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CONVERGENCE OF (0,1,2,3) INTERPOLATION ON AN ARBITRARY SYSTEM OF NODES*1)

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Abstract

Estimations of lower bounds for the fundamental functions of (0,1,2,3) interpolation are given. Based on this result conditions for convergence of (0,1,2,3) interpolation and for Grűnwald-type theorem are essentially simplified and improved.

Key words: Hermite interpolation, Hermite-Fejér interpolation, Convergence.

1. Introduction

This note deals with convergence of (0,1,2,3) interpolation on an arbitrary system of nodes. Fisrt we introduce some definitions and notations.

Let

$$1 \geq x_{1n} > x_{2n} > \cdots > x_{nn} \geq -1, \quad n = 1, 2, ... \tag{1.1}$$

Given a fixed even integer m, let $A_{jk} \in \boldsymbol{P}_{mn-1}$ (the set of polynomials of degree at most $mn-1$) satisfy

$$A_{jk}^{(p)}(x_q) = \delta_{jp}\delta_{kq}, \quad j, p = 0, 1, ..., m-1; \ k, q = 1, 2, ..., n. \tag{1.2}$$

Then the (0,1,...,m–1) Hermite-Fejér type interpolation for $f \in C[-1,1]$ is defined by

$$H_{nm}(f, x) := \sum_{k=1}^{n} f(x_k) A_{0k}(x), \quad n = 1, 2, \cdots, \tag{1.3}$$

and the (0,1,...,m–1) Hermite interpolation for $f \in C^{m-1}[-1,1]$ is defined by

$$H_{nm}^{*}(f, x) := \sum_{j=0}^{m-1} \sum_{k=1}^{n} f^{(j)}(x_k) A_{jk}(x), \quad n = 1, 2, \cdots \tag{1.4}$$

(cf. [6]). We also need a well known fact:

$$\|H_{nm}\| := \sup_{\|f\| \leq 1} \|H_{nm}(f)\| = \left\| \sum_{k=1}^{n} |A_{0k}| \right\|, \quad n = 1, 2, \cdots, \tag{1.5}$$

where $\|\cdot\|$ stands for the uniform norm on $[-1, 1]$.

Here H_{n2} is the classical Hermite-Fejér interpolation investigated in many papers (cf. [1]). H_{n4} is the so called Krylov-Stayermann interpolation, on which although there have been quite a few papers (cf. [6] and its references), almost all of them consider only interpolation based on the special system of nodes, like zeros of Jacobi polynomials. In this note we will give estimations of lower bounds for the fundamental functions of (0,1,2,3) interpolation. Based on this result conditions for convergence of (0,1,2,3) interpolation and for Grűnwald-type theorem are essentially simplified and improved. We will put these results in the next section and some conjectures in the last section.

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2. Results

Let

$$\omega_n(x) = (x-x_1)(x-x_2)\dots(x-x_n),$$

$$\ell_k(x) = \frac{\omega_n(x)}{\omega_n'(x_k)(x-x_k)}, \quad k = 1,2,\dots,n,$$

$$b_{ik} = \frac{1}{i!}\left[\ell_k(x)^{-m}\right]^{(i)}_{x=x_k}, \qquad j = 0,1,\dots,m-1, \quad k = 1,2,\dots,n,$$

$$B_{jk}(x) = \sum_{i=0}^{m-j-1} b_{ik}(x-x_k)^i, \qquad j = 0,1,\dots,m-1, \quad k = 1,2,\dots,n.$$

Then [4]

$$A_{jk}(x) = \frac{1}{j!}(x-x_k)^j B_{jk}(x)\ell_k(x)^m, \qquad j = 0,1,\dots,m-1, \quad k = 1,2,\dots,n. \tag{2.1}$$

In particular, for $m=4$ we have (cf. [2, (4.1)])

$$\begin{cases} B_{1k}(x) & = [10\ell_k'(x_k)^2 - 2\ell_k''(x_k)](x-x_k)^2 - 4\ell_k'(x_k)(x-x_k) + 1, \\ B_{2k}(x) & = -4\ell_k'(x_k)(x-x_k) + 1, \\ B_{3k}(x) & = 1. \end{cases} \tag{2.2}$$

The following lemma will play a basic role in this note.

