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LONG TIME ASYMPTOTIC BEHAVIOR OF SOLUTIONS OF EXPLICIT DIFFERENCE SCHEME FOR SEMILINEAR PARABOLIC EQUATIONS*[1)]

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Abstract

In this paper we prove that the solution of explicit difference scheme for a class of semilinear parabolic equations converges to the solution of difference schemes for the corresponding nonlinear elliptic equations in H^1 norm as $t \to \infty$. We get the long time asymptotic behavior of the discrete solutions which is interested in comparing to the case of continuous solutions.

Key words: Asymptotic behavior, Explicit difference scheme, Semilinear prarbolic equations.

1. Introduction

Let Ω be a bounded domain in R^2, $\Omega = \{0 \le x \le l, 0 \le y \le l\}$, and assume $f(x,y) \in L^\infty(\Omega), u_0(x,y) \in H^2(\Omega) \cap H_0^1(\Omega), \phi(u) \in C^1(R^1)$ satisfies

$$0 \le \phi^{'}(u) \le \mu_1 \mid u \mid^k + \mu_2,$$

where k, μ_1 and μ_2 are positive constants.

We consider the following initial-boundary value problem:

$$\begin{cases} \dfrac{\partial u}{\partial t} = \triangle u - \phi(u) + f(x,y) \quad in \quad \Omega \times R_+ \\ u \mid_{\partial\Omega} = 0 \\ u \mid_{t=0} = u_0(x,y), \quad (x,y) \in \Omega. \end{cases} \tag{1.1}$$

where $\triangle u = \dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2}$ is Laplace's Operator.

By the usual approach([1],[2],[3],[4])we can get the global existence of the solution of (1.1). Furthermore, the solution of (1.1) converges to the solution of the following nonlinear elliptic equations (1.2) as $t \to \infty$.

$$\begin{cases} \triangle u - \phi(u) + f(x,y) = 0 \qquad in \quad \Omega \\ u \mid_{\partial\Omega} = 0. \end{cases} \tag{1.2}$$

In [6],[7], the authors discussed the explicit scheme for (1.1) as $f(x,y) = 0$ and only the estimate in L^2 for discrete solution was obtained. In this paper we consider the asymptotic behavior of

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discrete solution of explicit difference scheme for (1.1) which is very interested in comparing to the case of continuous solution. We also prove that the solution of explicit difference scheme for (1.1) converges to the solution of difference scheme for (1.2) as $t \to \infty$, the estimate in H^1 for discrete solution is gotten.

2. Finite Difference Scheme

Let $h, \Delta t_n$ be the space stepsize and the time stepsize respectively, $h = \frac{l}{J}$, where J is an integer. We denote the discrete function which take value w_{ij}^n at $(x_i, y_j, n\triangle t)$ $(x_i = ih, y_j = jh, 0 \le i, j \le J, n = 0, 1, 2, \cdots)$ by w_h^n, define

$$\triangle_h^1 w_{ij}^n = \frac{w_{i+1,j}^n + w_{i-1,j}^n - 2w_{ij}^n}{h^2},$$

$$\triangle_h^2 w_{ij}^n = \frac{w_{i,j+1}^n + w_{i,j-1}^n - 2w_{ij}^n}{h^2}.$$

We introduce the following notations:

$$\|w_h^n\|^2 = \sum_{0\le i,j\le J} (w_{ij})^2 h^2,$$

$$\|\delta w_h^n\|^2 = \sum_{0\le i,j\le J-1} [(w_{i+1,j}^n - w_{ij}^n)^2 + (w_{i,j+1}^n - w_{ij}^n)^2],$$

$$\|\triangle w_h^n\|^2 = \sum_{1\le i,j\le J-1} (\triangle_h^1 w_{ij}^n + \triangle_h^2 w_{ij}^n)^2 h^2.$$

