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REGULARIZATION OF SINGULAR SYSTEMS BY OUTPUT FEEDBACK[*1)]

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Abstract

Problem of regularization of a singular system by derivative and proportional output feedback is studied. Necessary and sufficient conditions are obtained under which a singular system can be regularized into a closed-loop system that is regular and of index at most one. The reduced form is given that can easily explore the system properties as well as the feedback to be determined. The main results of the present paper are based on orthogonal transformations. Therefore, they can be implemented by numerically stable ways.

Key words: Regularization, Singular systems, Output feedback.

1. Introduction

In this paper, we consider a linear time-invariant system

$$\begin{cases} E\dot{x} = Ax + Bu \\ y = Cx \end{cases} \tag{1}$$

with the matrices $E, A \in R^{n\times n}$, $B \in R^{n\times m}$, $C \in R^{p\times n}$, and E is singular. The behavior of a singular system depends critically on the eigenstructure of the pencil (E, A). The pencil (E, A) and the corresponding system (1) are said to be regular if

$$\det(\alpha E - \beta A) \neq 0 \quad \text{for some } (\alpha, \beta) \in C^2 \tag{2}$$

and they are said to have index at most one if the dimension of the largest nilpotent block (which corresponds to an infinite pole) in the Kronecker canonical form of the pencil (E, A) is less than or equal to one[3,12].

A regular system of index at most one can be tansformed and seperated into a purely dynamical and a algebraic part. The algebraic part can be eliminated to give a standard system of (possibly) reduced order. Higher index singular systems can't be reduced to standard systems, and impulses can arise in the response if the control is not sufficiently smooth. The system can even lose causality[6].

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For a state feedback regularization of a singular system ($C = I$), numerous studies have been done. However, as we know, only [20, 21] investigated the output feedback regularization of singular systems and presented two numerical procedures for the feedback matrices. The results in [20, 21] can not be used straightly to study the output feedback eigenvalue assignment of the singular system because of that the reduced forms they used, for regularization of a singular system did not characterized.

The main objective of this study is to regularize a singular system by output feedback. We present some necessary and sufficient conditions for the existence of feedback matrices F, G such that the closed loop system is regular, of index at most one. The reduced form we use is a generalized controllable and observable canonical form of singular systems, so our results could form the basis of researching the eigenvalue assignment of a singular system by proportional and derivative output feedback.

This paper is arranged as folllows. Section 2 reviews some basic results and describes a reduced form relating to singular systems. Section 3 and Section 4 present and prove, respectively, necessary and sufficient conditions for output feedback regularization problem of a singular system based on the reduced form of Section 2.

2. Preliminaries

In this paper, we denote

$$r_e = \operatorname{rank}(E),\ r_{eb} = \operatorname{rank}[E\ B],\ r_{ec} = \operatorname{rank}\begin{bmatrix} E \\ C \end{bmatrix},\ r_{ebc} = \operatorname{rank}\begin{bmatrix} E & B \\ C & 0 \end{bmatrix}$$
$$S_{ebc} = \{r | r \text{ is an integer with } r_{eb} + r_{ec} - r_{ebc} \le r \le \min(r_{eb}, r_{ec})\}$$

We also use the following concepts.

Definition 2.1. *Let $E, A \in R^{n\times n}$, $B \in R^{n\times m}$, $[E, A]$ is regular, System* (1) *is completely controllable* (C*-controllable*) *if and only if*

$C0$: rank $[\alpha E - \beta A\ B] = n, \quad \forall(\alpha, \beta) \in C^2 \backslash \{0, 0\}$.

Definition 2.2. *Let $E, A \in R^{n\times n}$, $B \in R^{n\times m}$, $[E, A]$ is regular. System* (1) *is strongly controlable* (S*-controlable*) *if and only if*

C_1 : rank $[\lambda E - A\ B] = n$, *$\forall \lambda \in C$;*

C_2 : rank $[E\ AS_E\ B] = n$, *where the columns of S_E span $N(E)$, the nullspace of E.*

Definition 2.3. *Let $E, A \in R^{n\times n}$, $C \in R^{p\times n}$, $[E, A]$ be regular. System* (1) *is C-observable, S-observable if and only if $[E^T, A^T, C^T]$ satisfies conditions $C0, C1$ and $C2$, respectively.*

Obviously, if system (1) is C-controllable (C-observable), it is S-controllable (S-observable) too.

2.1 Two Basic Lemmas

We cite two well-known results. The first one will serve as a handy tool to determine if matrix pencil $[E, A]$ is regular and is index at most one. The second one indicates the allowable range of the rank of matrix $E + BGC$.

Lemma 2.4[12]**.** *Let $E, A \in R^{n\times n}$, then pencil $[E, A]$ is regular and has index at most one if and only if*

$$\deg\ (\ \det\ (\lambda E - A)) = \text{rank}\ (E),$$

where $\deg\ (f(\lambda))$ *denotes the degree of polynomial* $f(\lambda)$.

Lemma 2.5[20]**.** *Let $E \in R^{n\times n}$, $B \in R^{n\times m}$, $C \in R^{p\times n}$, then for any integer r satisfying*

$$r_{eb} + r_{ec} - r_{ebc} \le r \le \min(r_{eb}, r_{ec})$$

there exists $G_0 \in R^{m\times p}$ such that

$$\text{rank}\ (E + BG_0C) = r.$$

Or equivalently,

$$\{\text{rank}\ (E + BGC) | G \in R^{m\times p}\} = S_{ebc}.$$

2.2 Reduced Form

In this subsection we present a reduced form for system (1) that can be computed by using orthogonal transformations. This reduced form is used to establish the main result in Section 3.

Lemma 2.6[19]**.** *If $E, A \in R^{n\times n}$, $B \in R^{n\times m}$ and $C \in R^{p\times n}$, then there exist orthogonal matrices $U_1^{(1)}, V_1^{(1)} \in R^{n\times n}$ such that*

$$U_1^{(1)}EV_1^{(1)} = \begin{bmatrix} E_{11}^{(1)} & E_{12}^{(1)} & E_{13}^{(1)} \\ & E_{22}^{(1)} & E_{23}^{(1)} \\ & & E_{33}^{(1)} \end{bmatrix}, \quad U_1^{(1)}AV_1^{(1)} = \begin{bmatrix} A_{11}^{(1)} & A_{12}^{(1)} & A_{13}^{(1)} \\ & A_{22}^{(1)} & A_{23}^{(1)} \\ & & A_{33}^{(1)} \end{bmatrix},$$

$$U_1^{(1)}B = \begin{bmatrix} B_{11}^{(1)} \\ 0 \\ 0 \end{bmatrix}, \quad CV_1^{(1)} = [C_{11}^{(1)}\ C_{12}^{(1)}\ C_{13}^{(1)}]$$

where $E_{ij}^{(1)}, A_{ij}^{(1)} \in R^{\tilde{n}_i\times\tilde{n}_j}$, $C_{1j} \in R^{p\times\tilde{n}_j}$, $i, j = 1, 2, 3$. $B_{11}^{(1)} \in R^{\tilde{n}_1\times m}$, $E_{22}^{(1)}$ is nonsingular, $E_{33}^{(1)}$ and $A_{33}^{(1)}$ are upper triangular, the diagonal entires of $E_{33}^{(1)}$ are all zero, and $(E_{11}^{(1)}, A_{11}^{(1)}, B_{11}^{(1)})$ is C-controllable, i.e.