Lemma 1. *Let $m=4$. Then*

$$B_{1k}(x) \ge |B_{2k}(x)|, \quad B_{1k}(x) \ge \frac{1}{2}|B_{3k}(x)| = \frac{1}{2}, \qquad x \in \mathbb{R}, \quad k = 1,2,\dots,n, \tag{2.3}$$

and

$$\sum_{k=1}^n |A_{2k}(x)| \le \frac{1}{2}\sum_{k=1}^n (x-x_k)A_{1k}(x), \quad \sum_{k=1}^n |A_{3k}(x)| \le \frac{2}{3}\sum_{k=1}^n (x-x_k)A_{1k}(x), \qquad x \in [-1,1]. \tag{2.4}$$

Proof. By a well known result (cf. [5, p. 976])

$$\ell_k'(x_k)^2 - \ell_k''(x_k) = \sum_{i\ne k} \frac{1}{(x_k-x_i)^2} > 0$$

it follows from (2.2) that

$$B_{1k}(x) \ge 8[\ell_k'(x_k)(x-x_k)]^2 - 4\ell_k'(x_k)(x-x_k) + 1.$$

Using a simple symbol $y := \ell_k'(x_k)(x-x_k)$ we get

$$B_{1k}(x) \ge 8y^2 - 4y + 1$$

and

$$B_{2k}(x) = -4y + 1.$$

Hence

$$B_{1k}(x) - B_{2k}(x) \ge 8y^2 \ge 0,$$
$$B_{1k}(x) + B_{2k}(x) \ge 2(2y-1)^2 \ge 0,$$
$$B_{1k}(x) - \frac{1}{2} \ge \frac{1}{2}(4y-1)^2 \ge 0,$$

which is equivalent to (2.3).

By (2.1) and (2.3)

$$\sum_{k=1}^n |A_{2k}(x)| = \frac{1}{2}\sum_{k=1}^n |(x-x_k)^2 B_{2k}(x)\ell_k(x)^4|$$
$$\le \frac{1}{2}\sum_{k=1}^n (x-x_k)^2 B_{1k}(x)\ell_k(x)^4 = \frac{1}{2}\sum_{k=1}^n (x-x_k)A_{1k}(x).$$

Similarly for $|x| \leq 1$

$$\sum_{k=1}^{n} |A_{3k}(x)| = \frac{1}{6}\sum_{k=1}^{n} |(x-x_k)^3 B_{3k}(x)\ell_k(x)^4|$$
$$\leq \frac{2}{3}\sum_{k=1}^{n}(x-x_k)^2 B_{1k}(x)\ell_k(x)^4 = \frac{2}{3}\sum_{k=1}^{n}(x-x_k)A_{1k}(x).$$

In [2] we proved a theorem of Grűnwald-type for (0,1,...,m−1) Hermite-Fejér type interpolation for any even m.

Theorem A. [2, Theorem] *Let m be even. If for fixed positive numbers ρ and n_0*

$$B_{0k}(x) \geq \rho |B_{jk}(x)|, \qquad j = 1,2,...,m-1, \quad k = 1,2,...,n, \quad n \geq n_0, \quad |x| \leq 1, \tag{2.5}$$

then

$$\lim_{n\to\infty} \|H_{nm}(f) - f\| = 0 \tag{2.6}$$

holds for all $f \in C[-1,1]$.

Now using Lemma 1 we can essentially simplify the condition (2.5) for $m = 4$.

Theorem 1. *Let $m = 4$. If for fixed positive numbers ρ and n_0*

$$B_{0k}(x) \geq \rho B_{1k}(x), \qquad k = 1,2,...,n, \quad n \geq n_0, \quad |x| \leq 1, \tag{2.7}$$

then

$$\lim_{n\to\infty} \|H_{n4}(f) - f\| = 0 \tag{2.8}$$

holds for all $f \in C[-1,1]$.

