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MULTISCALE FINITE ELEMENT METHOD FOR SUBDIVIDED PERIODIC ELASTIC STRUCTURES OF COMPOSITE MATERIALS*1)

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Abstract

In this paper, from the view of point of macro- and meso- scale coupling, we discuss the mechanical behaviour for subdivided periodic elastic structures of composite materials. A multiscale numerical method and its error estimate are reported. Finally, numerical experiments results supports strongly the theoretical ones presented in the paper.

Key words: Multi-scale asymptotic method, Finite element method, Composite medium.

1. Introduction

For a kind of elastic structures of composite materials of which the geometric and physical parameters are of some periodicity, e.g., common laminated plate and shell, and fibre reinforced, particle reinforced, and woved composite materials and so forth , we can regard them as the periodic structures with a unit cell.

Generally speaking, it is extremely difficult to compute directly the above elastic structures by using usual FEM, due to the complicated geometric configurations and highly oscillatory physical parameters. To overcome this crucial difficulty, I.Babuska, J.L.Lions et al. [1] proposed early homogenization method. However, homogenization method can only reflect the macroscopic average properties of elastic structures, but does not describe the local mechanical behaviour. To this end, J.L.Lions and O.A.Oleinik et al. [2,5] obtained complete asymptotic expansions for the Dirichlet boundary value problems of the second order elliptic equation and the linear elastic structures of composite materials in perforated domains, respectively. Jun-zhi Cui and Li-qun Cao et al. [6,7] obtained the complete asymptotic expansions for the Dirichlet boundary value problems of the second order elliptic equation and the linear elastic system with rapidly oscillating coefficients in domains formed by entirely basic configurations, respectively. In present paper, we will propose the multiscale FEM for subdivided elastic structures of composite materials.

The organization of this paper is as follows. In section 2 , we shall obtain multiscale asymptotic expansion and truncation error estimates for subdivided elastic structures of composite materials. Section 3 is devoted to the FE computation of periodic solutions $N_\alpha(\xi)$ and the modified homogenized linear elastic system $\overline{U}^0(x)$. In section 4 , a multiscale FE scheme and total error estimates are given. Finally, numerical experiments results are reported and are coincident with the theoretical ones.

In what follows summation over repeated Latin indices from 1 to n is assumed. If the vectors u, v or matrices A, B have elements belonging to a Hilbert space $\mathcal{H}$ with a scarlar product $(\cdot,\cdot)_{\mathcal{H}}$, we use the following notations:

$$(u,v)_{\mathcal{H}} = (u_i, v_i)_{\mathcal{H}}, \quad \|u\|_{\mathcal{H}} = (u,u)_{\mathcal{H}}^{1/2}$$
$$(A,B)_{\mathcal{H}} = (a_{ij}, b_{ij})_{\mathcal{H}}, \quad \|A\|_{\mathcal{H}} = (A,A)_{\mathcal{H}}^{1/2}$$

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and write $u, v \in \mathcal{H}$; $\quad A, B \in \mathcal{H}$ instead of $u, v \in \mathcal{H}^2$; $\quad A, B \in \mathcal{H}^{n^2}$.

2. Multiscale Asymptotic Expansion and Truncation Error Estimates

Without loss of generality, we discuss only the elastic structure as shown in Figure 2.1. Let $\bar{\Omega} = \bar{\Omega}_1 \cup \bar{\Omega}_2, \quad \Omega_1 \cap \Omega_2 = \emptyset, \quad \bar{\Omega}_1 = \bigcup\limits_{z \in T_\varepsilon} \varepsilon(z + \bar{Q})$ be the elastic composite structures formed by entirely basic configurations, $T_\varepsilon = \{z \in Z^n : \varepsilon(z+Q) \subset \Omega\}, Q = \{\xi : \quad 0 < \xi_j < 1, \quad j = 1, 2, \cdots n\}$

To begin with, introduce the following notations:

Displacement boundary Γ_u, force boundary Γ_σ, $\partial\Omega = \Gamma_u \bigcup \Gamma_\sigma$, $mes(\Gamma_u) > 0$, $\Gamma^* = \partial\Omega_1 \bigcap \partial\Omega_2$, body force $f = (f_1, \cdots f_n)^T$; boundary force $\phi(x) = (\phi_1, \cdots \phi_n)^T$; strains $\varepsilon_{ij} = \frac{1}{2}(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i})$; stresses σ_{ij}; constitutive relation $\sigma_{ij} = C_{ij}^{pq}(x, \frac{x}{\varepsilon})\varepsilon_{pq}$, where

$$C^{pq} = \Big(C_{ij}^{pq}(x, \frac{x}{\varepsilon})\Big) = \begin{cases} A^{pq} = \Big(a_{ij}^{pq}(\frac{x}{\varepsilon})\Big) & \text{if} \quad x \in \overline{\Omega}_1 \\ B^{pq} = \Big(b_{ij}^{pq}\Big) & \text{if} \quad x \in \Omega_2 \end{cases} \tag{2.1}$$