$$\text{rank}\ [\alpha E_{11}^{(1)} - \beta A_{11}^{(1)}\ B_{11}^{(1)}] = \tilde{n}_1, \quad \forall(\alpha, \beta) \in C^2\backslash\{0, 0\}.$$

Applying Lemma 2.6 to $[(E_{11}^{(1)})^T, (A_{11}^{(1)})^T, (C_{11}^{(1)})^T]$, we have

Theorem 2.7. *If $E, A \in R^{n\times n}$, $B \in R^{n\times m}$ and $C \in R^{p\times n}$, then there exist orghogonal matrices $U^*, V^* \in R^{n\times n}$ such that*

$$E^* = U^*EV^* = \begin{bmatrix} E_{11} & & & E_{14} & E_{15} \\ E_{21} & E_{22} & & E_{24} & E_{25} \\ E_{31} & E_{32} & E_{33} & E_{34} & E_{35} \\ & & & E_{44} & E_{45} \\ & & & & E_{55} \end{bmatrix},$$

$$A^* = U^*AV^* = \begin{bmatrix} A_{11} & & & A_{14} & A_{15} \\ A_{21} & A_{22} & & A_{24} & A_{25} \\ A_{31} & A_{32} & A_{33} & A_{34} & A_{35} \\ & & & A_{44} & A_{45} \\ & & & & A_{55} \end{bmatrix},$$

$$B^* = U^*B = \begin{bmatrix} B_{11} \\ B_{21} \\ B_{31} \\ 0 \\ 0 \end{bmatrix}, \quad C^* = CV^* = [C_{11}\ 0\ 0\ C_{14}\ C_{15}]$$

where $E_{ij}, A_{ij} \in R^{n_i \times n_j}$, $B_{i1} \in R^{n_i \times m}$, $C_{1j} \in R^{p \times n_j}$, $i, j = 1, \cdots, 5$. E_{22}, E_{24} *are nonsingular,* $E_{33}^T, A_{33}^T, E_{55}$ *and* A_{55} *are upper triangular, the diagonal entires of* E_{33} *and* E_{55} *are all zero. Moreover,*

$$\text{rank} \begin{bmatrix} \alpha E_{11} - \beta A_{11} & & & B_{11} \\ \alpha E_{21} - \beta A_{21} & \alpha E_{22} - \beta A_{22} & & B_{21} \\ \alpha E_{31} - \beta A_{31} & \alpha E_{32} - \beta A_{32} & \alpha E_{33} - \beta A_{33} & B_{31} \end{bmatrix} = n_1 + n_2 + n_3,$$
$$\forall (\alpha, \beta) \in C^2 \backslash \{0, 0\} \tag{3}$$

$$\text{rank} \begin{bmatrix} \alpha E_{11} - \beta A_{11} \\ C_{11} \end{bmatrix} = n_1, \quad \forall (\alpha, \beta) \in C^2 \backslash \{0, 0\}. \tag{4}$$

Equalities (3) and (4) give that $[E_{11}, A_{11}, B_{11}]$ and $[E_{11}, A_{11}, C_{11}]$ are, respectively, C-controllable and C-observable, and for any matrices $F, G \in R^{m \times p}$, the eigenvalues of pencils $[E_{ii}, A_{ii}]$, $i = 2, \cdots, 5$ are also the eigenvalues of pencil $[E+BGC, A+BFC]$, so $[E^*, A^*, B^*, C^*]$ is a generalized controllable and observable canonical form of system (1).

$[E^*, A^*, B^*, C^*]$ can be computed by a numerically stable procedure[19].

3. Main Results

A part from regularizing the system, these feedback matrices F, G make the pencil $[E+BGC, A+BFC]$ being regular and of index at most one with the expected number of the finite poles.

Theorem 3.1. *Let* $E, A \in R^{n \times n}$, $B \in R^{n \times m}$, $C \in R^{p \times n}$. *Assume* E^*, A^*, B^*, C^* *are given in Theorem* 2.7 *and* $r_{ec} \le r_{eb}$, *then*

(i) *There exists* $G \in R^{m \times p}$ *such that pencil* $[E + BGC, A]$ *is regular with index at most one, and*

$$\text{rank } (E + BGC) = r_{ec}$$

if and only if

$$\begin{cases} E_{33} = 0,\ A_{33} \text{ is nonsingular}, & \text{if } n_3 > 0 \\ E_{55} = 0,\ A_{55} \text{ is nonsingular}, & \text{if } n_5 > 0 \\ r_{ec} = n_1 + n_2 + n_4 \end{cases} \tag{5}$$

(ii) *There exist* $F, G \in R^{m\times p}$ *such that the pencil* $[E+BCG, A+BFC]$ *is regular with index at most one and* rank $(E+BGC)=r_{ec}$ *if and only if* (5) *holds.*

Theorem 3.2. *Let* $E, A \in R^{n\times n}$, $B \in R^{n\times m}$, $C \in R^{p\times n}$. *Assume* E^*, A^*, B^*, C^* *are given in Theorem* 2.7 *and* $r_{eb} \le r_{ec}$, *then*

(iii) *There exists matrix* $G \in R^{m\times p}$ *such that the pencil* $[E+BGC, A]$ *is regular and with index at most one as well*

$$\text{rank } (E+BGC) = r_{eb}$$

if and only if

$$\begin{cases} n_3 = 0 \\ E_{55} = 0, \quad A_{55} \textit{ is nonsingular if } n_5 > 0. \end{cases} \tag{6}$$

(iv) *There exist* $F, G \in R^{m\times p}$ *such that pencil* $[E+BGC, A+BFC]$ *is regular with index at most one and* rank $(E+BGC)=r_{eb}$ *if and only if* (6) *is true.*

Theorem 3.1 (i) and Theorem 3.2 (iii) give two necessary and sufficient conditions for derivative output feedback regularization of singular system (1). These two conditions are very important, since they are also suitable for combined derivative and proportional output feedback, this is shown in the following two theorems.