The explicit difference equation associate with (1.1) is:

$$\frac{u_{ij}^{n+1} - u_{ij}^n}{\Delta t_n} = \triangle_h^1 u_{ij}^n + \triangle_h^2 u_{ij}^n - \phi(u_{ij}^n) + f_{ij} \tag{2.1.1}$$

for $i, j = 1, \cdots, J-1$ and $n = 1, 2, \cdots$, where $f_{ij} = f(x_i, y_j)$. The boundary condition of (2.1) is of the form

$$u_{i,0}^n = u_{i,J}^n = u_{0,j}^n = u_{J,j}^n = 0, \quad 0 \le i, j \le J. \tag{2.1.2}$$

and the initial condition is

$$u_{ij}^0 = u_0(x_i, y_j), \quad 0 \le i, j \le J. \tag{2.1.3}$$

The difference equation corresponding to (1.2) is:

$$\triangle_h^1 u_{ij}^* + \triangle_h^2 u_{ij}^* - \phi(u_{ij}^*) + f_{ij} = 0 \tag{2.2.1}$$

for $i, j = 1, \cdots, J-1$. The boundary condition of (2.2) is of the form

$$u_{i,0}^* = u_{i,J}^* = u_{0,j}^* = u_{J,j}^* = 0, 0 \le i, j \le J. \tag{2.2.2}$$

Let the discrete function u_h^n and u_h^* be the solution of difference equation (2.1) and (2.2) respectively. For u_h^* we have the same notations to the previous. For $n = 0, 1, 2, \cdots$, the discrete function v_h^n is defined as $v_{ij}^n = u_{ij}^n - u_{ij}^*, i, j = 0, 1, \cdots, J$. Then v_h^n satisfy

$$\frac{v_{ij}^{n+1} - v_{ij}^n}{\Delta t_n} = \triangle_h^1 v_{ij}^n + \triangle_h^2 v_{ij}^n - [\phi(u_{ij}^n) - \phi(u_{ij}^*)] \tag{2.3}$$

for $i, j = 1, \cdots, J-1$ and $n = 0, 1, 2, \cdots$ Obviously, $v_{i,0}^n = v_{i,J}^n = v_{0,j}^n = v_{J,j}^n = 0, 0 \leq i, j \leq J, n = 0, 1, 2, \cdots$

3. Preliminary Results

Lemma 1. *For any discrete function $w_h = \{w_{ij} \mid i, j = 0, 1, \cdots, J\}$ satisfying the homogeneous discrete boundary condition $w_{i,0} = w_{J,0} = w_{0,j} = w_{J,j} = 0, 0 \leq i, j \leq J$, we have*

$$\|w_h\| \leq k_1 \|\delta w_h\|, \quad \|\delta w_h\| \leq k_1 \|\triangle w_h\|,$$

$$\|\delta w_h\| \leq \frac{8}{h^2} \|w_h\|^2, \quad \|\triangle w_h\| \leq \frac{8}{h^2} \|\delta w_h\|^2,$$

where k_1 is a constant independent of w_h. The proof is analogous to [5].

Lemma 2. *Let the discrete function $u_h^* = \{u_{ij}^* \mid 0 \leq i, j \leq J\}$ be the solution of the difference equation (2.2), there is a constant k_2 independent of h such that*

$$\max_{0 \leq i,j \leq J} \mid u_{ij}^* \mid \leq k_2.$$

Proof. We choose constants $a > \dfrac{\|f\|_{L^\infty} + \mid \phi(0) \mid}{4}$ such that

$$u_{ij}^* = \psi_{ij} + a[x_i(l - x_i) + y_j(l - y_j)],$$

then

$$\triangle_h^1 \psi_{ij} + \triangle_h^2 \psi_{ij} - \phi(u_{ij}^*) + f_{ij} - 4a = 0, \tag{3.1}$$

We assume that $\psi_{i_0,j_0} = \max_{1 \leq i,j \leq J-1} \psi_{ij}, 1 \leq i_0, j_0 \leq J-1$. If $\psi_{i_0,j_0} > 0$, it is obvious that $u_{i_0,j_0}^* > 0$, and we have

$$\triangle_h^1 \psi_{i_0,j_0} + \triangle_h^2 \psi_{i_0,j_0} \leq 0. \tag{3.2}$$

By (3.1) and (3.2), $\phi(u_{i_0,j_0}^*) - \phi(0) = \triangle_h^1 \psi_{i_0,j_0} + \triangle_h^2 \psi_{i_0,j_0} - \phi(0) + f_{i_0 j_0} - 4a < 0$, this contradicts to the fact $u_{i_0,j_0}^* > 0$, whence,

$$\max_{1 \leq i,j \leq J-1} \psi_{ij} \leq 0.$$

It yields

$$\max_{1 \leq i,j \leq J-1} u_{ij}^* \leq \max_{1 \leq i,j \leq J-1} \{a[x_i(l - x_i) + y_j(l - y_j)]\} \leq \frac{l^2}{2} a.$$

Similarly, we have

$$\min_{1 \leq i,j \leq J-1} u_{ij}^* \geq -\frac{l^2}{2} a,$$

and this completes the proof.

