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WAVELET RATIONAL FILTERS AND REGULARITY ANALYSIS*

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Abstract

In this paper, we choose the trigonometric rational functions as wavelet filters and use them to derive various wavelets. Especially for a certain family of wavelets generated by the rational filters, the better smoothness results than Daubechies' are obtained.

Key words: Wavelet, Filter, Rational filter, Regularity.

1. Introduction

We denote by $\phi(x)$ a scaling function which satisfies

$$\phi(x) = \sum_{k\in Z} h_k \phi(2x-k) \qquad (\text{Z is the integer set}). \tag{1}$$

The Fourier transform of equation (1) is

$$\hat{\phi} = H(\omega/2)\hat{\phi}(\omega/2) \tag{2}$$

where $\hat{\phi}(\omega)$ is the Fourier transform of $\phi(x)$ and

$$H(\omega) = \frac{1}{2}\sum_{k\in Z} h_k e^{-ik\omega}.$$

We call $H(\omega)$ a filter.It satisfies

$$H(0) = 1, \quad |\, H(\omega)\,|^2 + |\, H(\omega+\pi)\,|^2 = 1 \tag{3}$$

When the expansion coefficient sequence $\{\, h_k\}$ of $H(\omega)$ is given, the wavelets corresponding to the $H(\omega)$ can be derived. For the $H(\omega)$ which is a trigonometric polynomial (in this case, we call $H(\omega)$ a polynomial filter), Daubechies has given the methods generating wavelets as well as the estimates of regularity[1][2].

In this paper, we choose $H(\omega)$ to be a trigonometric rational function to generate wavelets and give relative methods and theorems. For I-type rational filters (see the

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second section),they include Daubechies' filters. And for II-type rational filters,they include B spline wavelet filters and have linear phases.Especially for a certain family of wavelets generated by the rational filters, the better smoothness results than Daubechies' are obtained.

2. Rational Filters

For a filter

$$H(\omega)=(\frac{1+e^{-i\omega}}{2})^N F(e^{-i\omega})$$

where $F(e^{-i\omega}) = \sum_{k\in Z} f_k e^{-ik\omega}$, Daubechies has given the conditions of existence of wavelets[1]:

$$\text{(I)} \qquad \sup_{\omega\in R} \mid F(e^{-i\omega}) \mid < 2^{N-1}$$

$$\text{(II)} \qquad \sum_{k\in Z} \mid f_k \mid\mid k \mid^{\epsilon} < \infty \quad \text{for a certain} \quad \epsilon > 0$$

On the basis of the two conditions, we will study how to construct wavelets by a rational filter.

Definition For a filter $H(\omega) = ((1+e^{-i\omega})/2)^N F(e^{-i\omega})$, when $F(z)$ is a rational function or the modulus of a rational function, we call $H(\omega)$ a rational filter.

Assume $P(z)$ and $Q(z)$ are relatively prime polynomials with real coefficients. Then $\mid P(e^{-i\omega})/Q(e^{-i\omega}) \mid$ is a rational function in $\cos\omega$. Riesz' lemma allow us to conclude that there is a real coefficient rational function $F(z)$ such that

$$\mid F(e^{-i\omega}) \mid^2 = \mid \frac{P(e^{-i\omega})}{Q(e^{-i\omega})} \mid . \tag{4}$$

Let

$$\mid F(e^{-i\omega}) \mid^2 = \frac{S(y)}{T(y)}, \qquad y=\cos^2\frac{\omega}{2} \tag{5}$$

where $S(y)$ and $T(y)$ are positive polynomials in the intervel [0,1]. For the given $S(y)$ and $T(y)$, the following two types of the rational filters can be determined by (5):

$$H_I(\omega)=(\frac{1+e^{-i\omega}}{2})^N F(e^{-i\omega}), \qquad H_{II}(\omega)=(\frac{1+e^{-i\omega}}{2})^N \mid F(e^{-i\omega}) \mid .$$

They are respectively called I-type rational filters and II-type rational filters. For II-type rational filters,$H_{II}(-\omega) = e^{iN\omega} H_{II}(\omega)$. This implies that II-type rational filters have linear phases.We have known that the function $F(e^{-i\omega})$s in the filters of orthogonal B spline wavelets are the moduli of the trigonometric rational functions[3].Therefore,the wavelets derived by II-type rational filters can include B spline wavelets.