Theorem 3.3. *Let* $E, A \in R^{n\times n}$, $B \in R^{n\times m}$, $C \in R^{p\times n}$. *Assume* E^*, A^*, B^*, C^* *are given in Theorem* 2.7 *and* $r_{ec} \le r_{eb}$. *If condition* (5) *holds, then for any integer* r *satisfying*

$$r_{eb} + r_{ec} - r_{ebc} \le r \le r_{ec}, \tag{7}$$

there exist $F, G \in R^{m\times p}$ *such that pencil* $[E+BGC, A+BFC]$ *is regular with index at most one, and*

$$\text{rank } (E+BGC) = r.$$

Theorem 3.4. *Let* $E, A \in R^{n\times n}$, $B \in R^{n\times m}$, $C \in R^{p\times n}$. *Assume* E^*, A^*, B^*, C^* *are given in Theorem* 2.7 *and* $r_{eb} \le r_{ec}$. *If condition* (6) *holds, then for any integer* r *satisfying*

$$r_{eb} + r_{ec} - r_{ebc} \le r \le r_{eb}, \tag{8}$$

there exist $F, G \in R^{m\times p}$ *such that pencil* $[E+BGC, A+BFC]$ *is regular with index at most one, and*

$$\text{rank } (E+BGC) = r.$$

Assume S_{ebc} is defined in Section 2 and denote

$$\begin{aligned} S = \{[F,G] | F, G \in R^{m\times n},\ & \text{pencil}[E+BGC, A+BFC] \\ & \text{is regular and with index at most one}\}, \\ S_0 = \{G | [F,G] \in S \text{ for} & \text{ some } F \in R^{m\times p}\}, \end{aligned}$$

then merging Theorem 3.1 (ii) and Theorem 3.3, Theorem 3.2 (iv) and Theorem 3.4, respectively, we have

Theorem 3.5. *Let $E, A \in R^{n\times n}$, $B \in R^{n\times m}$, $C \in R^{p\times n}$. Assume E^*, A^*, B^*, C^* are given in Theorem 2.7 and $r_{ec} \leq r_{eb}$. Then*

$$\text{rank } (E + BGC)|G \in S_0\} = S_{ebc}$$

if and only if (5) is true.

Theorem 3.6. *Let $E, A \in R^{n\times n}$, $B \in R^{n\times m}$, $C \in R^{p\times n}$. Assume E^*, A^*, B^*, C^* are given in Theorem 2.7 and $r_{eb} \leq r_{ec}$. Then*

$$\{\text{rank } (E + BGC)|G \in S_0\} = S_{ebc}$$

if and only if (6) is true.

Related to proportional output feedback, we have

Theorem 3.7. *Let $E, A \in R^{n\times n}$, $B \in R^{n\times m}$, $C \in R^{p\times n}$. Assume E^*, A^*, B^*, C^* are given in Theorem 2.7. Then there exists $F \in R^{m\times p}$ such that pencil $[E, A + BFC]$ is regular and has index at most one if and only if*

$$\begin{cases} E_{33} = 0, \quad A_{33} \text{ is nonsingular}, \quad \text{if } n_3 > 0 \\ E_{55} = 0, \quad A_{55} \text{ is nonsingular}, \quad \text{if } n_5 > 0 \\ r_e = n_2 + n_4 + \text{rank } (E_{11}). \end{cases} \tag{9}$$

It is easy to see that the 7 theorems above give necessary and sufficient conditions for regularization of singular system (1) by derivative output feedback, combined derivative and proportional output feedback and proportional output feedback, respectively. It is also obvious that these conditions are very easy to be checked.

[20] gives a sufficient conditions (i.e. $C1, O1, C2, O2$ therein) to ensure the existness of feedback matrices F and G such that the closed-loop sytem is regular with an index at most one and

$$t_1 \leq \text{rank } (E + BGC) \leq t_1 + t_2, \quad \text{for some positive integers } t_1, t_2$$

In this paper, we obtain necessary and sufficient conditions for the regularization of singular system (1) by derivative and/or proportional output feedback. These conditions ensure that there exists derivative and/or proportional output feedback, such that the closed-loop system is regular with an index at most one as well as

$$r_{eb} + r_{ec} - r_{ebc} \leq \text{rank } (E + BGC) \leq \min(r_{eb}, r_{ec}).$$

Specially, these lower and upper bounds are reachable.

From [20], it follows

$$r_{eb} = t_1 + t_2 + t_3, \quad r_{ec} = t_1 + t - 2 + t_5, \quad r_{ebc} = t_1 + 2t_2 + t_3 + t_5$$

where t_3, t_5 are positive integers defined in [20], therefore

$$t_1 + t_2 = r_{eb} - t_3 = r_{ec} - t_5$$

and

$$t_1 + t_2 \le \min(r_{eb}, r_{ec}).$$

Hence, our upper bound $\min(r_{eb}, r_{ec})$ is greater than that shown in [20].

4. Proofs of Main Results

Before coming to prove the main result given in Section 3, we need some lemmas.

Lemma 4.1. *Given $D \in R^{s\times s}$, $H \in R^{q\times s}$, $K \in R^{s\times t}$, $L \in R^{q\times t}$. If*

$$\mathit{rank}\ [D\ K] = s \tag{10}$$

then for any integer r satisfying

$$\mathit{rank}\begin{bmatrix} D & K \\ H & L \end{bmatrix} - \mathit{rank}\begin{bmatrix} K \\ L \end{bmatrix} \le r \le s, \tag{11}$$

there exists a matrix $W \in R^{t\times s}$ such that

$$\mathit{rank}\begin{bmatrix} D + KW \\ H + LW \end{bmatrix} = \mathit{rank}\ (D + KW) = Y. \tag{12}$$

Proof. Let SVD of K be

$$\widetilde{U}_1 K P_1 = \begin{bmatrix} \Sigma_1 & 0 \\ 0 & 0 \end{bmatrix},$$

where $\Sigma_1 \in R^{\tilde{s}\times\tilde{s}}$ is a positive diagonal matrix, denote

$$\widetilde{U}_1 D = \begin{bmatrix} D_1 \\ D_2 \end{bmatrix}, \quad LP_1 = [L_1\ L_2],$$
$$D_1 \in R^{\tilde{s}\times s},\ D_2 \in R^{(s-\tilde{s})\times s},\ L_1 \in R^{q\times\tilde{s}},\ L_2 \in R^{q\times(t-\tilde{s})},$$

then (10) gives

$$\mathrm{rank}\ (D_2) = s - \tilde{s}.$$

Hence, there exists an orthogonal matrix V_1 such that

$$D_2 V_1 = [0\ D_{22}], \quad D_{22} \in R^{(s-\tilde{s})\times(s-\tilde{s})},\ \det\ (D_2) \ne 0.$$

Define

$$D_1 V_1 = [D_{11}\ D_{12}], \quad HV_1 = [\widetilde{H}\ \widehat{H}],$$
$$H_1 = \widetilde{H} - L_1\Sigma_1^{-1}D_{11}, \quad H_2 = \widehat{H} - L_1\Sigma_1^{-1}D_{12},$$
$$D_{11} \in R^{\tilde{s}\times\tilde{s}}, \quad \widetilde{H}, H_1 \in R^{q\times\tilde{s}}.$$

Let SVD of L_2 be

$$\widetilde{U}_3 L_2 \widetilde{P}_2 = \begin{bmatrix} \Sigma_2 & 0 \\ 0 & 0 \end{bmatrix}$$

where $\Sigma_2 \in R^{\tilde{q}\times\tilde{q}}$ is a positive diagonal matrix, and denote

$$\widetilde{U}_3 H_1 = \begin{bmatrix} H_{11} \\ H_{21} \end{bmatrix}, \quad \widetilde{U}_3 H_2 = \begin{bmatrix} H_{12} \\ H_{22} \end{bmatrix}, \begin{matrix} H_{11} \in R^{\tilde{q}\times\tilde{s}} \\ H_{12} \in R^{\tilde{q}\times(s-\tilde{s})} \end{matrix}$$