Proof. In this case (2.3) and (2.7) imply (2.5) with replacing ρ by $\rho/2$. Then we apply Theorem A to conclude that (2.8) holds for all $f \in C[-1,1]$.

Accually we get the equivalence of conditions (2.7) and (2.5) (with different values of ρ) for $m = 4$. Clearly (2.7) is more convenient for use than (2.5). The following theorem plays the same role.

Theorem 2. *Let $m = 4$. If*

$$\|H_{n4}\| = \bigcirc(1) \tag{2.9}$$

and

$$\lim_{n\to\infty} \left\| \sum_{k=1}^{n} |A_{1k}| \right\| = 0, \tag{2.10}$$

then (2.8) holds for all $f \in C[-1,1]$.

Proof. First, noting that $|x - x_k| \leq 2$ for $|x| \leq 1$, by (2.1) and (2.3) we have

$$|A_{2k}(x)| \leq |A_{1k}(x)|, \quad |A_{3k}(x)| \leq \frac{4}{3}|A_{1k}(x)|, \qquad k = 1,2,...,n, \quad |x| \leq 1. \tag{2.11}$$

Thus (2.10) implies

$$\lim_{n\to\infty} \left\| \sum_{j=1}^{3}\sum_{k=1}^{n} |A_{jk}| \right\| = 0.$$

The remainder of the proof is to apply

Theorem B. [7, Statement 2.1] *Let m be an even integer. If*

$$\|H_{nm}\| = \bigcirc(1) \tag{2.12}$$

and

$$\lim_{n\to\infty} \left\| \sum_{j=1}^{m-1}\sum_{k=1}^{n} |A_{jk}| \right\| = 0,$$

then (2.6) holds for all $f \in C[-1,1]$.

The two functions $f_1(x) := x$ and $f_2(x) := x^2$ will play the same role for H_{n4} as H_{n2} [3]. Precisely we have the following, here $R(P,x) := |H_{nm}(P,x) - P(x)|$.

Theorem 3. *Let $m = 4$. Then for any polynomial P and $4n - 1 \geq deg\ (P)$*

$$R(P,x) \leq (\|P'\| + 3\|P''\| + 2\|P'''\|)R(f_1,x) + (2\|P''\| + \|P'''\|)R(f_2,x), \quad |x| \leq 1. \tag{2.13}$$

Proof. Let $|x| \le 1$. By (1.4)

$$1 = \sum_{k=1}^{n} A_{0k}(x),$$

$$x = \sum_{k=1}^{n} [x_k A_{0k}(x) + A_{1k}(x)],$$

$$x^2 = \sum_{k=1}^{n} [x_k^2 A_{0k}(x) + 2x_k A_{1k}(x) + 2A_{2k}(x)].$$

Using these formulas we get

$$\sum_{k=1}^{n} (x - x_k) A_{0k}(x) = \sum_{k=1}^{n} A_{1k}(x) \tag{2.14}$$

and

$$\sum_{k=1}^{n} (x - x_k)^2 A_{0k}(x) = 2\sum_{k=1}^{n} [(x - x_k) A_{1k}(x) - A_{2k}(x)]. \tag{2.15}$$

(2.14) gives

$$\left| \sum_{k=1}^{n} A_{1k}(x) \right| = R(f_1, x). \tag{2.16}$$

(2.15) and (2.4) implies

$$\sum_{k=1}^{n} (x - x_k)^2 A_{0k}(x) \ge \sum_{k=1}^{n} [2(x - x_k) A_{1k}(x) - (x - x_k) A_{1k}(x)] = \sum_{k=1}^{n} (x - x_k) A_{1k}(x).$$