For a I-type rational filter, we may use power series expansion to obtain sequence $\{h_k\}$. By the property of power series, we know that the condition (II) can be satisfied.

For a II-type rational filter, since $\mid F(e^{-i\omega}) \mid$ is a periodic function with period 2π, we may use the expansion of the Fourier series to obtain sequence $\{h_k\}$. If $\mid F(e^{-i\omega}) \mid$ has the second continuous derivative, then the condition (II) can also be satisfied.

Now, substituting (5) into the equation $\mid H(\omega) \mid^2 + \mid H(\omega+\pi) \mid^2 = 1$ leads to

$$y^N S(y)T(1-y) + (1-y)^N S(1-y)T(y) = T(y)T(1-y). \tag{6}$$

Denote $M(y) = S(1-y)T(y)$ and $N(y) = T(y)T(1-y)$. Then (6) can be rewritten as

$$y^N M(1-y) + (1-y)^N M(y) = N(y).$$

Since $S(y)/T(y) = M(1-y)/N(1-y)$,we have

$$\mid F(e^{-i\omega}) \mid^2 = \frac{M(1-y)}{y^N M(1-y) + (1-y)^N M(y)}, \quad y = \cos^2 \frac{\omega}{2}. \tag{7}$$

When $M(y)$ is symmetric about $y = 1/2$, equation (7) becomes

$$\mid F(e^{-i\omega}) \mid^2 = \frac{1}{y^N + (1-y)^N}, \quad y = \cos^2 \frac{\omega}{2}. \tag{8}$$

In this case,$\sup_{w\in R} \mid F(e^{-i\omega}) \mid = \sup_{y\in[0,1]} \frac{1}{\sqrt{y^N+(1-y)^N}} = 2^{\frac{N-1}{2}}$ so that the $F(e^{-i\omega})$ determined by (8) satisfies condition (I).

Theorem 1. *For a positive polynomial $M(y)$ on [0,1], denote*

$$K = \min_{y\in[0,1]} (M(y)/M(1-y)).$$

If $K > 3^{-(N-1)}$,then the $F(e^{-i\omega})$ determined by (7) satisfies the condition (I).

Proof. For $y \in$[0,1],

$$\begin{aligned} \frac{M(1-y)}{y^N M(1-y)+(1-y)^N M(y)} &= \frac{1}{y^N+(1-y)^N M(y)/M(1-y)} \\ &\leq \frac{1}{y^N+K(1-y)^N} \\ &\leq \frac{1}{\min_{y\in[0,1]}[y^N+K(1-y)^N]} \\ &= \frac{(1+K^{\frac{1}{N-1}})^{N-1}}{K} \end{aligned}$$

When $K > 3^{-(N-1)}$,we have

$$1 + K^{\frac{1}{N-1}} < 4K^{\frac{1}{N-1}}$$

so that

$$\frac{(1+K^{\frac{1}{N-1}})^{N-1}}{K} < 2^{2(N-1)}$$

The theorem is proved.

If choose $M(y) = P_N(y) + q(y) + q(1-y)$, where $q(y)$ is a polynomial and

$$P_N(y) = \sum_{j=0}^{N-1} \binom{N-1+j}{j} y^j \tag{9}$$

since $y^N P_N(1-y)+(1-y)^N P_N(y)\equiv 1$,then we have

$$| F(e^{-i\omega}) |^2=\frac{P_N(1-y)+q(y)+q(1-y)}{1+[q(y)+q(1-y)][y^N+(1-y)^N]},\quad y=\cos^2\frac{\omega}{2}. \tag{10}$$

Theorem 2. *If $N\geq 2$ and $q(y)\geq 0$ for $y\in[0,1]$,then the $F(e^{-i\omega})$ determined by* (10) *satisfies the condition* (I).