Since there exist orthogonal matrices $\widetilde{U}_4, \widetilde{V}_4$ such that

$$\widetilde{U}_4 H_{21} \widetilde{V}_2 = \begin{bmatrix} H_{21}^1 & 0 \\ 0 & 0 \end{bmatrix}$$

where $H_{21}^1 \in R^{\tilde{p}\times\tilde{p}}$ is nonsingular, so, if

$$\begin{aligned} & D_{11}\widetilde{V}_2 = [D_{11}^1 \ D_{11}^2], \ H_{11}\widetilde{V}_2 = [H_{11}^1 \ H_{11}^2], \quad D_{11}^1 \in R^{\tilde{s}\times\tilde{p}}, \ H_{11}^1 \in R^{\tilde{q}\times\tilde{p}}, \\ & \widetilde{U}_4 H_{22} = \begin{bmatrix} H_{22}^1 \\ H_{22}^2 \end{bmatrix}, \quad H_{22}^1 \in R^{\tilde{p}\times(s-\tilde{s})}, \end{aligned}$$

and

$$\begin{aligned} & U_1 = \begin{bmatrix} \widetilde{U}_1 & \\ & I_q \end{bmatrix}, \quad U_2 = \begin{bmatrix} I_s & \\ (-L_1\Sigma_1^{-1}, 0) & I_q \end{bmatrix}, U_3 = \begin{bmatrix} I_3 & \\ & \widetilde{U}_3 \end{bmatrix}, \\ & U_4 = \begin{bmatrix} I_{s+\tilde{q}} & \\ & \widetilde{U}_4 \end{bmatrix}, \quad V_2 = \begin{bmatrix} \widetilde{V}_2 & \\ & I_{s-\tilde{s}} \end{bmatrix}, \quad P_2 = \begin{bmatrix} I_{\tilde{s}} & \\ & \widetilde{P}_2 \end{bmatrix}, \end{aligned}$$

then

$$U_4U_3U_2U_1 \begin{bmatrix} D \\ H \end{bmatrix} V_1V_2 = \begin{bmatrix} D_{11}^1 & D_{11}^2 & D_{12} \\ 0 & 0 & D_{22} \\ H_{11}^1 & H_{11}^2 & H_{12} \\ H_{21}^1 & 0 & H_{22}^1 \\ 0 & 0 & H_{22}^2 \end{bmatrix},$$

$$U_4U_3U_2U_1 \begin{bmatrix} K \\ L \end{bmatrix} P_1P_2 = \begin{bmatrix} \Sigma_1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & \Sigma_2 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

and for any $W \in R^{s\times t}$, we have

$$\begin{cases} \text{rank} \begin{bmatrix} D+KW \\ H+LW \end{bmatrix} = s - \tilde{s} + \tilde{p} + \text{rank} \begin{bmatrix} D_{11}^2 & \Sigma_1 W_{12} \\ H_{11}^2 & \Sigma_2 W_{22} \end{bmatrix} \\ \text{rank}\ (D+KW) = s - \tilde{s} + \text{rank}\ [D_{11}^1 + \Sigma_1 W_{11} \quad D_{11}^2 + \Sigma_1 W_{12}] \end{cases}$$

where

$$P_2^T P_1^T W V_1 V_2 = \begin{bmatrix} W_{11} & W_{12} & W_{13} \\ W_{21} & W_{22} & W_{23} \\ W_{31} & W_{32} & W_{33} \end{bmatrix}, \tag{13}$$

therefore for any integer r satisfying (11), if we choose

$$\begin{cases} [W_{11} \;\; W_{12}] = \Sigma_1^{-1}\left(\begin{bmatrix} I_{r+\tilde{s}-s} & \\ & 0 \end{bmatrix} - [D_{11}^1 \; D_{11}^2]\right) \\ W_{22} = -\Sigma_2^{-1} H_{11}^2 \end{cases} \tag{14}$$

then matrix W given by (13–14) satisfies (12).

Lemma 4.2. *Given*

$$\widetilde{E} = \begin{bmatrix} \widetilde{E}_{11} & \widetilde{E}_{15} \\ \widetilde{E}_{31} & \widetilde{E}_{35} \end{bmatrix}, \; \widetilde{B} = \begin{bmatrix} \widetilde{B}_{11} \\ \widetilde{B}_{31} \end{bmatrix}, \widetilde{C} = [\widetilde{C}_{11} \; 0]$$

with $\widetilde{E}_{11} \in R^{n_1 \times n_1}$, $\widetilde{E}_{15} \in R^{n_1 \times n_5}$, $\widetilde{E}_{31} \in R^{n_3 \times n_1}$, $\widetilde{E}_{35} \in R^{n_3 \times n_5}$, $\widetilde{B}_{11} \in R^{n_1 \times m}$, $\widetilde{B}_{31} \in R^{n_3 \times m}$, $\widetilde{C}_{11} \in R^{p \times n_1}$, $\widetilde{C} \in R^{p \times (n_1+n_5)}$ *and*

$$rank \begin{bmatrix} \widetilde{E}_{11} \\ \widetilde{C}_{11} \end{bmatrix} = n_1, \quad rank \begin{bmatrix} \widetilde{E}_{11} & \widetilde{B}_{11} \\ \widetilde{E}_{31} & \widetilde{B}_{31} \end{bmatrix} = n_1 + n_3, \tag{15}$$

$$rank \begin{bmatrix} \widetilde{E}_{11} & \widetilde{E}_{15} \\ \widetilde{E}_{31} & \widetilde{E}_{35} \\ \widetilde{C}_{11} & 0 \end{bmatrix} = n_1 \tag{16}$$

then for any integer r *satisfying*

$$rank \; [\widetilde{E} \; \widetilde{B}] + rank \begin{bmatrix} \widetilde{E} \\ \widetilde{C} \end{bmatrix} - rank \begin{bmatrix} \widetilde{E} & \widetilde{B} \\ \widetilde{C} & 0 \end{bmatrix} \le r \le n_1 \tag{17}$$

there exists $\widetilde{G} \in R^{m \times p}$ *such that*

$$rank \; (\widetilde{E} + \widetilde{B}\widetilde{G}\widetilde{C}) = rank \; (\widetilde{E}_{11} + \widetilde{B}_{11}\widetilde{G}\widetilde{C}_{11}) = r.$$