Lemma 3. *Let the discrete function $u_h^n = \{u_{ij}^n \mid 0 \leq i, j \leq J, n = 0, 1, 2, ...\}$ be the solution of difference equation (2.1), $a = \dfrac{\|f\|_{L^\infty} + \mid \phi(0) \mid}{4}$,*

$$g_n = \max_{0 \leq i,j \leq J} \mid u_{ij}^n \mid + \frac{al^2}{2}, \quad r_n = \max_{|\tau| \leq g_n} \phi'(\tau),$$

for given $\epsilon \in (0, 1)$, if $\Delta t_n, h$ satisfy

$$\Delta t_n \leq \begin{cases} \min\{\dfrac{(1-\epsilon)h^2}{4(1+\epsilon)}, \dfrac{2\epsilon}{(1+\epsilon)r_n}\}, & r_n > 0, \\ \dfrac{(1-\epsilon)h^2}{4(1+\epsilon)}, & r_n = 0. \end{cases} \tag{3.3}$$

there exists a positive constant k_3 independent of $h, n, \Delta t_n$ such that

$$\max_{i,j,n} \mid u_{ij}^n \mid \leq k_3.$$

Proof. We define $u_{ij}^n = w_{ij}^n + ax_i(l-x_i) + ay_j(l-y_j)$, then

$$\frac{w_{ij}^{n+1} - w_{ij}^n}{\Delta t_n} = \triangle_h^1 w_{ij}^n + \triangle_h^2 w_{ij}^n - \phi(u_{ij}^n) + f_{ij} - 4a. \tag{3.4}$$

Since $u_{ij}^n \geq w_{ij}^n$, we have $\phi(u_{ij}^n) \geq \phi(w_{ij}^n)$, thus from (3.4)

$$\begin{aligned} w_{ij}^{n+1} \leq & \frac{\Delta t_n}{h^2}(w_{i+1,j}^n + w_{i-1,j}^n + w_{i,j+1}^n + w_{i,j-1}^n) + (1 - \frac{4\Delta t_n}{h^2})w_{ij}^n \\ & + (-\phi(w_{ij}^n) + \phi(0))\Delta t_n + (-\phi(0) + f_{ij} - 4a)\Delta t_n \\ & \leq \frac{\Delta t_n}{h^2}(w_{i+1,j}^n + w_{i-1,j}^n + w_{i,j+1}^n + w_{i,j-1}^n) \\ & + (1 - \frac{4\Delta t_n}{h^2})w_{ij}^n + (-\phi(w_{ij}^n) + \phi(0))\Delta t_n \\ & = \frac{\Delta t_n}{h^2}(w_{i+1,j}^n + w_{i-1,j}^n + w_{i,j+1}^n + w_{i,j-1}^n) \\ & + (1 - \frac{4\Delta t_n}{h^2} - \phi'(\xi_{ij}^n)\Delta t_n)w_{ij}^n, \end{aligned} \tag{3.5}$$

where $\mid \xi_{ij}^n \mid \leq \mid w_{ij}^n \mid \leq g_n$. It is obvious that

$$0 \leq \phi'(\xi_{ij}^n)\Delta t_n \leq r_n \Delta t_n \leq \frac{2\epsilon}{1+\epsilon},$$

which yields that

$$1 - \frac{4\Delta t_n}{h^2} - \phi'(\xi_{ij}^n)\Delta t_n \geq 0,$$

Therefore by (3.5),

$$w_{ij}^{n+1} \leq [1 - \phi'(\xi_{ij}^n)\Delta t_n] \max_{i,j} w_{ij}^n. \tag{3.6}$$

