsilon h^{p+1} + h^{p+1}), \tag{3.10a}$$

$$\|\Delta Y^{(n)}\| \le D_4(\frac{\epsilon}{h}(\|\Delta y^{(n-1)}\| + h^{p+1}) + \|\Delta x^{(n-1)}\| + \epsilon\|\Delta y^{(n-1)}\| + \epsilon h^{p+1} + h^{p+1}). \tag{3.10b}$$

By (3.4c) and (3.6) we have

$$\Delta x^{(n)} = \bar{C}_{22}\Delta x^{(n-1)} + h\bar{C}_{21}F_X\Delta X^{(n)} + w_n, \tag{3.11}$$

where

$$\|w_n\| \le D_5(h\|\Delta Y^{(n)}\| + h^{p+1}).$$

Inserting (3.9) into (3.11) we get

$$\begin{aligned} \Delta x^{(n)} \quad &= (\bar{C}_{22} + h\bar{C}_{21}F_X(\bar{I}_s - h\bar{C}_{11}F_X)^{-1}\bar{C}_{12})\Delta x^{(n-1)} + w_n \\ &+h\bar{C}_{21}F_X(\bar{I}_s - h\bar{C}_{11}F_X)^{-1}h\bar{C}_{11}F_Y\tfrac{h}{\epsilon}(\tilde{I}_s - \tfrac{h}{\epsilon}\tilde{C}_{11}G_Y)^{-1}\tilde{C}_{11}G_X\Delta X^{(n)} \\ &+h\bar{C}_{21}F_X(\bar{I}_s - h\bar{C}_{11}F_X)^{-1}(h\bar{C}_{11}F_Y(\tilde{I}_s - \tfrac{h}{\epsilon}\tilde{C}_{11}G_Y)^{-1}(\tilde{C}_{12}\Delta y^{(n-1)} \\ &+\mathcal{O}(h^{p+1})) + \mathcal{O}(h^{p+1})). \end{aligned} \tag{3.12}$$

Let $Q = C_{22} - C_{21}C_{11}^{-1}C_{12}$, by $\rho(Q) < 1$ and the properties of Kronecker products([7]), we have $\rho(\tilde{Q}) = \rho(Q \otimes I_N) < 1$. Therefore, there exists a norm $\|\cdot\|_*$ in R^{rN} such that the corresponding matrix norm $\|\tilde{Q}\|_* = sup_{\|z\|_*=1}\|\tilde{Q}z\|_* < 1$. It follows from (3.7), (3.10), (3.12), the condition (1.2a), Lemma 3.1-3.2 and the equivalence of norms that for $h \leq h_0$

$$\|\Delta x^{(n)}\|_{\bar{G}} \leq (1 + \mathcal{O}(h))\|\Delta x^{(n-1)}\|_{\bar{G}} + D_6(\epsilon\|\Delta y^{(n-1)}\|_* + h^{p+1} + \epsilon h^{p+1}). \qquad (3.13a)$$

By (3.10b) and (3.5b) we obtain

$$\|\Delta y^{(n)}\|_* \leq (\|\tilde{Q}\|_* + \mathcal{O}(\tilde{\epsilon}))\|\Delta y^{(n-1)}\|_* + D_7(\|\Delta x^{(n-1)}\|_{\bar{G}} + h^{p+1} + \epsilon h^p). \qquad (3.13b)$$

Let $\alpha = \|\tilde{Q}\|_*$, we rewrite (3.13) as

$$\begin{pmatrix} \|\Delta x^{(n)}\|_{\bar{G}} \\ \|\Delta y^{(n)}\|_* \end{pmatrix} \leq \begin{pmatrix} 1 + \mathcal{O}(h) & \mathcal{O}(\epsilon) \\ \mathcal{O}(1) & \alpha + \mathcal{O}(\tilde{\epsilon}) \end{pmatrix} \begin{pmatrix} \|\Delta x^{(n-1)}\|_{\bar{G}} \\ \|\Delta y^{(n-1)}\|_* \end{pmatrix} + \psi \begin{pmatrix} h \\ 1 \end{pmatrix}.$$

where $\psi = \mathcal{O}(h^p) + \mathcal{O}(\epsilon h^p)$. By means of the technique in ([6], Lemma 2.9, pp.430-431), we easily obtain for $\epsilon \leq D_0 h^2$, $h \leq h_0$ and $nh \leq T$

$$\|\Delta x^{(n)}\|_{\bar{G}} \leq D_8(\|\Delta x^{(0)}\|_{\bar{G}} + \epsilon\|\Delta y^{(0)}\|_* + h^p + \epsilon h^p), \qquad (3.14a)$$

$$\|\Delta y^{(n)}\|_* \leq D_8(\|\Delta x^{(0)}\|_{\bar{G}} + (\tilde{\epsilon} + \alpha^n)\|\Delta y^{(0)}\|_* + h^p + \epsilon h^p). \qquad (3.14b)$$

The conclusion now follows from (3.4e), (3.4f), (3.14) and the equivalence of norms.