Proof. Let SVD of $\widetilde{C}_{11}$ be

$$Q_1 \widetilde{C}_{11} \widetilde{V}_1 = \begin{bmatrix} \Sigma_c & 0 \\ 0 & 0 \end{bmatrix}$$

where $\Sigma_c \in R^{\tilde{p} \times \tilde{p}}$ is a positive diagonal matrix, and denote

$$\begin{aligned} \widetilde{E}_{11}\widetilde{V}_1 &= [E_{11}^1 \; E_{11}^2], \quad E_{11}^1 \in R^{n_1 \times \tilde{p}} \\ \widetilde{E}_{31}\widetilde{V}_1 &= [E_{31}^1 \; E_{31}^2], \quad E_{31}^1 \in R^{n_3 \times \tilde{p}} \end{aligned}$$

then (15) gives that E_{11}^2 is full column rank, so there exists an orthogonal matrix $\widetilde{U}_1 \in R^{n_1 \times n_1}$ such that

$$\widetilde{U}_1 E_{11}^2 = \begin{bmatrix} 0 \\ \widetilde{\Sigma}_e \end{bmatrix}$$

where $\widetilde{\Sigma}_e \in R^{(n_1-\tilde{p}) \times (n_1 - \tilde{p})}$ is nonsingular. Define

$$\widetilde{U}_1 \widetilde{E}_{11}^1 = \begin{bmatrix} \widetilde{E}_{11}^1 \\ \widetilde{E}_{11}^3 \end{bmatrix}, \quad \widetilde{U}_1 \widetilde{E}_{15} = \begin{bmatrix} \widetilde{E}_{15}^1 \\ \widetilde{E}_{15}^2 \end{bmatrix}, \quad \widetilde{U}_1 \widetilde{B}_{11} = \begin{bmatrix} \widetilde{B}_{11}^1 \\ \widetilde{B}_{11}^2 \end{bmatrix}$$

where $\widetilde{E}_{11} \in R^{\tilde{p}\times\tilde{p}}$, $\widetilde{E}_{15}^1 \in R^{\tilde{p}\times n_3}$, $\widetilde{B}_{11}^1 \in R^{\tilde{p}\times m}$, let $\widetilde{U}_2$ be an orthogonal matrix such that

$$\widetilde{U}_2 \begin{bmatrix} \widetilde{\Sigma}_e \\ E_{31}^2 \end{bmatrix} = \begin{bmatrix} \widehat{\Sigma}_e \\ 0 \end{bmatrix}$$

with $\widehat{\Sigma}_2 \in R^{(n_1-\tilde{p})\times(n_1-\tilde{p})}$ being nonsingular, and set

$$\begin{aligned} &\widetilde{U}_2 \begin{bmatrix} \widetilde{E}_{11}^3 \\ E_{31}^1 \end{bmatrix} = \begin{bmatrix} \widehat{E}_{11}^3 \\ \widehat{E}_{31}^1 \end{bmatrix}, \quad \widetilde{U}_2 \begin{bmatrix} \widetilde{E}_{15}^2 \\ \widetilde{E}_{35} \end{bmatrix} = \begin{bmatrix} \widehat{E}_{15}^2 \\ \widehat{E}_{35} \end{bmatrix}, \quad \begin{matrix} \widehat{E}_{11}^3 \in R^{(n_1-\tilde{p})\times\tilde{p}} \\ \widehat{E}_{15}^2 \in R^{(n_1-\tilde{p})\times n_5} \end{matrix} \\ &\widetilde{U}_2 \begin{bmatrix} \widetilde{B}_{11}^2 \\ \widetilde{B}_{31} \end{bmatrix} = \begin{bmatrix} \widehat{B}_{11}^2 \\ \widehat{B}_{31} \end{bmatrix}, \quad \widehat{B}_{11} \in R^{(n_1-\tilde{p})\times m}, \\ &V_1 = \begin{bmatrix} \widetilde{V}_1 & \\ & I_{n5} \end{bmatrix}, \quad U_1 = \begin{bmatrix} \widetilde{U}_1 & \\ & I_{n3} \end{bmatrix}, \quad U_2 = \begin{bmatrix} I_{\tilde{p}} & \\ & \widetilde{U}_2 \end{bmatrix} \end{aligned}$$

then

$$U_2U_1\widetilde{E}V_1 = \begin{bmatrix} \widetilde{E}_{11}^1 & 0 & \widetilde{E}_{15}^1 \\ \widehat{E}_{11}^3 & \widehat{\Sigma}_e & \widehat{E}_{15}^2 \\ \widehat{E}_{31}^1 & 0 & \widehat{E}_{35} \end{bmatrix}, \quad U_2U_1\widetilde{B} = \begin{bmatrix} \widetilde{B}_{11}^1 \\ \widehat{B}_{11}^2 \\ \widehat{B}_{31} \end{bmatrix}, \quad Q_1\widetilde{C}V_1 = \begin{bmatrix} \Sigma_c & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

and for any $\widetilde{G} \in R^{m\times p}$:

$$\widetilde{G}Q_1^T = [\widehat{G}_{11} \ \widehat{G}_{12}]$$

we have

$$\begin{cases} \text{rank } (\widetilde{E} + \widetilde{B}\widetilde{G}\widetilde{C}) = n_1 - \tilde{p} + \text{rank} \begin{bmatrix} \widetilde{E}_{11}^1 + \widetilde{B}_{11}^1\widehat{G}_{11}\Sigma_c & \widetilde{E}_{15}^1 \\ \widehat{E}_{31}^1 + \widehat{B}_{31}\widehat{G}_{11}\Sigma_c & \widehat{E}_{35} \end{bmatrix} \\ \text{rank } (\widetilde{E}_{11} + \widetilde{B}_{11}\widetilde{G}\widetilde{C}_{11}) = n_1 - \tilde{p} + \text{rank } (\widetilde{E}_{11}^1 + \widetilde{B}_{11}^1\widehat{G}_{11}\Sigma_c). \end{cases} \tag{18}$$

Note that from (15) and (16) it follows

$$\begin{cases} \widetilde{E}_{15}^1 = 0, \quad \widehat{E}_{35} = 0 \\ \text{rank} \begin{bmatrix} \widetilde{E}_{11}^1 & \widetilde{B}_{11}^1 \\ \widehat{E}_{31}^1 & \widehat{B}_{31} \end{bmatrix} = \tilde{p} + n_3 \end{cases} \tag{19}$$

and Σ_c is positive diagonal, so, (17–19) and Lemma 4.1 complete the proof of Lemma 4.2.