On the other hand, we define $u_{ij}^n = \omega_{ij}^n - ax_i(l-x_i) - ay_j(l-y_j)$, then

$$\frac{\omega_{ij}^{n+1} - \omega_{ij}^n}{\Delta t_n} = \triangle_h^1 \omega_{ij}^n + \triangle_h^2 \omega_{ij}^n - \phi(u_{ij}^n) + f_{ij} + 4a, \tag{3.7}$$

Since $u_{ij}^n \leq \omega_{ij}^n$, we have $\phi(u_{ij}^n) \leq \phi(\omega_{ij}^n)$, thus from (3.7)

$$\begin{aligned} \omega_{ij}^{n+1} \geq & \frac{\Delta t_n}{h^2}(\omega_{i+1,j}^n + \omega_{i-1,j}^n + \omega_{i,j+1}^n + \omega_{i,j-1}^n) + (1 - \frac{4\Delta t_n}{h^2})\omega_{ij}^n \\ & + (-\phi(\omega_{ij}^n) + \phi(0))\Delta t_n + (-\phi(0) + f_{ij} + 4a)\Delta t_n \\ & \geq \frac{\Delta t_n}{h^2}(\omega_{i+1,j}^n + \omega_{i-1,j}^n + \omega_{i,j+1}^n + \omega_{i,j-1}^n) \\ & + (1 - \frac{4\Delta t_n}{h^2})\omega_{ij}^n + (-\phi(\omega_{ij}^n) + \phi(0))\Delta t_n \\ & = \frac{\Delta t_n}{h^2}(\omega_{i+1,j}^n + \omega_{i-1,j}^n + \omega_{i,j+1}^n + \omega_{i,j-1}^n) \\ & + (1 - \frac{4\Delta t_n}{h^2} - \phi'(\theta_{ij}^n)\Delta t_n)\omega_{ij}^n, \end{aligned} \tag{3.8}$$

where $|\ \theta_{ij}^n\ | \leq |\ \omega_{ij}^n\ | \leq g_n$. It is evident that

$$0 \leq \phi'(\theta_{ij}^n)\Delta t_n \leq r_n \Delta t_n \leq \frac{2\epsilon}{1+\epsilon},$$

hence

$$1 - \frac{4\Delta t_n}{h^2} - \phi'(\theta_{ij}^n)\Delta t_n \geq 0,$$

therefore by (3.8),

$$\omega_{ij}^{n+1} \geq [1 - \phi'(\theta_{ij}^n)\Delta t_n] \min_{i,j} \omega_{ij}^n, \tag{3.9}$$

By (3.6) and (3.9), we have

$$\max\{0, \max_{i,j} w_{ij}^{n+1}\} \leq \max\{0, \max_{i,j} w_{ij}^n\} \leq \cdots \leq \max\{0, \max_{i,j} w_{ij}^0\},$$

$$\min\{0, \min_{i,j} \omega_{ij}^{n+1}\} \geq \min\{0, \min_{i,j} \omega_{ij}^n\} \geq \cdots \geq \min\{0, \min_{i,j} \omega_{ij}^0\}.$$

This completes the proof.

Corollary 1. *Let the discrete function $u_h^n = \{u_{ij}^n \mid 0 \leq i,j \leq J, n = 0,1,2,\cdots\}$ be the solution of difference equation (2.1),*

$$a = \frac{\|f\|_{L^\infty} + |\ \phi(0)\ |}{4}, b = \|u_0\|_{L^\infty}, r = \max_{\tau \leq al^2+b} \phi'(\tau)$$

For given $\epsilon \in (0,1)$, if $\Delta t_n, h$ satisfy

$$\Delta t_n \leq \begin{cases} \min\{\dfrac{(1-\epsilon)h^2}{4(1+\epsilon)}, \dfrac{2\epsilon}{(1+\epsilon)r}\}, & r > 0, \\ \dfrac{(1-\epsilon)h^2}{4(1+\epsilon)}, & r = 0. \end{cases} \tag{3.10}$$

then

$$\max_{1 \leq i,j \leq J-1} |\ u_{ij}^n\ | \leq \frac{al^2}{2} + b, \qquad n = 0,1,2,\cdots$$

A simple computation shows that

Lemma 4. *Let $T_n = \sum_{k=0}^{n-1} \Delta t_k$, and suppose the sequence $\{a_n\}$ satisfies*

$$a_{n+1} \leq e^{-c_1 \Delta t_n} a_n + c_2 e^{-c_3 T_n} \Delta t_n,$$

where $a_n \geq 0, \forall n \geq 0, c_i > 0, i = 1,2,3$, then there exist $c_4 > 0$ and $\sigma > 0$ such that

$$a_n \leq c_4 e^{-\sigma T_n}.$$

4. Asymptotic Behavior of Explicit Difference Solutions

In this section, we intend to study the asymptotic behavior of solutions of (2.1).