Remark 3.4. (2.2a) and (2.2b) constitute a nonlinear system. The Jacobian of the system is of the form

$$\begin{pmatrix} \bar{I}_s - h\bar{C}_{11}Z_X & \mathcal{O}(h) \\ \mathcal{O}(1) & \frac{\epsilon}{h}\tilde{I}_s - \tilde{C}_{11}Z_Y \end{pmatrix},$$

where

$$Z_X = blockdiag(f_x(X_1^{(n)}, Y_1^{(n)}), \cdots, f_x(X_s^{(n)}, Y_s^{(n)})),$$
$$Z_Y = blockdiag(g_y(X_1^{(n)}, Y_1^{(n)}), \cdots, g_y(X_s^{(n)}, Y_s^{(n)})).$$

By Lemma 3.1 and condition (1.2a), we have for $h \leq h_0$

$$\|(\bar{I}_s - h\bar{C}_{11}Z_X)^{-1}\| \leq d_1.$$

We can show similarly as (3.7) for any given $h > 0$

$$\|(\frac{\epsilon}{h}\tilde{I}_s - \tilde{C}_{11}Z_Y)^{-1}\| \leq d_4.$$

Hence, the nonlinear system (2.2a)-(2.2b) possesses a locally unique solution.

Remark 3.5. Specializing Theorem 3.3 to the case of Runge-Kutta methods, we obtain immediately the related result obtained by Xiao [13].

4. Examples

Example 4.1. Consider s stage Runge-Kutta methods denoted by

$$\begin{array}{c|c} c & A \\ \hline & b^T \end{array} \qquad (4.1)$$

The methods (4.1) can be written as general linear methods with

$$C_{11} = A, \quad C_{12} = (1, \cdots, 1)^T \in R^s, \quad C_{21} = b^T, \quad C_{22} = (1) \in R^1, \quad \beta = (1) \in R^1.$$

Consequently $\rho(C_{22} - C_{21}C_{11}^{-1}C_{12}) =| 1 - b^T A^{-1} e |$. It is well known that both Radau IA and Radau IIA methods are algebraically stable, diagonally stable and satisfy $1 - b^T A^{-1} e = 0$. We have verified that the eigenvalues of A of the methods have positive real part for $s \leq 5$. We note that s stage Radau IA methods is of stage order $p = s - 1$ and Radau IIA methods is of stage order $p = s$. Hence, Radau IA and Radau IIA methods all satisfy the assumption conditions in Theorem 3.3, and $p = s-1, s (s \le 5)$, respectively.

We also can verify that 2 stage Lobatto IIIC method satisfies the assumption conditions in Theorem 3.3.

Example 4.2. Consider 2 step 1 stage Runge-Kutta methods [9]:

$$\begin{aligned} &C_{11} = (c), \quad C_{12} = \left(\tfrac{2a}{1+a}, \tfrac{1-a}{1+a}\right), \quad C_{21} = \begin{pmatrix} 0 \\ 1+a \end{pmatrix}, \quad C_{22} = \begin{pmatrix} 0 & 1 \\ a & 1-a \end{pmatrix}, \\ &\beta = (0,1), \quad \mu = (u), \quad \nu = (1,2)^T, \quad \eta = 2, \quad w_0 = (1,1)^T, \quad w_k(\nu) = (1, 2^k)^T, \end{aligned} \tag{4.2}$$

where $0 < a \le 1, c > \frac{1+3a}{2(1+a)}$ and $u = c + \frac{1-a}{1+a}$. The methods have been proved in [9] to be of stage order $p = 1$, algebraically stable, diagonally stable and satisfy $\rho(C_{22} - C_{21}C_{11}^{-1}C_{12}) < 1$. We note that the eigenvalue c of C_{11} is positive. Hence, the methods (4.2) satisfy the assumption conditions in Theorem 3.3.