Lemma 4.3. *Given*

$$\widetilde{E} = [\widetilde{E}_{11} \ \widetilde{E}_{15}], \quad \widetilde{B} = \widetilde{B}_{11}, \quad \widetilde{C} = [\widetilde{C}_{11} \ \widetilde{C}_{15}]$$

with $\widetilde{E}_{11} \in R^{n_1\times n_1}$, $\widetilde{E}_{15} \in R^{n_1\times n_5}$, $\widetilde{B}_{11} \in R^{n_1\times m}$, $\widetilde{C}_{11} \in R^{p\times n_1}$, $\widetilde{C}_{15} \in R^{p\times n_5}$ *and*

$$\textit{rank } [\widetilde{E}_{11} \ \widetilde{B}_{11}] = n_1, \quad \textit{rank } \begin{bmatrix} \widetilde{E}_{11} \\ \widetilde{C}_{11} \end{bmatrix} = n_1 \tag{20}$$

then for any integer r *satisfying*

$$rank[\widetilde{E}\ \widetilde{B}] + rank\ \begin{bmatrix} \widetilde{E} \\ \widetilde{C} \end{bmatrix} - rank\ \begin{bmatrix} \widetilde{E} & \widetilde{B} \\ \widetilde{C} & 0 \end{bmatrix} \leq r \leq n_1,$$

there exists $\widetilde{G} \in R^{m\times p}$ *such that*

$$rank\ (\widetilde{E} + \widetilde{B}\widetilde{G}\widetilde{C}) = rank\ (\widetilde{E}_{11} + \widetilde{B}_{11}\widetilde{G}\widetilde{C}_{11}).$$

Proof. Let SVD of $\widetilde{B}_{11}$ be

$$U_1\widetilde{B}_{11}P_1 = \begin{bmatrix} \Sigma_b & 0 \\ 0 & 0 \end{bmatrix}$$

where $\Sigma_b \in R^{\tilde{m}\times\tilde{m}}$ is a positive diagonal matrix, set

$$U_1\widetilde{E} = \begin{bmatrix} E_{11}^1 & E_{15}^1 \\ E_{11}^2 & E_{15}^2 \end{bmatrix}, \quad \begin{matrix} E_{11}^1 \in R^{\tilde{m}\times n_1} \\ E_{15}^1 \in R^{\tilde{m}\times n_5} \end{matrix}$$

Since (20) implies that E_{11}^2 is full row rank, so there exists an orthogonal matrix V_1 such that

$$\begin{aligned} U_1\widetilde{E}V_1 &= \begin{bmatrix} \widetilde{E}_{11}^1 & \widehat{E}_{11}^1 & E_{15}^1 \\ 0 & \widehat{E}_{11}^2 & E_{15}^2 \end{bmatrix}, \\ \widetilde{C}V_1 &= [\widetilde{C}_{11}^1\ \widetilde{C}_{11}^2\ \widetilde{C}_{15}] \end{aligned}$$

where $\widehat{E}_{11}^2 \in R^{(n_1-\tilde{m})\times(n_1-\tilde{m})}$ is nonsingular, $\widetilde{C}_{11} \in R^{p\times(n_1-\tilde{m})}$. Let V_2 be an orthogonal matrix such that

$$U_1\widetilde{E}V_1V_2 = \begin{bmatrix} \widetilde{E}_{11}^1 & \widehat{E}_{11}^2 & \widehat{E}_{15}^1 \\ 0 & \Sigma_e & 0 \end{bmatrix}$$

with $\Sigma_e \in R^{(n_1-\tilde{m})\times(n_1-\tilde{m})}$ being nonsingular. If we set

$$\widetilde{C}V_1V_2 = [\widetilde{C}_{11}^1\ \widehat{C}_{11}^2\ \widehat{C}_{15}]$$

then for any $\widetilde{G} \in R^{m\times p}$, denote

$$P_1^T\widetilde{G} = \begin{bmatrix} \widehat{G}_{11} \\ \widehat{G}_{21} \end{bmatrix},$$

we have

$$\begin{cases} \text{rank}\ (\widetilde{E} + \widetilde{B}\widetilde{G}\widetilde{C}) = n_1 - \tilde{m} + \text{rank}\ [\widetilde{E}_{11}^1 + \Sigma_b\widehat{G}_{11}\widetilde{C}_{11}^1 \quad \widehat{E}_{15}^1 + \Sigma_b\widehat{G}_{11}\widehat{C}_{15}] \\ \text{rank}(\widetilde{E}_{11} + \widetilde{B}_{11}\widetilde{G}\widetilde{C}_{11}) = n_1 - \tilde{m} + \text{rank}\ (\widetilde{E}_{11}^1 + \Sigma_b\widehat{G}_{11}\widetilde{C}_{11}^1) \end{cases} \tag{21}$$

Since (20) gives

$$\text{rank}\ \begin{bmatrix} \widetilde{E}_{11}^1 \\ \widetilde{C}_{11}^1 \end{bmatrix} = \tilde{m} \tag{22}$$

considering Σ_b is positive diagonal, so (21–22) and Lemma 4.1 complete the proof of Lemma 4.3.

Now, we prove Theorems 3.1–3.7.

The proof of Theorem 3.1. *For any $F, G \in R^{m\times p}$, we have*

$$det(\lambda(E+BGC)-(A+BFC)) = det(\lambda(E_{11}+B_{11}GC_{11})-(A_{11}+B_{11}FC_{11})) \times det(\lambda E_{22}-A_{22})det(\lambda E_{44}-A_{44})det\ (-A_{33})det\ (-A_{55}) \tag{23}$$

so if

$$\begin{cases} n_3 = 0, & or\ n_3 > 0,\ det\ (A_{33}) \neq 0 \\ n_5 = 0, & or\ n_5 > 0,\ det\ (A_{55}) \neq 0 \end{cases}$$

then we have

$$deg\ (det(\lambda(E+BGC)-(A+BFC))) = deg\ (det(\lambda(E_{11}+B_{11}GC_{11})- (A_{11}+B_{11}FC_{11}))) + n_2 + n_4 \leq rank(E+BGC) \tag{24}$$

Besides (23) *and* (24)*, we also have*

$$rank\ (E+BGC) \geq rank\ (E_{11}+B_{11}GC_{11}) + n_2 + n_4 + rank(E_{33}) + rank(E_{55}) \tag{25}$$

and

$$r_{ec} \geq n_1 + n_2 + n_4 + rank(E_{33}) + rank(E_{55}) \tag{26}$$

Necessity of (i). If $G \in R^{m\times p}$ such that pencil $[E+BGC, A]$ is regular with index at most one, and

$$\text{rank}\ (E+BGC) = r_{ec}.$$

then

$$\begin{cases} \det(\lambda(E+BGC)-A) \neq 0 \\ \deg\ (\det(\lambda(E+BGC)-A)) = r_{ec}. \end{cases} \tag{27}$$

Hence, merging (23–27) gives condition (5).

Sufficiency of (i). Assume (5) true. From Lemma 2.5, there exists $G \in R^{m\times p}$ such that $E_{11}+B_{11}GC_{11}$ is nonsingular, moreover

$$\deg\ (\det(\lambda(E+BGC)-A)) = n_1 + n_2 + n_4 = r_{ec}.$$

Because $r_{ec} \geq \text{rank}(E+BGC)$, the equality above and Lemma 2.4, (24) as well as (25) imply that rank $(E+BGC) = r_{ec}$, moreover, pencil $[E+BGC, A]$ is regular and has index at most one.