By Lemma 2 and Lemma 3 (Corollary 1),

$$|\ \phi(u_{ij}^n) - \phi(u_{ij}^*)\ |^2 \leq r(\phi(u_{ij}^n) - \phi(u_{ij}^*))v_{ij}^n, \tag{4.1}$$

If $h, \Delta t_n$ satisfy (3.10), it follows from (2.3),(4.1) and Lemma 1 that

$$\begin{aligned}
&\|v_h^{n+1}\|^2 - \|v_h^n\|^2 + 2\Delta t_n \|\delta v_h^n\|^2 + 2\Delta t_n \sum_{1\le i,j\le J-1} [\phi(u_{ij}^n) - \phi(u_{ij}^*)] v_{ij}^n h^2 \\
&= \|v_h^{n+1} - v_h^n\|^2 \\
&\le \Delta t_n^2 [(1+\epsilon)\|\triangle v_h^n\|^2 + (1+\frac{1}{\epsilon}) \sum_{i,j} (\phi(u_{ij}^n) - \phi(u_{ij}^*))^2 h^2] \\
&\le \Delta t_n^2 (1+\epsilon) \frac{8}{h^2} \|\delta v_h^n\|^2 + 2\Delta t_n \sum_{i,j} (\phi(u_{ij}^n) - \phi(u_{ij}^*)) v_{ij}^n h^2].
\end{aligned}$$

Hence

$$\begin{aligned}
&\|v_h^{n+1}\|^2 - \|v_h^n\|^2 + 2\Delta t_n \|\delta v_h^n\|^2 + 2\Delta t_n \sum_{1\le i,j\le J-1} [\phi(u_{ij}^n) - \phi(u_{ij}^*)] v_{ij}^n h^2 \\
&\le 2(1-\epsilon)\Delta t_n \|\delta v_h^n\|^2 + 2\Delta t_n \sum_{i,j} [\phi(u_{ij}^n) - \phi(u_{ij}^*)] v_{ij}^n h^2
\end{aligned}$$

thus we have

$$\|v_h^{n+1}\|^2 - \|v_h^n\|^2 + 2\epsilon \Delta t_n \|\delta v_h^n\|^2 \le 0,$$

from Lemma 1 we deduce that

$$\|v_h^{n+1}\|^2 - \|v_h^n\|^2 + C\Delta t_n \|v_h^n\|^2 \le 0, \tag{4.2}$$

(4.2) implies that

Lemma 5. *Let the discrete function u_h^n and u_h^* be the solution of difference equation (2.1) and (2.2) respectively. If $\Delta t_n, h$ satisfy (3.10), there exist positive constants k_4 and α independent of $h, \Delta t_n, n$ such that*

$$\|u_h^n - u_h^*\|^2 \le k_4 e^{-\alpha T_n},$$

where $T_n = \sum_{k=0}^{n-1} \Delta t_k$.

By (2.3),

$$\begin{aligned}
&\|\delta v_h^{n+1}\|^2 - \|\delta v_h^n\|^2 + 2\Delta t_n \|\triangle v_h^n\|^2 \\
&= 2\Delta t_n \sum_{1\le i,j\le J-1} [\phi(u_{ij}^n) - \phi(u_{ij}^*)](\triangle_h^1 v_{ij}^n + \triangle_h^2 v_{ij}^n) h^2 \\
&+ \|\delta(v_h^{n+1} - v_h^n)\|^2,
\end{aligned} \tag{4.3}$$