$p, q, i, j = 1, 2, \cdots n$

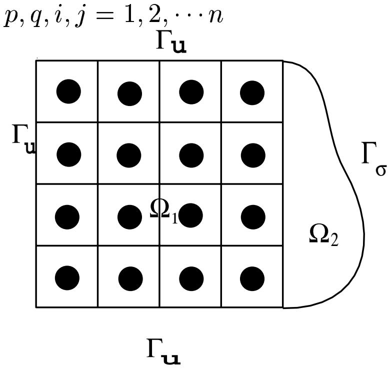

Figure 2.1

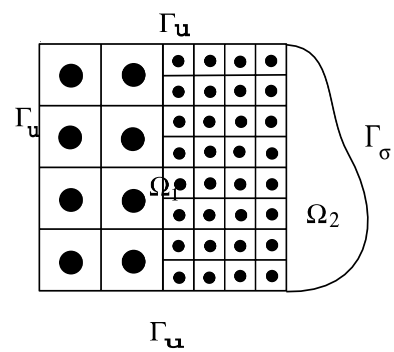

Figure 2.2

Definition 2.1. *We say that a family of matrices* $A^{pq}(\frac{x}{\varepsilon}) = A^{pq}(\xi), \quad \xi = \varepsilon^{-1}x, \quad p, q = 1, 2, \cdots n$, *belongs to class* $E(\mu_1, \mu_2)$, *if their elements* $a_{ij}^{pq}(\xi)$ *are bounded measurable functions satisfying the following conditions:*

(1) $a_{ij}^{pq}(\xi)$ *are 1-periodic in* ξ

(2)

$$a_{ij}^{pq}(\xi) = a_{ji}^{qp}(\xi) = a_{pj}^{iq}(\xi); \tag{2.2}$$

(3) $\mu_1 \eta_{ip}\eta_{ip} \leq a_{ij}^{pq}(\xi)\eta_{ip}\eta_{jq} \leq \mu_2 \eta_{ip}\eta_{ip}$

where η_{ip} *is an arbitray symmetric matrix with real elements,* μ_1, μ_2=*const*> 0

Let

$$b_{ij}^{pq} = \lambda\delta_{ip}\delta_{jq} + \mu\delta_{ij}\delta_{pq} + \mu\delta_{iq}\delta_{jp} \tag{2.3}$$

where $\lambda > 0, \quad \mu > 0$ are the Lame constants, δ_{ij} is the Kronecker notation.

$$\textbf{Equations of equilibrium} : -\sigma_{pq,q} = f_p, \quad p = 1, 2 \cdots n, \quad \text{in} \quad x \in \Omega, \tag{2.4}$$

$$\text{i.e.} \qquad -\mathcal{L}_\varepsilon U^\varepsilon \equiv -\frac{\partial}{\partial x_p}(C^{pq}(x, \frac{x}{\varepsilon})\frac{\partial U^\varepsilon(x)}{\partial x_q}) = f(x), \quad x \in \Omega \tag{2.5}$$

where

$$U^\varepsilon(x) = \begin{cases} u^\varepsilon(x) & x \in \overline{\Omega}_1 \\ w(x) & x \in \Omega_2 \end{cases} \tag{2.6}$$

$$\textbf{Displacement boundary condition} : \quad U^\varepsilon(x) = u^\varepsilon(x) = \bar{u}(x) \qquad x \in \Gamma_u \tag{2.7}$$

$$\textbf{Force boundary condition} : \quad \sigma_\varepsilon(U^\varepsilon) = \sigma(w) = \nu_p B^{pq}\frac{\partial w}{\partial x_q} = \phi(x) \qquad x \in \Gamma_\sigma \tag{2.8}$$

$$\textbf{Interface conditions} : \quad u^\varepsilon(x)|_{\Gamma^*} = w(x)|_{\Gamma^*}, \quad \sigma_\varepsilon(u^\varepsilon)|_{\Gamma^*} = -\sigma(w)|_{\Gamma^*} \tag{2.9}$$

Next, we turn to discuss multiscale asymptotic expansion for problem (2.5).

For $x \in \Omega_1$, formally set

$$u^\varepsilon(x) \cong \sum_{l=0}^{+\infty} \varepsilon^l \sum_{|\alpha|=l} N_\alpha(\xi) D^\alpha u^0(x) \tag{2.10}$$

In constrast to usual expression, here for the sake of convenience, we use the following notation

$$D^\alpha v = \frac{\partial^l v}{\partial x_{\alpha_1} \cdots \partial x_{\alpha_l}}$$

$\alpha = (\alpha_1, \cdots \alpha_l)$, $\quad |\alpha| = l$, $\quad \alpha_i$ takes the values $1, \cdots, n$.

Using the similar method of [7], define periodic solutions $N_0(\xi)$, $\quad N_{\alpha_1 \cdots \alpha_l}(\xi)$, $l \geq 1$, $\alpha_j = 1, 2, \cdots n$ in the following way:

$N_0(\xi) = I$, $\quad$ I is a unit matrix.