(ii). If pencil $(E+BGC, A)$ is regular and has index at most one with

$$\text{rank}(E+BGC) = r_{ec}$$

then

$$\deg\ (\det(\lambda(E+BGC)-A))=r_{ec}$$

and (i), (24), (26) and (27) give

$$r_{ec}=n_1+n_2+n_4, \quad \text{rank}\ (E_{11}+B_{11}GC_{11})=n_1$$

furthermore, if $F \in R^{m\times p}$, then

$$\deg\ (\det(\lambda(E+BGC)-(A+BFC)))=n_1+n_2+n_4=r_{ec}=\text{rank}\ (E+BGC).$$

Finally, the pencil $[E+BGC, A+BFC]$ is regular and has index at most one.

Inversely, if pencil $[E+BGC, A+BFC]$ is regular and has index at most one as well as and rank $(E+BGC)=r_{ec}$, then

$$\deg\ (\det(\lambda(E+BGC)-(A+BFC)))=\text{rank}(E+BGC)=r_{ec}.$$

Using (24), (26) and (27), we have

$$\begin{cases} r_{ec}=n_1+n_2+n_4 \\ \deg\ (\det(\lambda(E_{11}+B_{11}GC_{11})-(A_{11}+B_{11}FC_{11})))=n_1, \end{cases}$$

hence

$$\begin{cases} \text{rank}(E_{11}+B_{11}GC_{11})=n_1 \\ \deg\ (\det(\lambda(E+BGC)-A))=n_1+n_2+n_4=\text{rank}\ (E+BGC), \end{cases}$$

which imply that pencil $(E+BGC, A)$ is regular and has index at most one.

As a consequence, we known that if rank $(E+BGC)=r_{ec}$, then pencil $[E+BGC, A+BFC]$ is regular and has index at most one if and only if pencil $[E+BGC, A]$ is regular and has index at most one. Hence, (i) gives (ii).

The proof of Theorem 3.3. Take a pair of orthogonal matrices $\widetilde{U}^*, \widetilde{V}^*$ such that

$$\widetilde{U}^*E^*\widetilde{V}^*=\begin{bmatrix} E_{11} & 0 & 0 & \widetilde{E}_{14} & \widetilde{E}_{15} \\ \widetilde{E}_{21} & \widetilde{E}_{22} & 0 & \widetilde{E}_{24} & \widetilde{E}_{25} \\ \widetilde{E}_{31} & 0 & 0 & \widetilde{E}_{34} & \widetilde{E}_{35} \\ 0 & 0 & 0 & \widetilde{E}_{44} & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

$$\widetilde{U}^*B^*=\begin{bmatrix} \widetilde{B}_{11} \\ \widetilde{B}_{21} \\ \widetilde{B}_{31} \\ 0 \\ 0 \end{bmatrix}, \quad C^*\widetilde{V}^*=[C_{11}\ 0\ 0\ \widetilde{C}_{14}\ \widetilde{C}_{15}]$$

where $\widetilde{E}_{22} \in R^{n_2 \times n_2}$, $\widetilde{E}_{44} \in R^{n_4 \times n_4}$ are nonsingular, then (3–4) and (5) give

$$\text{rank } \begin{bmatrix} E_{11} & \widetilde{B}_{11} \\ \widetilde{E}_{31} & \widetilde{B}_{31} \end{bmatrix} = n_1 + n_3,$$

$$\text{rank } \begin{bmatrix} E_{11} \\ C_{11} \end{bmatrix} = n_1, \quad \text{rank } \begin{bmatrix} E_{11} & \widetilde{E}_{15} \\ \widetilde{E}_{31} & \widetilde{E}_{35} \\ C_{11} & \widetilde{C}_{15} \end{bmatrix} = n_1.$$

The last two equalities above imply that there exists $X \in R^{n_1 \times n_5}$ such that

$$[C_{11} \ \widetilde{C}_{15}] \begin{bmatrix} I_{n_1} & X \\ & I_{n_5} \end{bmatrix} = [C_{11} \ 0].$$

Therefore, we may assume without loss of generality that

$$\widetilde{C}_{15} = 0,$$

thus we also have

$$\text{rank } (E + BGC) = n_2 + n_4 + \text{rnak } \begin{bmatrix} E_{11} + B_{11}GC_{11} & \widetilde{E}_{15} \\ \widetilde{E}_{31} + \widetilde{B}_{31}GC_{11} & \widetilde{E}_{35} \end{bmatrix}.$$

From Lemma 4.2, we have that for any integer r satisfying (7), there exists $G \in R^{m \times p}$ such that

$$\text{rank } \begin{bmatrix} E_{11} + B_{11}G_{11} & \widetilde{E}_{15} \\ \widetilde{E}_{31} + \widetilde{B}_{31}GC_{11} & \widetilde{E}_{35} \end{bmatrix} = \text{rank } (E_{11} + B_{11}GC_{11}) = r - n_2 - n_4 \tag{28}$$

equivalently

$$\text{rank } (E + BGC) = r. \tag{29}$$

Let U_{11}^*, V_{11}^* be orthogonal matrices satisfying

$$U_{11}^*(E_{11} + B_{11}GC_{11})V_{11}^* = \begin{bmatrix} \Sigma_{11} & 0 \\ 0 & 0 \end{bmatrix}$$

where $\Sigma_{11} \in R^{(r-n_2-n_4)\times(r-n_2-n_4)}$ is nonsingular. Define

$$U_{11}^* A_{11} V_{11}^* = \begin{bmatrix} A_{11}^1 & A_{11}^2 \\ A_{11}^3 & A_{11}^4 \end{bmatrix}, \quad U_{11}^* B_{11} = \begin{bmatrix} B_{11}^1 \\ B_{11}^2 \end{bmatrix}, \ C_{11}V_{11}^* = [C_{11}^1 \ C_{11}^2]$$

with $A_{11}^1 \in R^{(r-n_2-n_4)\times(r-n_2-n_4)}$, $B_{11}^1 \in R^{(r-n_2-n_4)\times m}$, $C_{11}^1 \in R^{p\times(r-n_2-n_4)}$, then (3) and (4) imply

$$\text{rank } (B_{11}^2) = \text{rank } (C_{11}^2) = n_1 - (r - n_2 - n_4)$$

so there exists $F \in R^{m \times p}$ such that $A_{11}^4 + B_{11}^2 F C_{11}^2$ is nonsingular, F can be computd using SVD method. As to F and G, we have

$$\deg\ (\det(\lambda(E_{11} + B_{11}GC_{11}) - (A_{11} + B_{11}FC_{11}))) = \text{rank } (E_{11} + B_{11}GC_{11})$$

and

$$\begin{cases} \text{rank } (E+BGC)=r \\ \text{deg } (\det(\lambda(E+BGC)-(A+BFC)))=r \\ \text{deg } (\det(\lambda(E_{11}+B_{11}GC_{11})-(A_{11}+B_{11}FC_{11})))+n_2+n_4=r \end{cases} \tag{30}$$

therefore, (28)–(30) give that pencil $[E+BGC, A+BFC]$ is regular and has index at most one as well as

$$\text{rank } (E+BGC)=r.$$

Necessity of (iii). Let $F, G \in R^{m\times p}$ such that pencil $[E+BGC, A]$ is regular and has index at most one as well as

$$\text{rank } (E+BGC)=r_{eb},$$

then

$$\begin{cases} \det(\lambda(E+BGC)-A)\neq 0 \\ \text{deg } (\det(\lambda(E+BGC)-A))=r_{eb}. \end{cases} \tag{31}$$

From (3) and (24)

$$r_{eb}=n_1+n_2+n_3+n_4+\text{rank } (E_{55}) \tag{32}$$

we know that condition (6) holds.