From Lemma 1,

$$\begin{aligned}
\|\delta(v_h^{n+1} - v_h^n)\|^2 &\le \frac{8}{h^2} \Delta t_n^2 [\|\triangle v_h^n\|^2 + \sum_{1\le i,j\le J-1} (\phi(u_{ij}^n) - \phi(u_{ij}^*))^2 h^2 \\
&- 2 \sum_{1\le i,j\le J-1} [\phi(u_{ij}^n) - \phi(u_{ij}^*)](\triangle_h^1 v_{ij}^n + \triangle_h^2 v_{ij}^n) h^2] \\
&\le \frac{2(1-\epsilon)}{1+\epsilon} \Delta t_n [\|\triangle v_h^n\|^2 + \sum_{1\le i,j\le J-1} (\phi(u_{ij}^n) - \phi(u_{ij}^*))^2 h^2 \\
&- 2 \sum_{1\le i,j\le J-1} [\phi(u_{ij}^n) - \phi(u_{ij}^*)](\triangle_h^1 v_{ij}^n + \triangle_h^2 v_{ij}^n) h^2]
\end{aligned} \tag{4.4}$$

(4.3) and (4.4) yield

$$\begin{aligned}
&\|\delta v_h^{n+1}\|^2 - \|\delta v_h^n\|^2 + 2\Delta t_n \|\triangle v_h^n\|^2 \\
&\leq \frac{2(1-\epsilon)}{1+\epsilon}\Delta t_n [\|\triangle v_h^n\|^2 + \sum_{1\leq i,j\leq J-1} (\phi(u_{ij}^n) - \phi(u_{ij}^*))^2 h^2] \\
&+ \frac{6\epsilon - 2}{1+\epsilon}\Delta t_n \sum_{1\leq i,j\leq J-1} [\phi(u_{ij}^n) - \phi(u_{ij}^*)](\triangle_h^1 v_{ij}^n + \triangle_h^2 v_{ij}^n) h^2] \\
&\leq (2 - \frac{2\epsilon}{1+\epsilon})\Delta t_n \|\triangle v_h^n\|^2 + C\Delta t_n \sum_{1\leq i,j\leq J-1} [\phi(u_{ij}^n) - \phi(u_{ij}^*)]^2 h^2,
\end{aligned}$$

hence

$$\begin{aligned}
&\|\delta v_h^{n+1}\|^2 - \|\delta v_h^n\|^2 + \frac{2\epsilon}{1+\epsilon}\Delta t_n \|\triangle v_h^n\|^2 \\
&\leq C\Delta t_n \sum_{1\leq i,j\leq J-1} [\phi(u_{ij}^n) - \phi(u_{ij}^*)]^2 h^2,
\end{aligned} \tag{4.5}$$

if follows from Lemma 2 and Lemma 3 that

$$| \phi(u_{ij}^n) - \phi(u_{ij}^*) | \leq C | v_{ij}^n |,$$

therefore by (4.5),

$$\|\delta v_h^{n+1}\|^2 - \|\delta v_h^n\|^2 + \frac{2\epsilon}{1+\epsilon}\Delta t_n \|\triangle v_h^n\|^2 \leq C\Delta t_n \|v_h^n\|^2, \tag{4.6}$$

By Lemma 1 and Lemma 4, Lemma 5, the following conclusion is now an immmediate consequence of (4.6):

Theorem 1. *Let the discrete function u_h^n and u_h^* be the solution of difference equation (2.1) and (2.2) respectively, if $\Delta t_n, h$ satisfy (3.10), there exist constants $M_1 > 0, \beta > 0$ independent of $h, n, \Delta t_n$ such that*

$$\|\delta(u_h^n - u_h^*)\|^2 \leq M_1 e^{-\beta T_n},$$

where $T_n = \sum_{k=0}^{n-1} \Delta t_k$.

Summing up (4.6) for n from k to $k+s$ and using Theorem 1 and Lemma 5, we can easily obtain:

Corollary 2. *Let the discrete function u_h^n and u_h^* be the solution of difference equation (2.1) and (2.2) respectively, if $\Delta t_n, h$ satisfy (3.10), then for any positive integer s, there exist constants $M_2 > 0, \lambda > 0$ independent of $h, n, \Delta t$ such that*

$$\sum_{i=0}^{s} \|\triangle(u_h^{n+i} - u_h^*)\|^2 \Delta t_n \leq M_2 e^{-\lambda T_n},$$

where $T_n = \sum_{k=0}^{n-1} \Delta t_k$.

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