Example 4.3. Consider 2 step 2 stage Runge-Kutta methods [9]:

$$\begin{aligned} &C_{11} = ([\mu, \mu^2] - C_{12}U)V^{-1}, \quad C_{21} = \begin{pmatrix} 0 & 0 \\ \gamma_1 & \gamma_2 \end{pmatrix}, \quad C_{22} = \begin{pmatrix} 0 & 1 \\ \alpha_1 & \alpha_2 \end{pmatrix}, \\ &\beta = (0,1), \quad C_{12} = \begin{pmatrix} \frac{4a(\mu_2-1)}{2(1+a)\mu_2-(3+a)} & \frac{(1-a)(2\mu_2-3)}{2(1+a)\mu_2-(3+a)} \\ \frac{4a(\mu_1-1)}{2(1+a)\mu_1-(3+a)} & \frac{(1-a)(2\mu_1-3)}{2(1+a)\mu_1-(3+a)} \end{pmatrix}, \end{aligned} \tag{4.3}$$

where

$$\begin{gathered} \alpha = (\alpha_1, \alpha_2)^T = (a, 1-a)^T, \quad \mu = (\mu_1, \mu_2)^T = \left(u, \left(\tfrac{7+a}{3} - \tfrac{3+a}{2}u\right)/\left(\tfrac{3+a}{2} - (1+a)u\right)\right)^T, \\ \gamma = (\gamma_1, \gamma_2)^T = \tfrac{1}{\mu_2-\mu_1}\left((1+a)\mu_2 - \tfrac{3+a}{2}, -(1+a)\mu_1 + \tfrac{3+a}{2}\right)^T, \quad \nu = (1,2)^T, \\ \eta = r, \quad U = \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}, \quad V = \begin{pmatrix} 1 & 2\mu_1 \\ 1 & 2\mu_2 \end{pmatrix}, \quad w_0 = (1,1)^T \in R^r, \\ w_k(\nu) = (1, 2^k)^T \in R^r, \end{gathered}$$

here a and u are parameters and $a \in (0,1]$. For $u \ge 2$, Li [9] has proved the methods (4.3) are of stage order $p = 2$, algebraically stable, diagonally stable and satisfy $\rho(C_{22} - C_{21}C_{11}^{-1}C_{12}) < 1$. For $2 \le u \le 3$, we have verified that the eigenvalues of C_{11} have positive real part by means of computer search. Hence, for $0 < a \le 1$ and $2 \le u \le 3$, the methods (4.3) satisfy the assumption conditions in Theorem 3.3.

5. Numerical results

In order to illustrate the result obtained in previous section, we apply 2 stage Radau IIA, 2 stage Lobatto IIIC and the method (4.2) to problem (5.1), respectively. For method (4.2), we choose $a = 0.5, c = \frac{1+3a}{2(1+a)} + 0.5$. Let $\epsilon = 10^{-6}$. We denote by err_x and err_y the global errors of $x-$ and $y-$components at $T = 2$, respectively. Numerical results (i.e., err_x and err_y) are listed in Table 1. The exact solution is $x(2) \approx -3.4980578720409565 \times 10^{-4}, y(2) \approx 0.1349856126373868$, which is obtained by applying 3 stage Radau IIA method with $h = 0.0001$.

Example 5.1. Consider the nonlinear problem

$$\begin{cases} x'(t) = -1000x(t) + y^2(t) - e^{-\frac{t}{2}}, \quad t > 0, \\ \epsilon y'(t) = x(t) - y(t) + e^{-t}, \quad t > 0, \\ x(0) = 1, y(0) = 1. \end{cases} \tag{5.1}$$

Table 1. Numerical results for problem (5.1)

h	2 stage Radau IIA		2 stage Lobatto IIIC		method (4.2)	
	err_x	err_y	err_x	err_y	err_x	err_y
0.2	5.4×10^{-10}	1.2×10^{-9}	1.1×10^{-9}	1.3×10^{-8}	3.6×10^{-4}	1.9×10^{-3}
0.1	1.2×10^{-10}	2.7×10^{-10}	7.4×10^{-10}	6.3×10^{-9}	5.7×10^{-7}	4.7×10^{-4}
0.05	2.6×10^{-11}	6.5×10^{-11}	4.1×10^{-10}	3.0×10^{-9}	1.3×10^{-8}	1.2×10^{-4}

It is clear that the results given by Table 1 confirm Theorem 3.3.

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