Sufficiency of (iii). Assume (6) being true. From Lemma 2.5 and (24), there exists $G\in R^{m\times p}$ such that

$$\text{rank } (E_{11}+B_{11}GC_{11})=n_1,$$

Moreover

$$\text{deg } (\det(\lambda(E+BGC)-A))=n_1+n_2+n_4=r_{eb} \tag{33}$$

Using (24), (32) and Lemma 2.5, it follows

$$\begin{cases} r_{eb}=n_1+n_2+n_4 \\ \text{rank } (E+BGC)=r_{eb}. \end{cases} \tag{34}$$

Hence, (33) and (34) imply that pencil $[E+BGC, A]$ is regular and has index at most one.

(iv). If pencil $[E+BGC, A]$ is regular and has index at most one, and

$$\text{rank } (E+BGC)=r_{eb}$$

then (iii), (32) and (24) give

$$\begin{cases} r_{eb}=n_1+n_2+n_4 \\ \text{rank } (E_{11}+B_{11}GC_{11})=n_1. \end{cases} \tag{35}$$

Moreover, from (25), (24) and Lemma 2.4, we know that pencil $[E+BGC, A+BFC]$ is regular and has index at most one.

Inversity, if pencil $(E+BGC, A+BFC)$ is regular, has index at most one, and

$$\text{rank }(E+BGC)=r_{eb} \tag{36}$$

then, (24) and (32) imply

$$\begin{cases}\text{rank }(E_{11}+B_{11}GC_{11})=n_1\\ r_{eb}=n_1+n_2+n_4.\end{cases}$$

Therefore, using (36), (24) and Lemma 2.4, we get that encil $[E+BGC, A]$ is regular and has index at most one.

As a consequence, we obtain that if rank $(E+BGC)=r_{eb}$, then pencil $(E+BGC, A+BFC)$ is regular and has index at most one if and only if pencil $[E+BGC, A]$ is regular and has index at most one. Therefore, (iii) gives (iv).

The proof of Theorem 3.4. *Take a pair of orthogonal matrices $\widetilde{U}^*, \widetilde{V}^*$ such that*

$$\widetilde{U}^*E^*\widetilde{V}^*=\begin{bmatrix}E_{11} & 0 & \widetilde{E}_{14} & \widetilde{E}_{15}\\ E_{21} & E_{22} & \widetilde{E}_{24} & \widetilde{E}_{25}\\ 0 & 0 & \widetilde{E}_{44} & 0\\ 0 & 0 & 0 & 0\end{bmatrix}$$
$$\widetilde{U}^*B^*=B^*,\ C^*\widetilde{V}^*=[C_{11}\ 0\ \widetilde{C}_{14}\ \widetilde{C}_{15}]$$

with $\widetilde{E}_{44}\in R^{n_4\times n_4}$ nonsingular, then (3) and (4) give

$$\mathit{rank}\ [E_{11}\ B_{11}]=n_1,\quad \mathit{rank}\begin{bmatrix}E_{11}\\ C_{11}\end{bmatrix}=n_1$$

and

$$\mathit{rank}\ (E+BGC)=n_2+n_4+\mathit{rank}\ [E_{11}+B_{11}GC_{11}\ \widetilde{E}_{15}+B_{11}G\widetilde{C}_{15}],\quad \forall G\in R^{m\times p} \tag{37}$$

so, from Lemma 4.3, there exists $G\in R^{m\times p}$ such that

$$\begin{aligned}&\mathit{rank}\ [E_{11}+B_{11}GC_{11}\ \widetilde{E}_{15}+B_{11}G\widetilde{C}_{15}]\\ &\quad=\mathit{rank}\ (E_{11}+B_{11}GC_{11})=r-n_2-n_4.\end{aligned} \tag{38}$$

Similar to the proof of Theorem 3.3, we can choose $F\in R^{m\times p}$ such that

$$\mathit{deg}\ (\mathit{det}(\lambda(E_{11}+B_{11}GC_{11})-(A_{11}+B_{11}FC_{11})))=\mathit{rank}(E_{11}+B_{11}GC_{11})$$

then F and G satisfy that

$$\mathit{rank}\ (E+BGC)=\mathit{deg}\ (\mathit{det}(\lambda(E+BGC)-(A+BFC)))=r.$$

The above equality, (37) and (38) imply that pencil $(E+BGC, A+BFC)$ is regular and has index at most one with r finite poles.

Theorem 3.5 and Theorem 3.6 are resulted directly from Theorem 3.1 (ii) and Theorem 3.3, Theorem 3.2 (iv) and Theorem 3.4, respectively.

The proof of Theorem 3.7. (Necessity) *In this case, $[EA+BFC]$ is regular and has index at most one, Lemma 2.4 (23) and (24) give that*

$$\begin{cases} deg\ (det(\lambda E-(A+BGC)) = rank(E) \\ deg\ (det(\lambda E-(A+BGC)) = deg\ (det(\lambda E_{11}-(A_{11}+B_{11}FC_{11})) + n_2 + n_4. \end{cases}$$

But

$$\begin{cases} rank(E) \geq rank\ (E_{11}) + n_2 + n_4 \\ deg\ (det(\lambda E_{11}-(A_{11}+B_{11}FC_{11})) \leq rank\ (E_{11}) \end{cases}$$

Hence, merging (23) and two relations above, we obtain condition (9).

(**Sufficiency**) Similar to the proof of Theorem 3.3, we have $F \in R^{m\times p}$ satisfhing

$$\deg\ (\det(\lambda E_{11}-(A_{11}+B_{11}FC_{11})) = \operatorname{rank}(E_{11}).$$

For this F, using (24), it is easy to know that

$$\deg\ (\det(\lambda E-(A+BFC)) = \operatorname{rank}\ (E_{11}) + n_2 + n_4 = \operatorname{rank}(E)$$

which completes the proof of sufficiency.

It should be emphasized that the proofs of Theorem 3.1–Theorem 3.7 provide numerical procedures for constructing the feedback matrices F, G based on orthogonal transformations. Moreover, the constructive procedure of the expected feedback matrices F, G can consist of two separate steps: firstly, to construct a derivative feedback matrix G such that $E+BGC$ satisfies some rank conditions; secondly, to construct a proportional feedback matrix F such that $[E_{11}+B_{11}GC_{11}, A_{11}+B_{11}FC_{11}]$ is regular and of index at most one. This property is very useful for computation.

[20] studied output feedback regularization problem related to S-controllability S-observability of singular system (1) and present some interesting conditions. These conditions can also be obtained by using Theorem 3.1–Theorem 3.7. Since the derivation procedures are very simple, so we omit them here.

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