$$\begin{cases} \dfrac{\partial}{\partial \xi_p}(A^{pq}(\xi)\dfrac{\partial N_{\alpha_1}(\xi)}{\partial \xi_q}) = -\dfrac{\partial}{\partial \xi_p}(A^{p\alpha_1}(\xi)) & \text{in} \quad Q \\ N_{\alpha_1}(\xi) = 0 \quad \text{on} \quad \partial Q \end{cases} \tag{2.11}$$

$$\begin{cases} \dfrac{\partial}{\partial \xi_p}(A^{pq}(\xi)\dfrac{\partial N_{\alpha_1\alpha_2}(\xi)}{\partial \xi_q}) = -\dfrac{\partial}{\partial \xi_p}(A^{p\alpha_1}(\xi)N_{\alpha_2}(\xi)) \\ \qquad -A^{\alpha_1 q}(\xi)\dfrac{\partial N_{\alpha_2}(\xi)}{\partial \xi_q} - A^{\alpha_1\alpha_2}(\xi) + \widehat{A}^{\alpha_1\alpha_2} \quad \text{in} \quad Q \\ N_{\alpha_1\alpha_2}(\xi) = 0 \quad \text{on} \quad \partial Q \end{cases} \tag{2.12}$$

where

$$\widehat{A}^{\alpha_1\alpha_2} = \int_Q (A^{\alpha_1\alpha_2}(\xi) + A^{\alpha_1 q}(\xi)\frac{\partial N_{\alpha_2}(\xi)}{\partial \xi_q})d\xi \tag{2.13}$$

For $|\alpha| = l \geq 3$

$$\begin{cases} \dfrac{\partial}{\partial \xi_p}(A^{pq}(\xi)\dfrac{\partial N_{\alpha_1\cdots\alpha_l}(\xi)}{\partial \xi_q}) = -\dfrac{\partial}{\partial \xi_p}(A^{p\alpha_1}(\xi)N_{\alpha_2\cdots\alpha_l}(\xi)) \\ \qquad -A^{\alpha_1 q}(\xi)\dfrac{\partial N_{\alpha_2\cdots\alpha_l}(\xi)}{\partial \xi_q} - A^{\alpha_1\alpha_2}(\xi)N_{\alpha_3\cdots\alpha_l}(\xi) \quad \text{in} \quad Q \\ N_{\alpha_1\cdots\alpha_l}(\xi) = 0 \quad \text{on} \quad \partial Q \end{cases} \tag{2.14}$$

Define the homogenized elastic system corresponding to (2.5) as follows:

$$\begin{cases} -\widehat{\mathcal{L}}U^0(x) \equiv -\dfrac{\partial}{\partial x_p}(\widehat{C}^{pq}(x)\dfrac{\partial U^0(x)}{\partial x_q}) = f(x) \quad \text{in} \quad \Omega \\ U^0(x) = \bar{u}(x) \quad \text{on} \quad \Gamma_u \\ \hat{\sigma}(U^0) \equiv \nu_p \widehat{C}^{pq}\dfrac{\partial U^0(x)}{\partial x_q} = \nu_p B^{pq}\dfrac{\partial U^0(x)}{\partial x_q} = \phi(x) \quad \text{on} \quad \Gamma_\sigma \end{cases} \tag{2.15}$$

where

$$\widehat{C}^{pq}(x) = \left(\widehat{C}^{pq}_{ij}(x)\right) = \begin{cases} \widehat{A}^{pq} = \left(\hat{a}^{pq}_{ij}\right) & x \in \overline{\Omega}_1 \\ B^{pq} = \left(b^{pq}_{ij}\right) & x \in \Omega_2 \end{cases} \tag{2.16}$$

Remark 2.1. One can verify that $\widehat{\mathcal{L}}$ is a linear elastic operator, i.e. $\widehat{C}^{pq}(x) = (\widehat{C}^{pq}_{ij}(x))$, $p, q = 1, 2, \cdots n$ satisfy the conditions (2),(3) of Definition 2.1, see [5].

Remark 2.2. It is obvious to prove that $U^0(x) \in H^1(\Omega, \Gamma_u) = \left\{v \in H^1(\Omega), v|_{\Gamma_u} = 0\right\}$. Set $U^0(x)|_{\Omega_1} = u^0(x)$, $U^0(x)|_{\Omega_2} = w(x)$

Next let us consider the truncation function of $U^\varepsilon(x)$ and its error estimate.

Let

$$U^\varepsilon_s(x) = \begin{cases} u^\varepsilon_s(x) = u^0(x) + \sum\limits_{l=1}^{s} \varepsilon^l \sum\limits_{|\alpha|=l} N_\alpha(\xi)D^\alpha u^0(x), \quad x \in \overline{\Omega}_1 \\ w(x) \quad x \in \Omega_2 \end{cases} \tag{2.17}$$

Theorem 2.1. *Let $U^\varepsilon(x)$ be the weak solution of the linear elastic system (2.5), $u^0 \in H^{s+2}(\overline{\Omega}_1)$,then holds*

$$\|U^\varepsilon(x) - U^\varepsilon_s(x)\|_{H^1(\Omega,\Gamma_u)} \leq C\varepsilon^{1/2}\|u^0\|_{H^3(\Omega_1)}, \quad s = 1 \tag{2.18}$$

$$\|U^\varepsilon(x) - U^\varepsilon_s(x)\|_{H^1(\Omega,\Gamma_u)} \le C\varepsilon^{s-1}\|u^0\|_{H^{s+2}(\Omega_1)}, \quad s \ge 2 \tag{2.19}$$

where a constant $C > 0$ is independent of ε.