Proof. Since

$$P_N(y)<2^{2(N-1)}\qquad \text{for } \ y\in[0,1]$$

and

$$y^N+(1-y)^N\geq\frac{1}{2^{N-1}}\qquad \text{for } \ y\in[0,1],$$

when $N\geq 2$ and $q(y)\geq 0$ for $y\in[0,1]$, we have

$$\begin{aligned} 2^{2(N-1)}\{1+[q(y)+q(1-y)][y^N+(1-y)^N]\} &\geq 2^{2(N-1)}+2^{N-1}[q(y)+q(1-y)] \\ &> P_N(y)+q(y)+q(1-y) \end{aligned}$$

so that

$$\frac{P_N(1-y)+q(y)+q(1-y)}{1+[q(y)+q(1-y)][y^N+(1-y)^N]}<2^{2(N-1)}.$$

The theorem is proved.

When $q(y)=C/2$ in (10),

$$| F(e^{-i\omega}) |^2=\frac{P_N(1-y)+C}{1+C[y^N+(1-y)^N]},\qquad y=\cos^2\frac{\omega}{2}. \tag{11}$$

For $C=0$, (11) becomes

$$| F(e^{-i\omega}) |^2=P_N(1-y),\quad y=\cos^2\frac{\omega}{2}. \tag{12}$$

The I-type filters determined by (12) are just Daubechies' polynomial filters[1].

3. Regularity Analysis

The function space C^α is defined as $C^\alpha=\{f(x)\mid\int_R\hat{f}(\omega)(1+\mid\omega\mid^\alpha)d\omega<\infty\}$. Denote $B_k=\sup_{\omega\in R}\mid F(e^{-i\omega})F(e^{-i\omega\frac{\omega}{2}})\dots F(e^{-i\frac{\omega}{2^{k-1}}})\mid$ and ϕ_N the scaling function corresponding to the filter $H(\omega)=(\frac{1+e^{-i\omega}}{2})^N F(e^{-i\omega})$. On the basis of the following estimate[2]

$$| \hat{\phi}_N(\omega) |\leq D(1+|\omega|)^{-N+\log B_k/(k\log 2)}\qquad (D \ \text{is a constant}), \tag{13}$$

Daubechies has approached polynomial filters and given that

$$\phi_N(x)\in C^{(\mu-\epsilon)N}\qquad (\text{see } [2])$$

where $\mu_N\leq 1-\frac{\log 3}{2\log 2}+O(\frac{\log N}{N})\approx 0.2075+O(\frac{\log N}{N})$.

Now, we will point out that the larger μ_N for a certain family of rational filters can be got. In fact, by (11) ,we lead to

$$| F(e^{-i\omega}) |= \sqrt{\frac{P_N(1-y)+C}{1+C[y^N+(1-y)^N]}} < \sqrt{\frac{P_N(1)+C}{2^{-(N-1)}C}} < 2^{\frac{N-1}{2}}(1+\frac{P_N(1)}{2C}).$$

Hence,

$$\log | F(e^{-i\omega}) |< \frac{N-1}{2}\log 2+\log(1+\frac{P_N(1)}{2C}) < \frac{N-1}{2}\log 2+\frac{P_N(1)}{2C}. \qquad (14)$$

It follows that $\phi_N(x) \in C^{(\mu_N-\epsilon)N}$,here

$$\mu_N = 0.5 - \frac{1}{2N} + \frac{P_N(1)}{2NC\log 2}$$

and $\epsilon > 0$ is an arbitrary small quantity.

(14) is only an asymptotic estimation about constant C. However, for the $F(e^{-i\omega})$ determined by (8), $B_1 = 2^{\frac{N-1}{2}}$ so that $\phi_N(x) \in C^{(\mu_N-\epsilon)N}$,where $\mu_N = 0.5 - \frac{1}{2N}$.

4. Examples

Example 1. For $N = 3$, the I-type filter determined by (11) is

$$H_I(\omega) = (\frac{1+e^{-i\omega}}{2})^3 \frac{a_0+a_1e^{-i\omega}+a_2e^{-i2\omega}}{b_0+b_2e^{-i2\omega}} \qquad (15)$$

where

$$\begin{aligned} a_0 &= (\sqrt{1+C}+\sqrt{10+C}+\sqrt{5+2C+2\sqrt{10+11C+C^2}})/4 \\ a_1 &= (\sqrt{1+C}-\sqrt{10+C})/2 \\ a_2 &= \frac{3}{2(\sqrt{1+C}+\sqrt{10+C}+\sqrt{5+2C+2\sqrt{10+11C+C^2}})} \\ b_0 &= (\sqrt{1+C}+\sqrt{1+C/4})/2, \quad b_2 = \frac{3C}{16b_0}. \end{aligned}$$

In order to obtain sequence $\{h_k\}$, we expanse (15) to be a power series. For $C = 10000$, the scaling function $\phi(x)$ and wavelet function $\psi(x)$ corresponding to (15) are shown in Figure 1 . They are smoother than Daubechies' for $N = 3$.