But

$$\sum_{k=1}^{n} (x - x_k)^2 A_{0k}(x) = x^2 - 2x \sum_{k=1}^{n} x_k A_{0k}(x) + \sum_{k=1}^{n} x_k^2 A_{0k}(x)$$

$$= 2x \left[x - \sum_{k=1}^{n} x_k A_{0k}(x) \right] - \left[x^2 - \sum_{k=1}^{n} x_k^2 A_{0k}(x) \right] \le 2R(f_1, x) + R(f_2, x).$$

At last we obtain

$$\sum_{k=1}^{n} (x - x_k) A_{1k}(x) \le 2R(f_1, x) + R(f_2, x). \tag{2.17}$$

After these preliminaries we can prove (2.13). In fact,

$$\begin{aligned} R(P, x) &= \left| \sum_{j=1}^{3} \sum_{k=1}^{n} P^{(j)}(x_k) A_{jk}(x) \right| \\ &\le \left| \sum_{k=1}^{n} P'(x_k) A_{1k}(x) \right| + \|P''\| \sum_{k=1}^{n} |A_{2k}(x)| + \|P'''\| \sum_{k=1}^{n} |A_{3k}(x)| \end{aligned}$$

and by the mean value theoem for the derivative

$$\begin{aligned} \left| \sum_{k=1}^{n} P'(x_k) A_{1k}(x) \right| &\le \left| \sum_{k=1}^{n} P'(x) A_{1k}(x) \right| + \left| \sum_{k=1}^{n} [P'(x) - P'(x_k)] A_{1k}(x) \right| \\ &= \left| \sum_{k=1}^{n} P'(x) A_{1k}(x) \right| + \left| \sum_{k=1}^{n} P''(\xi_k)(x - x_k) A_{1k}(x) \right| \\ &\le \|P'\| \left| \sum_{k=1}^{n} A_{1k}(x) \right| + \|P''\| \sum_{k=1}^{n} (x - x_k) A_{1k}(x). \end{aligned}$$

Thus by (2.4), (2.16), and (2.17)

$$R(P,x) \le \|P'\| \left|\sum_{k=1}^{n} A_{1k}(x)\right| + \left(\frac{3}{2}\|P''\| + \frac{2}{3}\|P'''\|\right) \sum_{k=1}^{n} (x-x_k)A_{1k}(x)$$
$$\le (\|P'\| + 3\|P''\| + 2\|P'''\|)R(f_1,x) + (2\|P''\| + \|P'''\|)R(f_2,x).$$

As immediate consequences of Theorem 3 we state two corollaries.

Corollary 1. *Let $m = 4$. If (2.8) holds for $f = f_1$ and $f = f_2$, then (2.8) holds for every polynomial f.*

Corollary 2. *Let $m = 4$. (2.8) holds for every $f \in C[-1,1]$ if and only if (2.8) holds for $f = f_i$, $i = 1, 2$, and (2.9) is valid.*

In [2] we proved a result:

Theorem C. [2, Lemma 3] *Let m be an even integer. If*

$$\left\|\sum_{j=0}^{m-1}\sum_{k=1}^{n} |A_{jk}|\right\| = \bigcirc(1),$$

then

$$\lim_{n\to\infty} \|H^*_{nm}(f) - f\| = 0 \tag{2.18}$$

holds for all $f \in C^{m-1}[-1,1]$.

In comparison with this result the following theorem is interesting and somewhat surprising.

Theorem 4. *If (2.9) is valid then*

$$\lim_{n\to\infty} \|H^*_{n4}(f) - f\| = 0 \tag{2.19}$$

holds for every $f \in C^3[-1,1]$.

Proof. By Weierstrass Theorem for a given $\epsilon > 0$ there is a polynomial P such that

$$\|P^{(j)} - f^{(j)}\| \le \epsilon, \quad j = 0,1,2,3. \tag{2.20}$$

On the other hand, if $4n - 1 \ge \deg\ (P)$, then $P(x) = H^*_{n4}(P,x)$. Then by (2.9) $N := \sup_n \|H_{n4}\| < \infty$ and hence by (2.20)

$$\|H^*_{n4}(f) - f\| \le \|P - f\| + \|H^*_{n4}(P - f)\|$$
$$\le \epsilon + \left\|\sum_{j=0}^{3}\sum_{k=1}^{n} [P^{(j)}(x_k) - f^{(j)}(x_k)]A_{jk}(x)\right\|$$
$$\le \epsilon(N+1) + \left\|\sum_{k=1}^{n} [P'(x_k) - f'(x_k)]A_{1k}(x)\right\| + \epsilon\left\|\sum_{k=1}^{n} [|A_{2k}| + |A_{3k}|]\right\|$$
$$:= \epsilon(N+1) + S_1 + S_2.$$