Proof. The proof of (2.18) refers to §2.1, Chap.II of [5], and now we prove only that (2.19) is valid.

$$\mathcal{L}_\varepsilon(U^\varepsilon(x) - U^\varepsilon_s(x)) = \mathcal{L}_\varepsilon(w(x) - w(x)) = 0 \quad x \in \Omega_2 \tag{2.20}$$

if $x \in \overline{\Omega}_1$

$$\begin{aligned}
&\mathcal{L}_\varepsilon(U^\varepsilon(x) - U^\varepsilon_s(x)) = \mathcal{L}_\varepsilon(u^\varepsilon(x) - u^\varepsilon_s(x))\\
&= \varepsilon^{s-1}\sum_{\alpha_1,\cdots\alpha_{s+1}=1}^{n}\frac{\partial}{\partial\xi_p}(A^{pq}(\xi)\frac{\partial N_{\alpha_1,\cdots\alpha_{s+1}}}{\partial\xi_q})D^\alpha u^0(x)\\
&+\varepsilon^{s-1}\sum_{\alpha_1,\cdots,\alpha_{s+1}=1}^{n}\frac{\partial}{\partial\xi_p}(A^{pq}(\xi)\frac{\partial N_{\alpha_1\cdots\alpha_{s+1}}}{\partial\xi_q})D^\alpha u^0(x)\\
&+\varepsilon^{s-1}\sum_{\alpha_1,\cdots,\alpha_{s+1}=1}^{n}\frac{\partial}{\partial\xi_p}(A^{p\alpha_1}(\xi)N_{\alpha_2\cdots\alpha_{s+1}})D^\alpha u^0(x)\\
&+\varepsilon^{s}\sum_{\alpha_1,\cdots,\alpha_{s+2}=1}^{n}A^{\alpha_1 q}(\xi)\frac{\partial N_{\alpha_2\cdots\alpha_{s+2}}}{\partial\xi_q}D^\alpha u^0(x) = \varepsilon^{s-1}F_0(x,\varepsilon)
\end{aligned} \tag{2.21}$$

where $\|F_0\|_{L^2(\Omega)} \le C\|u^0(x)\|_{H^{s+2}(\Omega_1)}$

$$U^\varepsilon(x) - U^\varepsilon_s(x) = u^\varepsilon(x) - u^\varepsilon_s(x) = \bar{u}(x) - \bar{u}(x) = 0 \quad \text{on} \quad x \in \Gamma_u \tag{2.22}$$

$$\sigma_\varepsilon(U^\varepsilon(x) - U^\varepsilon_s(x)) = \sigma(w(x) - w(x)) = 0 \quad \text{on} \quad x \in \Gamma_\sigma \tag{2.23}$$

It follows from Korn's inequality that

$$\begin{aligned}
&\|U^\varepsilon(x) - U^\varepsilon_s(x)\|^2_{H^1(\Omega,\Gamma_u)} \le C\|e(U^\varepsilon - U^\varepsilon_s)\|^2_{L^2(\Omega)}\\
&\le C\int_\Omega (C^{pq}\frac{\partial(U^\varepsilon - U^\varepsilon_s)}{\partial x_q}, \frac{\partial(U^\varepsilon - U^\varepsilon_s)}{\partial x_p})dx\\
&= C\varepsilon^{s-1}\int_{\Omega_1}\Big(F_0, U^\varepsilon - U^\varepsilon_s\Big)dx\\
&\le C\varepsilon^{s-1}\|u^0\|_{H^{s+2}(\Omega_1)}\|U^\varepsilon(x) - U^\varepsilon_s(x)\|_{L^2(\Omega_1)}\\
&\le C\varepsilon^{s-1}\|u^0\|_{H^{s+2}(\Omega_1)}\|U^\varepsilon(x) - U^\varepsilon_s(x)\|_{H^1(\Omega,\Gamma_u)}
\end{aligned}$$

Therefore

$$\|U^\varepsilon(x) - U^\varepsilon_s(x)\|_{H^1(\Omega,\Gamma_u)} \le C\varepsilon^{s-1}\|u^0\|_{H^{s+2}(\Omega_1)}, \quad s \ge 2$$

3. Multiscale Finite Element Method

For convenience, here we discuss only 2-D problems.

3.1. FE computation of periodic solutions $N_\alpha(\xi)$

Let $\mathcal{J}^{h_0} = \{K\}$ be a regular family of triangulations of the unit cell Q, $h_0 = \max_K h_K$. Define a linear finite element space:

$$V_{h_0} = \{v = (v_{ij}) \in (C(\overline{Q}))^{2\times 2}: \quad v_{ij}|_{\partial Q} = 0, \quad v_{ij}|_K \in P_1(K)\} \subset H^1_0(Q) \tag{3.1}$$

The discrete variational forms corresponding to (2.11),(2.12)and (2.14) are as follows:

$$a(N^{h_0}_{\alpha_1}, v_{h_0}) = F_{\alpha_1}(v_{h_0}) \quad \forall v_{h_0} \in V_{h_0} \tag{3.2}$$

$$a(N^{h_0}_{\alpha_1\alpha_2}, v_{h_0}) = F_{\alpha_1\alpha_2}(v_{h_0}) \quad \forall v_{h_0} \in V_{h_0} \tag{3.3}$$

and

$$a(N^{h_0}_{\alpha_1\cdots\alpha_l}, v_{h_0}) = F_{\alpha_1\cdots\alpha_l}(v_{h_0}) \quad \forall v_{h_0} \in V_{h_0} \tag{3.4}$$

where the bilinear form

$$a(w, v_{h_0}) = \int_Q (A^{pq}\frac{\partial w}{\partial\xi_q}, \frac{\partial v_{h_0}}{\partial\xi_p})d\xi \tag{3.5}$$

the linear functionals

$$F_{\alpha_1}(v_{h_0}) = -\int_Q (A^{p\alpha_1}(\xi), \frac{\partial v_{h_0}(\xi)}{\partial\xi_p})d\xi \tag{3.6}$$

$$
\begin{aligned}
F_{\alpha_1\alpha_2}(v_{h_0}) = &-\int_Q (A^{p\alpha_1}(\xi)N^{h_0}_{\alpha_2}(\xi), \frac{\partial v_{h_0}(\xi)}{\partial \xi_p})d\xi \\
&+\int_Q (A^{\alpha_1 q}(\xi)\frac{\partial N^{h_0}_{\alpha_2}(\xi)}{\partial \xi_q} + A^{\alpha_1\alpha_2}(\xi) - \widehat{A}^{\alpha_1\alpha_2,h_0}, v_{h_0}(\xi))d\xi
\end{aligned}
\tag{3.7}
$$