Expmple 2. For $N = 3$, the I-type filter determined by (8) is

$$H_I(\omega) = (\frac{1+e^{-i\omega}}{2})^3 \frac{4}{3+e^{-i2\omega}} = \frac{1}{2}[\frac{1}{3}+e^{-i\omega}+\sum_{k=0}^{\infty}\frac{8}{9}(\frac{1}{3})^k e^{-i(2k+2)\omega}]. \qquad (16)$$

The II-type filter determined by (8) is

$$H_{II}(\omega) = (\frac{1+e^{-i\omega}}{2})^3 \mid F(e^{-i\omega}) \mid .$$

where $| F(e^{-i\omega}) | = \frac{2}{\sqrt{1+3\cos^2\omega}}$. By the Fourier series expansion,

$$| F(e^{-i\omega}) | = \sum_{k\in Z} f_k e^{-ik\omega}$$

here

$$f_{-k} = f_k = \frac{2}{\pi}\int_0^\pi \frac{\cos k\omega}{\sqrt{1+3\cos^2\omega}} d\omega, \quad k = 0, 1, \ldots$$

The scaling function $\phi(x)$ and the wavelet function $\psi(x)$ corresponding to the $H_{II}(\omega)$ are shown in Figure 2. $\phi(x)$ is symmetric and $\psi(x)$ is antisymmetric.

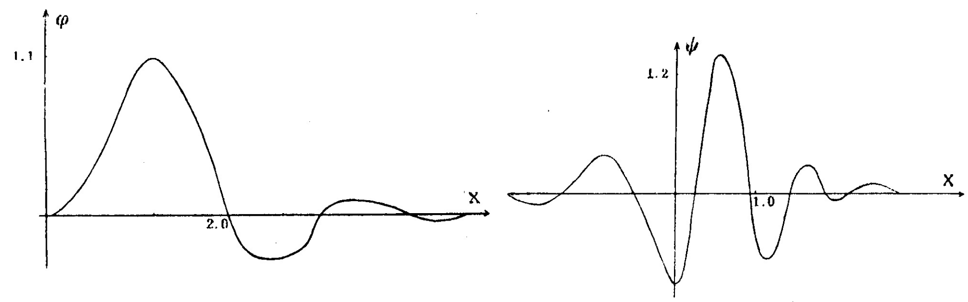

Fig. 1. The scaling function ϕ and wavelet function ψ corresponding to the I-type rational filter of Example 1 with C=10000.

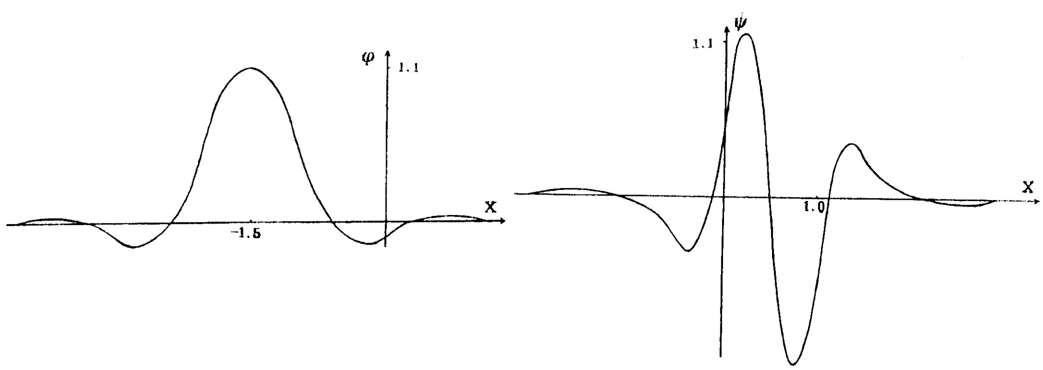

Fig. 2. The scaling function ϕ and wavelet function ψ corresponding to the II-type rational filter of Example 2.

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