By the mean value therem for the derivetive

$$S_1 \le \left\|\sum_{k=1}^{n} [P'(x) - f'(x)]A_{1k}(x)\right\| + \left\|\sum_{k=1}^{n} \{[P'(x) - f'(x)] - [P'(x_k) - f'(x_k)]\}A_{1k}(x)\right\|$$
$$= \left\|\sum_{k=1}^{n} [P'(x) - f'(x)]A_{1k}(x)\right\| + \left\|\sum_{k=1}^{n} [P''(\xi_k) - f''(\xi_k)](x - x_k)A_{1k}(x)\right\|$$
$$\le \epsilon\left\|\sum_{k=1}^{n} A_{1k}\right\| + \epsilon\left\|\sum_{k=1}^{n} (x - x_k)A_{1k}(x)\right\|.$$

According to (2.4)

$$S_2 \le 2\epsilon\left\|\sum_{k=1}^{n} (x - x_k)A_{1k}(x)\right\|.$$

Hence

$$\|H^*_{n4}(f) - f\| \le \epsilon(N+1) + \epsilon\left\|\sum_{k=1}^{n} A_{1k}\right\| + 3\epsilon\left\|\sum_{k=1}^{n} (x - x_k)A_{1k}(x)\right\|.$$

But by (2.16) and (2.17) we obtain

$$\left\| \sum_{k=1}^{n} A_{1k} \right\| = \|R(f_1)\| = \left\| x - \sum_{k=1}^{n} x_k A_{0k}(x) \right\| \le N + 1$$

and

$$\left\| \sum_{k=1}^{n} (x - x_k) A_{1k}(x) \right\| \le 2\|R(f_1)\| + \|R(f_2)\| \le 3(N + 1).$$

At last we get

$$\|H^*_{n4}(f) - f\| \le 11(N + 1)\epsilon,$$

which proves (2.19).

As a direct consequence of this theorem we state

Corollary 3. *If (2.8) is valid for every $f \in C[-1,1]$ then (2.19) holds for every $f \in C^3[-1,1]$.*

3. Conjectures

We believe that the main results in this note remain true for any even integer. More precisely we state the following conjectures.

Conjecture 1. *Let $m \ge 4$ be even. Then*

$$B_{1k}(x) \ge c|B_{jk}(x)|, \quad j = 2, ..., m - 1, \quad k = 1, 2, ..., n, \quad x \in \mathbb{R},$$

where c is a positive constant depending only on m.

Conjecture 2. *Let m be even. If for fixed positive numbers ρ and n_0*

$$B_{0k}(x) \ge \rho B_{1k}(x), \qquad k = 1, 2, ..., n, \quad n \ge n_0, \quad |x| \le 1,$$

then (2.6) holds for all $f \in C[-1,1]$.

Conjecture 3. *Let m be even. If (2.12) is valid and*

$$\lim_{n\to\infty} \left\| \sum_{k=1}^{n} |A_{1k}| \right\| = 0,$$

then (2.6) holds for all $f \in C[-1,1]$.

Conjecture 4. *Let m be even. Then for any polynomial P and $mn - 1 \ge deg\ (P)$*

$$R(P, x) \le c_1 R(f_1, x) + c_2 R(f_2, x), \quad |x| \le 1,$$

where c_1 and c_2 are positive numbers depending only on m and P.

Conjecture 5. *Let m be an even integer. If (2.12) is valid, then (2.18) holds for every $f \in C^{m-1}[-1,1]$.*

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