Note that $\widehat{A}^{pq,h_0}$ will be defined in (3.11), and

$$
\begin{aligned}
F_{\alpha_1\cdots\alpha_l}(v_{h_0}) = &-\int_Q (A^{p\alpha_1}(\xi)N^{h_0}_{\alpha_2\cdots\alpha_l}(\xi), \frac{\partial v_{h_0}(\xi)}{\partial \xi_p})d\xi \\
&+\int_Q (A^{\alpha_1 q}(\xi)\frac{\partial N^{h_0}_{\alpha_2\cdots\alpha_l}(\xi)}{\partial \xi_q} + A^{\alpha_1\alpha_2}(\xi)N^{h_0}_{\alpha_3\cdots\alpha_l}(\xi), v_{h_0}(\xi))d\xi
\end{aligned}
\tag{3.8}
$$

Since $V_{h_0} \subset H^1_0(Q)$, existence and uniqueness of periodic solutions $N_{\alpha_1}(\xi)$, $N_{\alpha_1\cdots\alpha_l}(\xi)$ associated with (3.2)- (3.4) can be easily proved by Korn's inequality and Lax-Milgram lemma.

One can prove that the following theorem without any difficulty.

Theorem 3.1. *Let $N_{\alpha_1\cdots\alpha_j}(\xi)$, $\quad 1 \le j \le l$, $\quad \alpha_j = 1,2$, $\quad l = 1,2,\cdots$ be the weak solutions of problems (2.11),(2.12)and (2.14) ,respectively. $N^{h_0}_{\alpha_1\cdots\alpha_j}(\xi)$ are FE solutions associated with (2.11), (2.12) and (2.14), respectively, if $N_{\alpha_1\cdots\alpha_j}(\xi) \in H^2(Q)$, $\quad j = 1,2\cdots l$, then holds*

$$
\|N_{\alpha_1\cdots\alpha_l} - N^{h_0}_{\alpha_1\cdots\alpha_l}\|_{H^1_0(Q)} \le Ch_0\Big(\sum_{j=1}^{l}\|N_{\alpha_1\cdots\alpha_j}\|_{H^2(Q)}\Big) \tag{3.9}
$$

where C=const > 0 is independent of h_0, $\quad N_{\alpha_1\cdots\alpha_j}$, $\quad 1 \le j \le l$.

3.2. FE computation of the modified homogenized linear elastic system

If we solve the above problem (2.15), in practice , we need to solve the modified boundary value problem as follows:

$$
\begin{cases}
-\widehat{\mathcal{L}}_{h_0}\overline{U}^0(x) \equiv -\dfrac{\partial}{\partial x_p}(\widehat{C}^{pq,h_0}(x)\dfrac{\partial \overline{U}^0(x)}{\partial x_q}) = f(x) & \text{in} \quad \Omega \\
\overline{U}^0(x) = \bar{u}(x) & \text{on} \quad \Gamma_u \\
\hat{\sigma}(\overline{U}^0) \equiv \nu_p \widehat{C}^{pq,h_0}\dfrac{\partial \overline{U}^0}{\partial x_q} = \nu_p B^{pq}\dfrac{\partial \overline{U}^0}{\partial x_q} = \phi(x) & \text{on} \quad \Gamma_\sigma
\end{cases}
\tag{3.10}
$$

where

$$
\widehat{A}^{pq,h_0} = \int_Q (A^{pq}(\xi) + A^{pm}(\xi)\frac{\partial N^{h_0}_q(\xi)}{\partial \xi_m})d\xi \tag{3.11}
$$

and

$$
\widehat{C}^{pq,h_0} = \Big(\widehat{C}^{pq,h_0}_{ij}(x)\Big) = \begin{cases} \widehat{A}^{pq,h_0} = \Big(\hat{a}^{pq,h_0}_{ij}\Big) & x \in \overline{\Omega}_1 \\ B^{pq} = \Big(b^{pq}_{ij}\Big) & x \in \Omega_2 \end{cases} \tag{3.12}
$$

Remark 3.1. One can verify that $\widehat{\mathcal{L}}_{h_0}$ is a linear elastic operator, see [7].

Theorem 3.3. *Let $U^0(x)$ be the weak solution of the homogenized linear elastic system (2.15), $\overline{U}^0(x)$ be the weak solution of the modified homogenized problem (3.10), and $\overline{U}^0_h(x)$ be the FE solution corresponding to $\overline{U}^0(x)$, $N_k(\xi) \in H^2(Q)$, $\quad f(x) \in L^2(\Omega), \phi \in H^1(\Omega)$,then holds:*

$$
\|U^0(x) - \overline{U}^0_h(x)\|_{H^1(\Omega,\Gamma_u)} \le C\Big\{h_0\|N_k\|_{H^2(Q)} + h(\|f\|_{L^2(\Omega)} + \|\phi\|_{H^1(\Omega)})\Big\} \tag{3.13}
$$

where C=const > 0 is independent of h, $\quad h_0$; h, $\quad h_0$ are the mesh parameters of domains Ω and Q ,respectively.

4. Higher Order Difference Quotients and Total Error Estimates

In this section , we shall discuss the approximate computations of partial derivatives for the vector valued displacement $u^0(x) = U^0(x)|_{\Omega_1}$.

To begin with, introduce higher-order difference quotients as follows:

$$
\delta_{x_i}u^0(M) = \frac{1}{\tau(M)}\sum_{T\in\sigma(M)}\left[\frac{\partial \bar{u}^0_h}{\partial x_i}\right]_T \tag{4.1}
$$

where $\sigma(M)$ is the set of elements on which M is one of the vertices;
$\tau(M)$ is the number of elements of $\sigma(M)$;
$\bar{u}^0_h(x) = \overline{U}^0_h(x)|_{\Omega_1}$ is the FE solution corresponding to $\bar{u}^0(x) = \overline{U}^0(x)|_{\Omega_1}$;
$[\frac{\partial \bar{u}^0_h}{\partial x_i}]_T$ is the value of the derivative $\frac{\partial \bar{u}^0_h}{\partial x_i}$ in the element T.
Similarly, we can define the higher-order difference quotients as follows, $l \geq 2$

$$\delta^l_{x_{\alpha_1}\cdots x_{\alpha_{l-1}} x_{\alpha_l}} u^0(M) = \frac{1}{\tau(M)} \sum_{T\in\sigma(M)} [\sum_{j=1}^{d} \delta^{l-1}_{x_{\alpha_1}\cdots x_{\alpha_{l-1}}} u^0(M_j) \frac{\partial \psi_j(x)}{\partial x_{\alpha_l}}]_T \tag{4.2}$$

To sum up, we give a multiscale FE scheme for computing $U^\varepsilon(x)$ as follows:

$$\overline{U}^{\varepsilon,h}_{s,h_0}(x) = \begin{cases} \bar{u}^0_h(x) + \sum\limits_{l=1}^{s} \varepsilon^l \sum\limits_{|\alpha|=l} N^{h_0}_\alpha(\xi) \delta^l_{x_{\alpha_1}\cdots x_{\alpha_l}} u^0(x), & x \in \overline{\Omega}_1 \\ w_h(x) \quad x \in \Omega_2 \end{cases} \tag{4.3}$$

where $\bar{u}^0_h(x) = \overline{U}^0_h(x)|_{\Omega_1}$, $w_h(x) = \overline{U}^0_h(x)|_{\Omega_2}$, $N^{h_0}_\alpha(\xi)$ are the FE solutions defined in (3.2) -(3.4) corresponding to $N_\alpha(\xi)$, respectively.
Combining with (2.19) and (3.9), we can prove that the following theorem:

Theorem 4.1. *Let $U^\varepsilon(x)$ be the weak solution of the original linear elastic system (2.5), and $\overline{U}^{\varepsilon,h}_{s,h_0}(x)$ be approximate solution calculated in (4.3), under the assumptions of Theorem 2.1, then holds*

$$\|U^\varepsilon(x) - \overline{U}^{\varepsilon,h}_{s,h_0}(x)\|_{H^1(\Omega,\Gamma_u)} \leq C(\varepsilon^{1/2} + h_0 + h), \quad s = 1 \tag{4.4}$$

$$\|U^\varepsilon(x) - \overline{U}^{\varepsilon,h}_{s,h_0}(x)\|_{H^1(\Omega,\Gamma_u)} \leq C(\varepsilon^{s-1} + h_0 + h), \quad s \geq 2 \tag{4.5}$$

where C=const > 0 is independent of ε, h_0, h , $\varepsilon > 0$ is the small parameter, h_0, h are the mesh parameters of domains Q, and Ω, respectively.

5. Numerical Experiments

Consider the plane stress problem as follows:

$$\begin{cases} \mathcal{L}_\varepsilon U^\varepsilon(x) \equiv -\frac{\partial}{\partial x_p}\Big(A^{pk}(\frac{x}{\varepsilon})\frac{\partial U^\varepsilon(x)}{\partial x_k}\Big) = f(x), \text{ in } \Omega \\ U^\varepsilon(x) = 0, \text{ on } \partial\Omega \end{cases} \tag{5.1}$$

where domain Ω and unit cell Q as shown in Figure 5.1 and Figure 5.2, respectively.

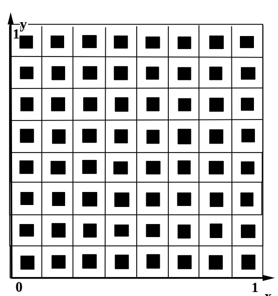

Figure 5.1: domain Ω

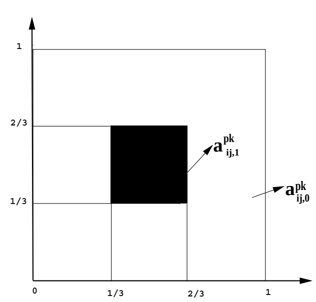

Figure 5.2: periodic cell Q

Assume that $\varepsilon = \frac{1}{8}$, $f(x) = 2.78 \times 10^6 Mpa$, and matrice functions $A^{pk}(\frac{x}{\varepsilon}) = (a^{pk}_{ij}(\frac{x}{\varepsilon}))$, $p, k, i, j = 1, 2$

$$a^{pk}_{ij,0} = \lambda^{(0)}\delta_{ip}\delta_{jk} + \mu^{(0)}\delta_{ij}\delta_{pk} + \mu^{(0)}\delta_{ik}\delta_{pj}$$

$$a^{pk}_{ij,1} = \lambda^{(1)}\delta_{ip}\delta_{jk} + \mu^{(1)}\delta_{ij}\delta_{pk} + \mu^{(1)}\delta_{ik}\delta_{pj}$$

It is well known that there exist the following relations:

$$\lambda^{(l)} = \frac{E^{(l)} \cdot \nu^{(l)}}{(1+E^{(l)})(1-2\nu^{(l)})}, \quad \mu^{(l)} = \frac{E^{(l)}}{2(1+\nu^{(l)})}, \quad l = 0, 1$$

where $E^{(l)}, \quad \nu^{(l)}$ are moduli of elasticity, and Poisson's ratio, respectively.

Here we choose $\nu^{(0)} = \nu^{(1)} = 0.25$.

Since it is difficult to find the analytic solution of the original problem (5.1), we have to replace $U^{\varepsilon}(x)$ with its FE solution in a very refined mesh. Here we implement the uniform rectangular subdivision of Ω, which is such that the discontinuities of the coefficients coincide with sides of the rectangule.

Table 5.1. Compare with computational amount

	original problem	unit cell problem	homogenized problem
number of elements	20736	5184	5184
number of nodes	21025	5329	5329

Case 1. $E^{(0)} = 10^8\ MPa, \quad E^{(1)} = 10^9\ MPa$;
Case 2. $E^{(0)} = 10^9\ MPa, \quad E^{(1)} = 10^7\ MPa$;
Case 3. $E^{(0)} = 10^9\ MPa, \quad E^{(1)} = 10^6\ MPa$.

Let the homogenized matrices

$$\widehat{A} = \begin{pmatrix} \hat{a}_{ij}^{11} & \hat{a}_{ij}^{12} \\ \hat{a}_{ij}^{21} & \hat{a}_{ij}^{22} \end{pmatrix}$$

where $(\hat{a}_{ij}^{pk}), \quad p, k = 1, 2$ are 2×2 matrices, respectively.

$$\textbf{Case 1} \quad \widehat{A} = \begin{bmatrix} 0.13254E+09 & 0.46060E-07 & 0.54508E-07 & 0.48206E+08 \\ 0.46060E-07 & 0.35521E+08 & 0.48206E+08 & -0.13372E-07 \\ 0.54508E-07 & 0.48206E+08 & 0.35521E+08 & -0.27393E-07 \\ 0.48206E+08 & -0.13372E-07 & -0.27393E-07 & 0.13254E+09 \end{bmatrix}$$

$$\textbf{Case 2} \quad \widehat{A} = \begin{bmatrix} 0.87279E+09 & 0.11618E-06 & 0.18670E-06 & 0.28056E+09 \\ 0.11618E-06 & 0.25473E+09 & 0.28056E+09 & 0.73887E-07 \\ 0.18670E-06 & 0.28056E+09 & 0.25473E+09 & 0.15438E-06 \\ 0.28056E+09 & 0.73887E-06 & 0.15438E-06 & 0.87279E+09 \end{bmatrix}$$

$$\textbf{Case 3} \quad \widehat{A} = \begin{bmatrix} 0.86451E+09 & 0.15937E-06 & 0.25034E-06 & 0.27722E+09 \\ 0.15937E-06 & 0.24999E+09 & 0.27722E+09 & 0.19517E-06 \\ 0.25034E-06 & 0.27722E+09 & 0.24999E+09 & 0.20257E-06 \\ 0.27722E+09 & 0.19517E-06 & 0.20257E-06 & 0.86451E+09 \end{bmatrix}$$

Table 5.2. Comparison of computation results

	$\frac{\Vert e_0\Vert_{L^2}}{\Vert U^0\Vert_{L^2}}$	$\frac{\Vert e_1\Vert_{L^2}}{\Vert U_1^{\varepsilon}\Vert_{L^2}}$	$\frac{\Vert e_2\Vert_{L^2}}{\Vert U_2^{\varepsilon}\Vert_{L^2}}$	$\frac{\Vert e_0\Vert_{H^1}}{\Vert U^0\Vert_{H^1}}$	$\frac{\Vert e_1\Vert_{H^1}}{\Vert U_1^{\varepsilon}\Vert_{H^1}}$	$\frac{\Vert e_2\Vert_{H^1}}{\Vert U_2^{\varepsilon}\Vert_{H^1}}$
Case1	0.027987	0.010283	0.010839	0.37562	0.089584	0.086748
Case2	0.06372	0.045796	0.033933	1.0378	0.65564	0.15333
Case3	0.37893	0.37612	0.058611	7.9292	6.4542	0.12409

Here, for the sake of convenience , $U^{\varepsilon}(x)$, $U^0(x)$ denote the FE solutions of the longitudinal dispalcements for the original linear elastic system (5.1) and the corresponding homogenized one respectively. $U_1^{\varepsilon}(x), \quad U_2^{\varepsilon}(x)$ are respectively the first-order and the second-order multiscale FE solutions for the longitudinal displacements calculated by multiscale FE scheme (4.3), let $e_0 = U^{\varepsilon} - U^0, \quad e_1 = U^{\varepsilon} - U_1^{\varepsilon}, \quad e_2 = U^{\varepsilon} - U_2^{\varepsilon}$.

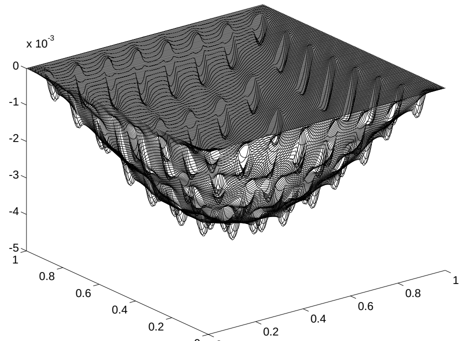

Figure 5.3a, Case2 :solution U^{ε}

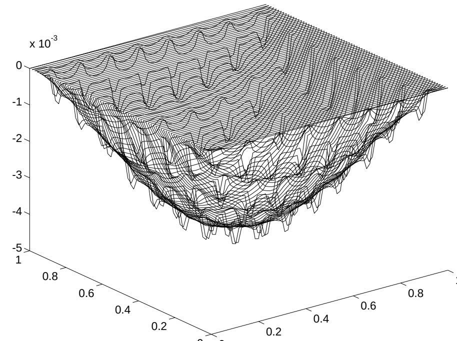

Figure 5.3b, Case2: MFEM U_2^{ε}

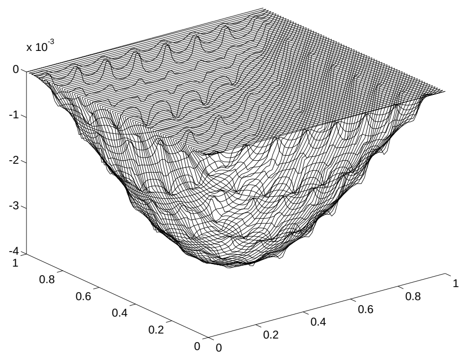

Figure 5.3c, Case 2: MFEM U_1^ε

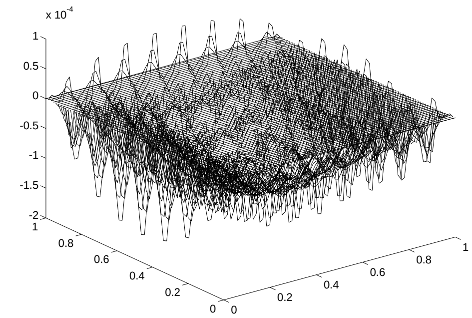

Figure 5.3d , Case2: $e_2 = U^\varepsilon - U_2^\varepsilon$

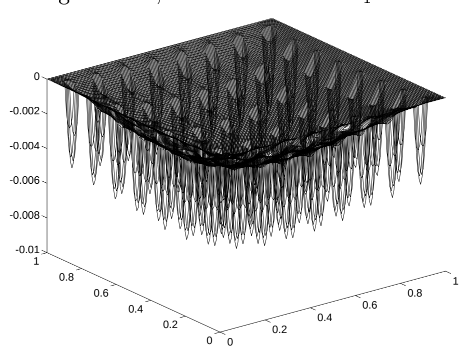

Figure 5.4 a, Case3 :solution U^ε

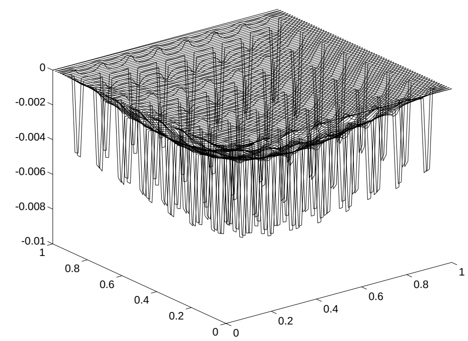

Figure 5.4b, Case3: MFEM U_2^ε

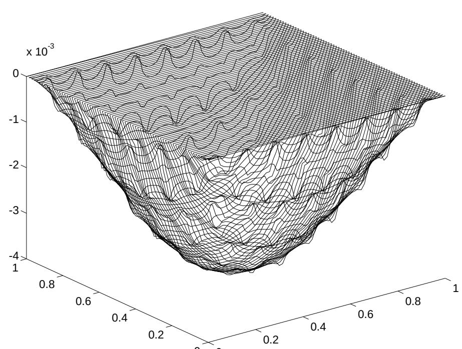

Figure 5.4 , Case 3: MFEM U_1^ε

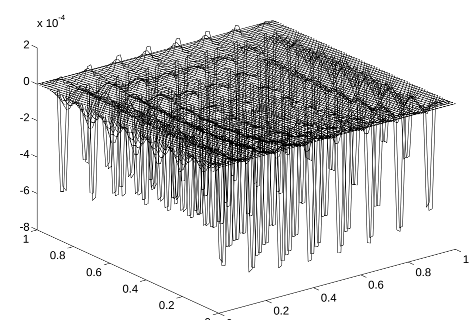

Figure 5.4d , Case3: $e_2 = U^\varepsilon - U_2^\varepsilon$

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